Dissipative relativistic fluid theories of divergence type

Robert Geroch
Enrico Fermi Institute, University of Chicago, Chicago, Illinois 60637

Lee Lindblom
Department of Physics, Montana State University, Bozeman, Montana 59717
(Received 30 October 1989)

We investigate the theories of dissipative relativistic fluids in which all of the dynamical equations can be written as total-divergence equations. Extending the analysis of Liu, Müller, and Ruggeri, we find the general theory of this type. We discuss various features of these theories, including the causality of the full nonlinear evolution equations and the nature and stability of the equilibrium states.

I. INTRODUCTION

The earliest attempts[1,2] to formulate theories that describe the evolution of a dissipative fluid in a manner consistent with relativity theory were based on the assumption that the usual fluid variables (a four-velocity and two thermodynamic variables) were the appropriate dynamical fields for such a fluid. All of these early theories are now known[3] to be physically unacceptable, because they fail to provide causal evolution equations, and they do not admit physically acceptable equilibrium states. More recent efforts[4–7] to formulate acceptable dissipative relativistic fluid theories have involved extending the set of dynamical variables to be the entire stress-energy tensor T^{ab} and the conserved particle-number current N^a of the fluid. A candidate theory of this type consists of the differential equations for the evolution of these two tensor fields.

Among these candidates for a relativistic theory of a dissipative fluid are those of divergence type—those in which all of the dynamical equations for T^{ab} and N^a can be written as total-divergence equations. These theories were proposed originally by Liu, Müller, and Ruggeri.[7] This class of theory is of particular interest for at least two reasons. First, discontinuous solutions of the fluid equations can be given mathematical meaning in theories of divergence type. As a consequence, it is possible to consider in these theories solutions that describe shock waves, domain boundaries, etc. Second, the equations for fluid theories of divergence type can be cast into a remarkably simple mathematical form. Because of this form, for example, it is straightforward to determine the conditions under which the full nonlinear evolution equations for the theory are causal.

In this paper we investigate several aspects of the dissipative relativistic fluid theories of divergence type. In Sec. II we obtain, by extending the analysis of Liu, Müller, and Ruggeri,[7] the general fluid theory of this type. The key object that determines the dynamical properties of such a theory is a single scalar "generating function" of the dynamical variables. In Appendix A we extend this analysis to more general systems of differential equations of divergence type. We explore several examples of these theories in Appendix B and Sec. III. In Appendix B we show how an ordinary relativistic perfect fluid can be formulated in terms of such a generating function. In Sec. III we show that the original dissipative relativistic fluid theory of Eckart[1] can be written as a theory of divergence type, and we find its generating function. We also give an example of a theory of this type whose evolution equations are causal (i.e., have a well-posed initial-value formulation with all fluid signals having speeds not exceeding that of light) for fluid states near equilibrium. This theory, and a similar but more complicated example proposed by Liu, Müller, and Ruggeri,[7] are known to be causal in an open set of states surrounding the equilibrium states. In contrast, the theories of the type proposed by Israel and Stewart[5,6] are not known to be symmetric (let alone causal) for any open set of fluid states.[8,9]

The last two sections are devoted to an analysis of the near-equilibrium states of a dissipative relativistic fluid theory of divergence type. In Sec. IV we show that the equilibrium states for such a fluid must have the same properties as those of the Eckart theory: rigid flow and constant (generalized) temperature. In Sec. V we show that causality of the underlying theory implies stability of the equilibrium states. This interesting relation between causality and stability is similar to a weaker relation for the dissipative fluid-theory proposed by Israel and Stewart.[5,6] In that case it is known[8,9] that the causality of the equations for linear perturbations off an equilibrium state implies the stability of that state.

II. A REVIEW OF DISSIPATIVE-FLUID THEORIES OF DIVERGENCE TYPE

Consider a fluid theory having the following three properties. (i) The dynamical variables can be taken to be the particle-number current N^a and the (symmetric) stress-energy tensor T^{ab}. (ii) The dynamical equations are, in addition to the conservation laws,

$$\nabla_m N^m = 0 \ , \tag{1}$$

$$\nabla_m T^{ma} = 0 \ , \tag{2}$$

an equation of the form

$$\nabla_m A^{mab} = I^{ab} \ . \tag{3}$$

Here, the tensors A^{mab} and I^{ab} are algebraic functions of the dynamical variables N^a and T^{ab}, and are trace-free and symmetric in the indices a and b. [The latter will ensure that Eqs. (1)–(3) give the correct number of dynamical equations.] (iii) There exists an entropy current s^a (an algebraic function of N^a and T^{ab}), which, as a consequence of the Eqs. (1)–(3), satisfies an entropy law of the form

$$\nabla_m s^m = \sigma \ , \tag{4}$$

where σ is some algebraic function of N^a and T^{ab}. Such theories are said to be of *divergence type* because the left sides of Eqs. (1)–(3) are all divergences. These properties generalize slightly those required by Liu, Müller, and Ruggeri[7] in that we do *not* require that A^{mab} be totally symmetric.

The general theory having these three properties is determined by specifying a single scalar "generating function" χ and the tensor I^{ab} as algebraic functions of a new set of dynamical variables[10] ζ, ζ_a, and ζ_{ab} (with the latter trace-free and symmetric). The dynamical equations for these new variables are Eqs. (1)–(3), with

$$N^a = \frac{\partial^2 \chi}{\partial \zeta \, \partial \zeta_a} \ , \tag{5}$$

$$T^{ab} = \frac{\partial^2 \chi}{\partial \zeta_a \partial \zeta_b} \ , \tag{6}$$

and

$$A^{abc} = \frac{\partial^2 \chi}{\partial \zeta_a \partial \zeta_{bc}} \ . \tag{7}$$

The entropy current in this theory is

$$s^a = \frac{\partial \chi}{\partial \zeta_a} - \zeta N^a - \zeta_b T^{ab} - \zeta_{bc} A^{abc} \tag{8}$$

and its source is

$$\sigma = -\zeta_{ab} I^{ab} \ . \tag{9}$$

These fields satisfy the entropy law, Eq. (4), as a consequence of Eqs. (1)–(3). The transformation between the old dynamical variables N^a and T^{ab} and the new dynamical variables ζ, ζ_a, and ζ_{ab} is given by Eqs. (5) and (6). This transformation will, in the generic case, be nondegenerate, and so the theory will in general have property (i).

To see that this is indeed the general theory having the three properties stated above, consider any theory having these properties. In order that the entropy law [Eq. (4)] be a consequence of Eqs. (1)–(3), the divergence of s^a must be some linear combination of the left sides of those equations. Let $-\zeta$, $-\zeta_a$, and $-\zeta_{ab}$ represent the coefficients in this linear combination, so that

$$\nabla_m s^m = -\zeta \nabla_m N^m - \zeta_a \nabla_m T^{ma} - \zeta_{ab} \nabla_m A^{mab} \tag{10}$$

for all possible choices of the fields. These coefficients will be taken as the new dynamical fields. Next, introduce the vector χ^a, an algebraic function of ζ, ζ_a, and ζ_{ab}, defined by

$$\chi^a = s^a + \zeta N^a + \zeta_b T^{ab} + \zeta_{bc} A^{abc} \ . \tag{11}$$

Compute the divergence of χ^a [using Eq. (10)] to obtain the equation

$$0 = \left[\frac{\partial \chi^m}{\partial \zeta} - N^m \right] \nabla_m \zeta + \left[\frac{\partial \chi^m}{\partial \zeta_a} - T^{ma} \right] \nabla_m \zeta_a + \left[\frac{\partial \chi^m}{\partial \zeta_{ab}} - A^{mab} \right] \nabla_m \zeta_{ab} \ . \tag{12}$$

Since Eq. (12) holds for all possible choices of the dynamical fields, the coefficients of the gradients of these fields must vanish. It follows that N^a, T^{ab}, and A^{abc} are given by the corresponding partial derivatives of the vector χ^a. Further, symmetry of the stress-energy tensor implies that the vector χ^a is "curl-free" with respect to ζ_b, and therefore may be written as the "ζ_a gradient" of a scalar field $\chi(\zeta, \zeta_a, \zeta_{ab})$:

$$\chi^a = \frac{\partial \chi}{\partial \zeta_a} \ . \tag{13}$$

Replacing χ^a in favor of this single generating function χ, Eqs. (5)–(7) follow. In Appendix A we discuss the extension of this argument to more general systems of differential equations of divergence type.

Thus, the general fluid theory having properties (i)–(iii) is determined by specifying the two algebraic functions χ and I^{ab} of the three variables ζ, ζ_a, and ζ_{ab}. The argument of the previous paragraph shows that a given fluid theory initially in the form of Eqs. (1)–(4) has a unique representation in terms of these new variables. These variables consequently have physical significance. We explore this significance through several examples in Sec. III and Appendix B.

In order to make more transparent the dynamical structure of Eqs. (1)–(3) it will be helpful to introduce ζ_A to represent the entire collection of dynamical variables: $\zeta_A = (\zeta, \zeta_a, \zeta_{ab})$. Similarly we introduce I^A to represent the dissipation-source tensor: $I^A = (0, 0, I^{ab})$. Thus an upper case index, such as A, covers a total of 14 dimensions. Equations (1)–(3) may be written in this notation as

$$\frac{\partial^2 \chi^m}{\partial \zeta_A \partial \zeta_B} \nabla_m \zeta_B \equiv M^{ABm} \nabla_m \zeta_B = I^A \ . \tag{14}$$

This first-order system of equations for ζ_B is called *symmetric* because M^{ABm} is symmetric in the indices A and B (a consequence, in turn, of the fact that partial derivatives commute). A symmetric system is *hyperbolic* in an open set of fluid states (i.e., has a well-posed initial-value formulation there) if $M^{ABm} w_m$ is negative-definite for *some* (possibly state-dependent) future-directed timelike w_m (Ref. 11). A symmetric system is *causal* in an open set of fluid states (i.e., hyperbolic with no fluid signals propagating faster than light) if $M^{ABm}w_m$ is negative definite for *all* future-directed timelike w_m.

These properties—hyperbolicity and causality—may or may not hold for a particular fluid theory, depending on the generating function for that theory. It is of course natural on physical grounds to demand causality. However, it is appropriate to demand causality only for the physical states of the fluid, and these are only some open set of the dynamical variables ζ, ζ_a, and ζ_{ab}. These variables could be restricted to those for which N^a is future-directed timelike, for example, or to those for which T^{ab} has positive energies or future-directed momenta. It is also natural on physical grounds to require that the entropy source σ be non-negative.

III. EXAMPLES

We now consider two examples of dissipative relativistic fluid theories of divergence type. These examples serve to illustrate the physical significance of the new dynamical variables, ζ, ζ_a, and ζ_{ab} that were introduced in Sec. II.

The first example is the Eckart theory,[1] the first dissipative relativistic fluid theory. The equations of this theory can be cast into the form of Eqs. (1)–(4), by setting

$$N^a = nu^a \ , \tag{15}$$

$$T^{ab} = (\rho+p)u^a u^b + pg^{ab} + 2u^{(a}q^{b)} + \tau^{ab} + \tau(g^{ab} + u^a u^b) \ , \tag{16}$$

$$A^{mab} = 2Tu^{(a}(g^{b)m} + u^{b)}u^m) \ , \tag{17}$$

$$I^{ab} = -\frac{T}{\eta_1}\tau^{ab} - \frac{2T}{3\eta_2}(g^{ab} + 4u^a u^b)\tau - \frac{2}{\kappa}q^{(a}u^{b)} \ , \tag{18}$$

$$s^a = snu^a + \frac{1}{T}q^a \ , \tag{19}$$

$$\sigma = \frac{\tau^2}{\eta_2 T} + \frac{q^a q_a}{\kappa T^2} + \frac{\tau^{ab}\tau_{ab}}{2\eta_1 T} \ . \tag{20}$$

Here the fluid four-velocity u^a is a future-directed unit timelike vector; the heat flow q^a satisfies $u_a q^a = 0$; the stress τ^{ab} satisfies $\tau^{ab}u_b = 0$ and $\tau^a_a = 0$; and, the viscosity coefficients η_1 and η_2 and the thermal conductivity κ are positive functions of n (the particle number density) and ρ (the energy density of the fluid). Tensor indices are raised and lowered with the spacetime metric g_{ab} (and its inverse g^{ab}); and parentheses surrounding indices indicate symmetrization, e.g., $\tau^{(ab)} = \frac{1}{2}(\tau^{ab} + \tau^{ba})$. The five thermodynamic variables n, ρ, T (the temperature), s (the entropy per particle), and p (the pressure) are related by the first law of thermodynamics:

$$d\rho = nT\, ds + \frac{\rho+p}{n}dn \ . \tag{21}$$

This equation implies that the relationship of these five thermodynamic variables is determined by a single function of two of the variables (the equation of state). Equations (15) and (16) can be considered to be the definitions of n, ρ, u^a, q^a, τ, and τ^{ab} in terms of N^a and T^{ab}. Then Eqs. (17)–(20) are simply the expressions for A^{mab}, I^{ab}, s^a, and σ as algebraic functions of N^a and T^{ab}.

Having written the Eckart theory in the form of Eqs. (1)–(4), the analysis of Sec. II shows that these equations are a manifestly symmetric system when expressed in terms of the new variables ζ, ζ_a, and ζ_{ab}, and that this system of equations arises from some generating function. It is not difficult to show that the generating function χ in this case is

$$\chi = \alpha(\zeta,\mu) - \frac{\zeta^{ab}\zeta_a\zeta_b}{\mu} \ , \tag{22}$$

where $\mu = \zeta^a\zeta_a$, and α is an arbitrary smooth function (which plays the role of the equation of state). By comparing Eqs. (5)–(8) with Eqs. (15)–(17) and (19) we find that the new dynamical variables ζ_A are related to the Eckart variables by the equations

$$\zeta = \frac{\rho+p}{nT} - s \ , \tag{23}$$

$$\zeta^a = \frac{u^a}{T} \ , \tag{24}$$

$$\zeta^{ab} = \frac{1}{2T^2}\left[\tau^{ab} - 2u^{(a}q^{b)} + \frac{\tau}{4}(g^{ab} + 4u^a u^b)\right] \ , \tag{25}$$

$$T^2 = -\frac{1}{\mu} \ , \tag{26}$$

$$nT = 2\frac{\partial^2\alpha}{\partial\mu\,\partial\zeta} \ , \tag{27}$$

$$p = 2\frac{\partial\alpha}{\partial\mu} \ , \tag{28}$$

$$\rho + p = -4\mu\frac{\partial^2\alpha}{\partial\mu^2} \ . \tag{29}$$

The five thermodynamic variables ρ, n, s, T, and p as defined in Eqs. (23) and (26)–(29) satisfy the first law of thermodynamics, Eq. (21), identically. We note that ζ is the thermodynamic variable whose gradient vanishes in the absence of heat flow in the Eckart theory; ζ_a is proportional to the particle number current of the fluid; and ζ_{ab} contains both the heat flow and viscous stress of the fluid.

While the Eckart theory is an example of a dissipative relativistic fluid theory of divergence type, its equations are not causal and so it cannot be considered to be a satisfactory physical theory. Indeed the quadratic form $M^{ABm}w_m Z_A Z_B$ with $Z_A = (Z, Z_a, Z_{ab})$ fails to be negative in the Eckart theory because $\partial^3\chi/\partial\zeta_m\partial\zeta_{ab}\partial\zeta_{cd}$ vanishes. But this theory can easily be modified to correct this defect. For our second example we consider the theory whose generating function is taken to be

$$\chi = \alpha(\zeta,\mu) + \beta(\zeta,\mu)\zeta^{ab}\zeta_a\zeta_b + \chi_2(\zeta,\zeta_a,\zeta_{ab}) \ , \tag{30}$$

where we have set

$$\chi_2 = \frac{\gamma(\zeta,\mu)}{\mu^2}(\mu g_{ab} - 2\zeta_a\zeta_b)(\mu g_{cd} - 2\zeta_c\zeta_d)\zeta^{ac}\zeta^{bd} \ . \tag{31}$$

Thus, we have modified the Eckart theory by generalizing the term linear in ζ_{ab} and adding the term χ_2, which is (when ζ_a is timelike) a positive-definite quadratic form in ζ_{ab}.

To investigate the causality of this theory, we must examine the quadratic form $M^{ABm}w_m Z_A Z_B$ for arbitrary future-directed timelike w_m. For simplicity, we limit our consideration here to states of the fluid having $\zeta_{ab}=0$. (As the analysis of Sec. IV shows, these are the equilibrium states of the fluid.) For these states ζ^a is proportional to the number current N^a and thus will be timelike for physical states of the fluid. The full quadratic form $M^{ABm}w_m Z_A Z_B$ consists of the block involving the variables (Z, Z_a), another block involving Z_{ab}, and the cross terms between these blocks. The (Z, Z_a) block of the quadratic form is negative (for fluid states having $\zeta_{ab}=0$) provided the perfect-fluid causality conditions (B6) and (B7) are satisfied. The Z_{ab} block of the quadratic form

$$w_m \frac{\partial^3 \chi_2}{\partial \zeta_m \partial \zeta_{ab} \partial \zeta_{cd}} Z^{ab} Z^{cd} \tag{32}$$

is negative for all Z^{ab} whenever w_m and ζ_a are future-directed timelike vectors and γ satisfies

$$\frac{\partial \gamma}{\partial \mu} \geq \left| \frac{2\gamma}{\mu} \right| , \tag{33}$$

$$\frac{\partial \gamma}{\partial \mu} > 0 . \tag{34}$$

It follows that the quadratic form $M^{ABm}w_m Z_A Z_B$ is negative definite for all fluid states having $\zeta_{ab}=0$ provided the perfect-fluid causality conditions (B6) and (B7) are satisfied and $\partial\gamma/\partial\mu$ is sufficiently large to ensure that the Z_{ab}-block dominates the cross terms. (It is straightforward to determine explicitly the required condition on $\partial\gamma/\partial\mu$, but it is rather complicated.) Since M^{ABm} is symmetric and a continuous function of ζ_A for all states of the fluid, it follows that the theory is causal for all sufficiently small ζ_{ab} (i.e., in some open neighborhood of the equilibrium states). The causal properties of this theory, and the more complicated example of Liu, Müller, and Ruggeri,[7] are considerably better understood, nevertheless, than those of any of the theories of the type proposed by Israel and Stewart.[5,6] In those theories it is not even known[8,9] whether the equations form a symmetric (let alone causal) system for any open set of fluid states.

IV. EQUILIBRIUM STATES

A dissipative physical system is thought of as being in an equilibrium state if its dynamics is time reversible. In terms of the dynamical variables of these fluid theories, we take *time reverse* to mean $N^a \to -N^a$ and $T^{ab} \to T^{ab}$. That is, a current reverses sign while a flux of a current remains unchanged. Thus, we call a solution N^a, T^{ab} of Eqs. (1)–(3) in some region of spacetime an *equilibrium solution* if $-N^a$, T^{ab} is also a solution in that region. Since the tensors A^{mab} and I^{ab} are algebraic functions of N^a and T^{ab}, it follows that $A^{mab}(-N^c, T^{de}) = -A^{mab}(N^c, T^{de})$ and $I^{ab}(-N^c, T^{de}) = I^{ab}(N^c, T^{de})$. Therefore a solution of Eqs. (1)–(3) is an equilibrium solution if and only if $I^{ab}=0$. Note in particular that an equilibrium solution has, by Eq. (9), vanishing entropy production density σ.

For the discussion of the equilibrium states that follows, we make two additional assumptions about the dissipative-fluid theory. First, the entropy production density is non-negative: $\sigma \geq 0$. Second, the theory is "generic" in the sense that the derivatives of the generating function χ and of the dissipation source tensor I^{ab} satisfy a number of inequalities, which will be specified shortly. We shall find the general equilibrium state for a fluid theory satisfying these two assumptions.

Consider a particular equilibrium solution of the fluid equations (1)–(3). Since the entropy production density $\sigma = -\zeta_{ab} I^{ab}$ is assumed non-negative for *all* states of the fluid, its value in an equilibrium state (i.e., zero) is its minimum. It follows that the first variation of σ under arbitrary variations in ζ, ζ_a, and ζ_{ab} must also vanish at this equilibrium state: $\delta\sigma = -\zeta_{ab}\delta I^{ab} = 0$. We now require that the theory be generic in the sense that under such variations one may achieve for δI^{ab} an arbitrary symmetric trace-free tensor.[12] The vanishing of the first variation $\delta\sigma$ then requires $\zeta_{ab}=0$. Thus, in an equilibrium state of the theory the dynamical field ζ_{ab} must vanish.

We next obtain expressions for the various fluid tensor fields when $\zeta_{ab}=0$. We denote the value of an algebraic function of the dynamical variables, $Q(\zeta, \zeta_a, \zeta_{ab})$, evaluated at $\zeta_{ab}=0$, by Q_0. For the generating function χ and its ζ_{ab} derivative we have

$$\chi_0 = \alpha(\zeta, \mu) , \tag{35}$$

$$\left[\frac{\partial \chi}{\partial \zeta_{ab}} \right]_0 = \beta(\zeta, \mu)(\zeta^a \zeta^b - \tfrac{1}{4}\mu g^{ab}) , \tag{36}$$

where α and β are algebraic functions of ζ and $\mu = \zeta_a \zeta^a$. This follows since the right sides of Eqs. (35) and (36) are, respectively, the most general scalar and trace-free tensor that can be constructed as algebraic functions of ζ, ζ_a, and g_{ab}. (For example, the Eckart theory discussed in Sec. III satisfies these equations with $\beta = -1/\mu$.) The tensors N^m, s^m, T^{ma}, and A^{mab}, evaluated at $\zeta_{ab}=0$, have the form

$$N_0^m = 2\frac{\partial^2 \alpha}{\partial \mu\, \partial \zeta} \zeta^m , \tag{37}$$

$$s_0^m = -2 \left[\zeta \frac{\partial^2 \alpha}{\partial \mu\, \partial \zeta} + 2\mu \frac{\partial^2 \alpha}{\partial \mu^2} \right] \zeta^m , \tag{38}$$

$$T_0^{ma} = 4\frac{\partial^2 \alpha}{\partial \mu^2} \zeta^m \zeta^a + 2\frac{\partial \alpha}{\partial \mu} g^{ma} , \tag{39}$$

$$A_0^{mab} = \tfrac{1}{2}\beta(4g^{m(a}\zeta^{b)} - \zeta^m g^{ab}) + \frac{1}{2}\frac{\partial \beta}{\partial \mu} \zeta^m (4\zeta^a \zeta^b - \mu g^{ab}) , \tag{40}$$

when evaluated in a state having $\zeta_{ab}=0$ using Eqs. (5)–(8) and (35)–(36). Note that the right sides of Eqs. (37)–(39) are precisely the forms of the perfect-fluid expressions for the particle number current, entropy current, and stress-energy tensor, respectively. (See, for example, Appendix B.) Identifying these expressions with the standard ones, we obtain formulas for the thermodynamic variables ρ, p, n, and s as various derivatives of α. The temperature T is determined by using these expressions and the first law of thermodynamics, Eq. (21). The resulting formulas are the same as those of the Eckart theory, Eqs. (23) and (26)–(29).

In addition to these algebraic conditions on the fundamental tensors in an equilibrium state, there also exist as a consequence of the dynamical equations (1)–(3), differential conditions. These are, in the present case, the vanishing of the divergence (on the index m) of each of the tensors in Eqs. (37), (39), and (40). The three scalar equations $\nabla_m N_0^m = \zeta_a \nabla_m T_0^{ma} = \zeta_a \zeta_b \nabla_m A_0^{mab} = 0$ each involve linear combinations of $\nabla_m \zeta^m$, $\zeta^m \nabla_m \zeta$, and $\zeta^m \nabla_m \mu$:

$$\frac{\partial^2\alpha}{\partial\mu\,\partial\zeta}\nabla_m\zeta^m + \frac{\partial^3\alpha}{\partial\mu\,\partial\zeta^2}\zeta^m\nabla_m\zeta + \frac{\partial^3\alpha}{\partial\mu^2\partial\zeta}\zeta^m\nabla_m\mu = 0 \; , \quad (41)$$

$$2\mu\frac{\partial^2\alpha}{\partial\mu^2}\nabla_m\zeta^m + \left[2\mu\frac{\partial^3\alpha}{\partial\mu^2\partial\zeta} + \frac{\partial^2\alpha}{\partial\mu\,\partial\zeta}\right]\zeta^m\nabla_m\zeta + 2\left[\mu\frac{\partial^3\alpha}{\partial\mu^3} + \frac{\partial^2\alpha}{\partial\mu^2}\right]\zeta^m\nabla_m\mu = 0 \; , \quad (42)$$

$$\left[3\mu^2\frac{\partial\beta}{\partial\mu} - \mu\beta\right]\nabla_m\zeta^m + 3\mu\left[\frac{\partial\beta}{\partial\zeta} + \mu\frac{\partial^2\beta}{\partial\mu\,\partial\zeta}\right]\zeta^m\nabla_m\zeta + \left[3\mu^2\frac{\partial^2\beta}{\partial\mu^2} + 6\mu\frac{\partial\beta}{\partial\mu} + 2\beta\right]\zeta^m\nabla_m\mu = 0 \; . \quad (43)$$

We now require that the theory be generic in the sense that the 3×3 matrix of coefficients of the derivative terms in these equations have nonvanishing determinant. It follows that the only solution is the vanishing of these derivative terms:

$$\nabla_m\zeta^m = \zeta^m\nabla_m\zeta = \zeta^m\nabla_m\mu = 0 \; . \quad (44)$$

The two vector equations $\nabla_m T_0^{ma} = \zeta_b \nabla_m A_0^{mab} = 0$ each involve [using Eqs. (44)] linear combinations of $\nabla_b\zeta$ and $\zeta^a(\nabla_a\zeta_b + \nabla_b\zeta_a)$:

$$2\frac{\partial^2\alpha}{\partial\mu^2}\zeta^a(\nabla_a\zeta_b + \nabla_b\zeta_a) + \frac{\partial^2\alpha}{\partial\mu\,\partial\zeta}\nabla_b\zeta = 0 \; , \quad (45)$$

$$\left[\beta + 2\mu\frac{\partial\beta}{\partial\mu}\right]\zeta^a(\nabla_a\zeta_b + \nabla_b\zeta_a) + \mu\frac{\partial\beta}{\partial\zeta}\nabla_b\zeta = 0 \; . \quad (46)$$

We now require that the theory be generic in the sense that the 2×2 matrix of coefficients of the derivative terms in these equations have nonvanishing determinant. It follows that the only solution is the vanishing of these derivative terms:

$$\zeta^a(\nabla_a\zeta_b + \nabla_b\zeta_a) = \nabla_b\zeta = 0 \; . \quad (47)$$

The single tensor equation $\nabla_m A_0^{mab} = 0$ reduces [using Eqs. (44) and (47)] to

$$\beta(\nabla_a\zeta_b + \nabla_b\zeta_a) = 0 \; . \quad (48)$$

We require, finally, that the theory be generic in the sense that β is nonvanishing. It follows that ζ_a is a Killing vector field [i.e., satisfies Eq. (48)].

To summarize, in an equilibrium state of the fluid ζ_a must be a Killing vector field and ζ must be a constant. We note that these are precisely the equilibrium conditions found for the standard Eckart theory.[13]

V. STABILITY OF THE EQUILIBRIUM STATES

In this final section we study the stability of the equilibrium states of a dissipative-fluid theory of divergence type. We demonstrate that equilibrium states are stable in any theory of this type having causal evolution equations.

Consider a smooth one-parameter family $\zeta_A(\lambda)$ of solutions of the fluid equations which for $\lambda=0$ is an equilibrium solution. (We assume for simplicity that the spacetime metric g_{ab} is independent of λ.) Denote as $\delta\zeta_A$ the derivative of this family with respect to λ, evaluated at $\lambda=0$. To determine the evolution of this perturbation, we differentiate the fluid equations (14) with respect to λ and evaluate the result at $\lambda=0$:

$$\frac{\partial^3\chi}{\partial\zeta_m\partial\zeta_A\partial\zeta_B}\nabla_m\delta\zeta_B + \frac{\partial^4\chi}{\partial\zeta_m\partial\zeta_A\partial\zeta_B\partial\zeta_C}\delta\zeta_C\nabla_m\zeta_B = \delta I^A \; . \quad (49)$$

The second term on the left side of Eq. (49) contains $\nabla_m\zeta_B$, the derivative of the equilibrium fields. But this field is very simple. We have $\nabla_m\zeta_B = (0, \nabla_m\zeta_b, 0)$ using the fact that, in equilibrium ζ is a constant and ζ_{ab} vanishes. It follows that the second term on the left side of Eq. (49) vanishes, for $\nabla_m\zeta_b$ is antisymmetric [from Eq. (48)] while the partial derivatives of χ are symmetric. Thus the evolution equation for a perturbation about an equilibrium state is

$$\frac{\partial^3\chi}{\partial\zeta_m\partial\zeta_A\partial\zeta_B}\nabla_m\delta\zeta_B = \delta I^A \; . \quad (50)$$

Note that the coefficient of the derivative on the left side in Eq. (50) is precisely the coefficient M^{ABm} that appears in Eq. (14). It follows immediately that a dissipative-fluid theory of divergence type is hyperbolic (respectively, causal) in a neighborhood of an equilibrium state if and only if the equation for perturbations off this equilibrium state is hyperbolic (respectively, causal).

In order to investigate the stability of these fluids we introduce the "energy current"

$$E^m = \frac{1}{2}\frac{\partial^3\chi}{\partial\zeta_m\partial\zeta_A\partial\zeta_B}\delta\zeta_A\delta\zeta_B \; . \quad (51)$$

The divergence of this current is easily computed using Eq. (50):

$$\nabla_m E^m = \delta\zeta_A\delta I^A = \delta\zeta_{ab}\delta I^{ab} \le 0 \; . \quad (52)$$

The inequality in Eq. (52) follows from the fact that $\delta\zeta_{ab}\delta I^{ab}$ is the second derivative of $-\sigma = \zeta_{ab} I^{ab}$, a field that achieves its maximum at an equilibrium state.

We now assume (i) that the fluid theory is causal and

(ii) that the background equilibrium state is in a spacetime that admits Cauchy surfaces. For a given Cauchy surface Σ, we define the "energy" associated with perturbations of the fluid by

$$E(\Sigma)=-\int_{\Sigma}E^{m}dS_{m}\ , \tag{53}$$

where dS_m is the proper three-volume element for Σ. By causality, this energy is positive for any nonvanishing perturbation of a fluid. But this energy is also nonincreasing into the future, for, from Eq. (53),

$$E(\Sigma_2)-E(\Sigma_1)=\int_{\Omega}\delta\zeta_{ab}\delta I^{ab}d\Omega\leq 0\ , \tag{54}$$

where Σ_2 is a Cauchy surface to the future of Σ_1 and Ω is the spacetime region in between. We conclude that the energy is bounded into the future.

Thus, the perturbations off an equilibrium state are stable in the sense that every perturbation must evolve keeping the norm $E(\Sigma)$ bounded.

VI. CONCLUSIONS

We suggest two possible generalizations of the analyses presented here. First, it seems likely that there exists an example of a dissipative-fluid theory of divergence type that is causal for all physical states of the fluid. Can one, for example, choose the functions β and γ in Eq. (30) to achieve this? Second, it seems likely that there exists a converse of the result obtained in Sec. V, relating causality to the stability of the fluid. Several references[9,15] have argued (in the context of other dissipative-fluid theories) that if a fluid is stable, then its equations must, essentially, be causal. But these arguments fail to be complete proofs because (i) they fail to show the existence of a sufficient number of linearly independent solutions to the perturbation equations and (ii) they fail to show that the solutions of the perturbation equations evolve toward equilibrium states in the appropriate sense. Is it possible to find an argument that corrects these defects?

Note added in proof. We have learned that the analysis leading to Eq. (13) has been published previously by S. Pinnisi in *Symposium on Kinetic Theory and Extended Thermodynamics,* edited by I. Müller and T. Ruggeri (Pitagora Editrice, Bologna, 1989).

ACKNOWLEDGMENTS

This research was supported by Grants Nos. PHY85-18490 and PHY84-16691 from the National Science Foundation.

APPENDIX A: GENERAL SYSTEMS OF EQUATIONS OF DIVERGENCE TYPE

The analysis of Sec. II can be applied to much more general systems of differential equations than the dissipative-fluid theories considered there.[14] Consider any system of first-order differential equations on tensor fields, which are in divergence form

$$\nabla_m M^{Am}=I^{A}\ , \tag{A1}$$

where the index A runs over the set of tensor equations of this system. Let there follow as a consequence of these equations, one additional equation of the form

$$\nabla_m s^{m}=\sigma\ . \tag{A2}$$

The argument given in Sec. II demonstrates that, in the generic case, there exists a function χ^m of a set of tensor fields ζ_B (with $\chi^m=s^m+\zeta_A M^{Am}$) in terms of which this system can be recast in the form

$$\frac{\partial^2\chi^m}{\partial\zeta_A\partial\zeta_B}\nabla_m\zeta_B\equiv M^{ABm}\nabla_m\zeta_B=I^{A}\ . \tag{A3}$$

In Sec. II we were able to go one step further and replace χ^m with the scalar generating function χ by setting $\chi^m=\partial\chi/\partial\zeta_m$. This is possible whenever one of the M^{Am}'s happens to be a symmetric second-rank tensor, e.g., whenever the conservation of stress energy is one of the equations.

APPENDIX B: PERFECT FLUIDS

We now consider the case of a perfect (i.e., nondissipative) relativistic fluid. The perfect-fluid equations are one example of the general system of equations of divergence type discussed in Appendix A. The fields in this theory are u^a, a future-directed unit timelike vector field, and n and ρ, two scalar fields. The equations for this theory are the vanishing of the divergence of $N^a=nu^a$ and $T^{ab}=(\rho+p)u^a u^b+pg^{ab}$, where p is any smooth algebraic function of n and ρ and g^{ab} is the (inverse) metric of spacetime. These equations, written out in detail, are

$$u^m\nabla_m n+n\nabla_m u^m=0\ , \tag{B1}$$

$$u^m\nabla_m\rho+(\rho+p)\nabla_m u^m=0\ , \tag{B2}$$

$$(\rho+p)u^m\nabla_m u^a+(g^{am}+u^a u^m)\nabla_m p=0\ . \tag{B3}$$

Next, let T and s be any smooth algebraic functions of n and ρ satisfying the first law of thermodynamics:

$$n^2Tds=nd\rho-(\rho+p)dn\ . \tag{B4}$$

(That there exist such functions follows from the fact that any one-form on a two-manifold is surface orthogonal.) It follows, as a consequence of Eqs. (B1), (B2), and (B4), that

$$\nabla_m(snu^m)=0\ . \tag{B5}$$

Thus, the equations for a perfect fluid are of the form (A1), and as a consequence of these equations there exists an additional equation of the form (A2). The analysis described in Appendix A can be applied, therefore, directly to this system. The field ζ_A consists in this case of one scalar field ζ and one vector field ζ_a. The generating function is an arbitrary function, $\chi=\alpha(\zeta,\mu)$, of ζ and $\mu=\zeta^a\zeta_a$. The source tensor I^A and the entropy production density σ vanish. Comparing Eq. (A3) with Eqs. (B1)–(B3) we obtain the same relationships between the standard perfect-fluid variables and the new variables ζ_A as those given by Eqs. (23), (24), and (26)–(29) (for the case of the Eckart fluid).

The system of equations for a perfect fluid have thus been cast into the form of Eq. (A3), which is manifestly symmetric. The system will also be causal if $M^{ABm}w_m$ is negative definite for every future-directed timelike w_m. The causality of these equations reduces therefore to a set of inequalities on the derivatives of the generating function $\alpha(\zeta,\mu)$. While it is straightforward to obtain these conditions on $\alpha(\zeta,\mu)$ directly, it is far simpler to write $M^{ABm}w_m$ first in terms of the standard fluid variables using Eqs. (23), (24), and (26)–(29). The necessary and sufficient conditions for causality are thus found to be

$$\rho+p>0\ , \qquad \text{(B6)}$$

$$\left[\frac{\partial\rho}{\partial p}\right]_\zeta > \left[\frac{\partial\rho}{\partial p}\right]_s \geq 1\ . \qquad \text{(B7)}$$

The inequality (B6) is simply an energy condition on the stress-energy tensor of the perfect fluid, while the second inequality in (B7) requires that the adiabatic sound speed be real and less than or equal to the speed of light. The first inequality in (B7) is equivalent to the condition that certain specific heats and compressibilities are positive;[8] these are the standard stability conditions for a perfect fluid.

It is of interest to compare this result on the causality of the form (A3) for the perfect-fluid equations with the answer to a slightly different question. What are the necessary and sufficient conditions for there to exist *some* causal form for the perfect-fluid equations? That is, when can Eqs. (B1)–(B3) be written in the form $M^{ABm}\nabla_m\psi_B=0$ (where ψ_B is some representation of the fluid variables) with $M^{ABm}w_m$ symmetric and negative definite for every future-directed timelike w_m? This question, it turns out, is easily answered. The key feature of this form of the equations is that the equations are labeled by the same type of index A as are the field variables. That is, we make an identification between equations and fields. Thus we wish to designate some linear combination of Eqs. (B1) and (B2) as the "n equation," some combination as the "ρ equation," and we take (B3) as the "u^a equation." That is, consider

$$k_1u^m\nabla_m n+k_2u^m\nabla_m\rho+[k_1n+k_2(\rho+p)]\nabla_m u^m=0\ , \qquad \text{(B8)}$$

$$k_3u^m\nabla_m n+k_4u^m\nabla_m\rho+[k_3n+k_4(\rho+p)]\nabla_m u^m=0\ , \qquad \text{(B9)}$$

$$(\rho+p)u^m\nabla_m u^a+(g^{am}+u^au^m)\times\left[\left[\frac{\partial p}{\partial n}\right]_\rho\nabla_m n+\left[\frac{\partial p}{\partial\rho}\right]_n\nabla_m\rho\right]=0\ . \qquad \text{(B10)}$$

One sees by inspection that this form of the equations for a perfect fluid is symmetric if and only if

$$k_2=k_3\ , \qquad \text{(B11)}$$

$$k_1n+k_2(\rho+p)=\left[\frac{\partial p}{\partial n}\right]_\rho\ , \qquad \text{(B12)}$$

$$k_3n+k_4(\rho+p)=\left[\frac{\partial p}{\partial\rho}\right]_n\ . \qquad \text{(B13)}$$

For causality, we must require that the M^{ABm} implicit in Eqs. (B8)–(B10) be negative definite when contracted with each future-directed timelike w_m. This condition is equivalent to certain algebraic inequalities on k_1, k_2, k_3, and k_4. These inequalities can be satisfied if and only if the fluid variables satisfy the conditions

$$\rho+p>0\ , \qquad \text{(B14)}$$

$$\left[\frac{\partial\rho}{\partial p}\right]_s\geq 1\ . \qquad \text{(B15)}$$

These are weaker than the conditions (B6) and (B7) that guarantee the causality of the perfect-fluid equations when written in terms of the special variables of Eq. (A3).

[1]C. Eckart, Phys. Rev. **58**, 919 (1940).
[2]L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon, Oxford, 1975), p. 499.
[3]W. A. Hiscock and L. Lindblom, Phys. Rev. D **31**, 725 (1985).
[4]I. Müller, Z. Phys. **198**, 329 (1967).
[5]W. Israel, Ann. Phys. (N.Y.) **100**, 310 (1976).
[6]W. Israel and J. M. Stewart, Ann. Phys. (N.Y.) **118**, 341 (1979).
[7]I. S. Liu, I. Müller, and T. Ruggeri, Ann. Phys. (N.Y.) **169**, 191 (1986).
[8]W. A. Hiscock and L. Lindblom, Ann. Phys. (N.Y.) **151**, 466 (1983).
[9]W. A. Hiscock and L. Lindblom, Contemp. Math. **71**, 181 (1988).
[10]I. S. Liu, Arch. Rational Mech. Anal. **46**, 131 (1972).
[11]See, for example, R. Courant and D. Hilbert, *Methods of Mathematical Physics. II. Partial Differential Equations* (Interscience, New York, 1962), p. 593.
[12]The Eckart theory is generic in this sense, as can be seen from Eq. (18). It is not difficult, however, to invent theories which fail this condition, e.g., let $I^{ab}=(4\zeta^a\zeta^b-\mu g^{ab})\zeta^m\zeta^n\zeta_{mn}$.
[13]See, for example, L. Lindblom, Astrophys. J. **208**, 873 (1976).
[14]K. O. Friedrichs and P. D. Lax, Proc. Natl. Acad. Sci. **68**, 1686 (1971).
[15]L. Lindblom, Astrophys. J. **267**, 402 (1983).