Oscillations and Stability of Rapidly Rotating Neutron Stars

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A method is described for obtaining numerical solutions of the pulsation equations of rapidly rotating inhomogeneous stellar models. This previously intractable problem has been solved by reexpressing the pulsation equations in terms of a single potential. These equations are solved and the points of the onset of secular instability to gravitational radiation are found. These results indicate that it is difficult to interpret the 0.5-ms period of SN 1987A cannot be interpreted as the rotation of a neutron star using current descriptions of neutron-star matter.

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The task of solving the equations for the normal modes of rapidly rotating Newtonian stellar models has remained an outstanding problem in astrophysics. In a variety of astrophysical contexts a need arises for knowledge of the fluid motions and pulsation frequencies associated with these normal models. This need has become especially pressing in light of the recent discovery of a pulsar with a 0.5-ms period[1] and two others with 1.6-ms periods.[2] For these objects, in particular, a knowledge of the frequencies of the normal modes is insufficient. The eigenfunctions are needed as well in order to determine the influence of gravitational radiation (GR) and viscosity on the evolution and stability of the star, and in order to determine the extent to which the pulsations can interact with an accretion disk. While in the case of rapidly rotating uniform-density stellar models the pulsation equations can be solved analytically,[3] in general the problem must be attacked numerically in realistic inhomogeneous stellar models. The equations have never been solved directly except in the special case of axisymmetric pulsations.[4] We solve this problem in general by transforming the equations into a form involving a single potential[5] δU instead of the Lagrangian displacement vector.[6] In this form the pulsation equations are easier to solve numerically. Our purpose here is to outline this method and to present numerical solutions that are relevant descriptions of the oscillations and stability of millisecond pulsars.

We define the potential $\delta U = \delta p/p - \delta\varphi$ in terms of the (Eulerian) perturbations in the pressure δp, gravitational potential $\delta\varphi$, and the density of the equilibrium star ρ. We assume that all perturbation quantities (in particular δU) have the following form: $\delta U = \delta U(r,\theta) e^{-i\omega t + im\phi}$, where r, θ, and ϕ are spherical coordinates, t is time, ω is a constant, and m is an integer. Euler's equation for small perturbations about a stationary axisymmetric star can be written in terms of this potential δU as

$$\delta v^a = Q^{ab}\nabla_b \delta U , \tag{1}$$

where δv^a is the perturbed fluid velocity, ∇_b is the Euclidean covariant derivative (i.e., partial derivatives in Cartesian coordinates), and Q^{ab} (with $\nabla_c Q^{ab}=0$) depends only on the frequency ω and the constant angular velocity Ω [in particular $Q_{ab}^{-1} = i(\omega - m\Omega) g_{ab} - 2\nabla_b v_a$, where g_{ab} is the Euclidean metric, $v_a = \varpi^2 \Omega \nabla_a \phi$ is the velocity of the star, and $\varpi = r\sin\theta$]. Using Eq. (1) to eliminate δv^a from the remaining equations, the conservation of mass and the perturbed gravitational potential equation reduce therefore to the following:

$$V_a(\rho Q^{ab}\nabla_b \delta U) - i(\omega - m\Omega)\rho (d\rho/dp)(\delta U + \delta\varphi) = 0 , \tag{2}$$

$$\nabla^a \nabla_b \delta\varphi + 4\pi G\rho (d\rho/dp)(\delta U + \delta\varphi) = 0 , \tag{3}$$

where G is the gravitation constant. These equations for δU and $\delta\varphi$ constitute a fourth-order eigenvalue problem (where the frequency ω plays the role of the eigenvalue) when supplemented with appropriate boundary conditions. The boundary condition on $\delta\varphi$ ensures that the mass of the star remains unchanged, while the boundary condition on δU ensures that the pressure at the perturbed surface of the star remains zero. These conditions can be expressed in the following forms:

$$\lim_{r\to\infty} r\,\delta\varphi = 0 , \tag{4}$$

$$i(\omega - m\Omega)(\delta U + \delta\varphi) - Q^{ab}\nabla_b \delta U \nabla_a(\varphi + \tfrac{1}{2}\Omega^2 \varpi^2) = 0 , \tag{5}$$

where φ is the gravitational potential of the background star. Equation (5) is to be imposed at the surface of the star. We point out that Eqs. (2) and (3) can be reduced to a single fourth-order equation for δU by solving Eq. (2) for $\delta\varphi$ and inserting the result into Eq. (3). We represent the resulting equation symbolically as

$$\mathcal{L}_\omega(\delta U) = 0 , \tag{6}$$

where $\mathcal{L}_\omega$ is a fourth-order differential operator depending on ω. When δU has the form $\delta U(r,\theta) e^{-i\omega t + im\phi}$, Eq. (6) is a *real* equation for $\delta U(r,\theta)$. We solve this equa-

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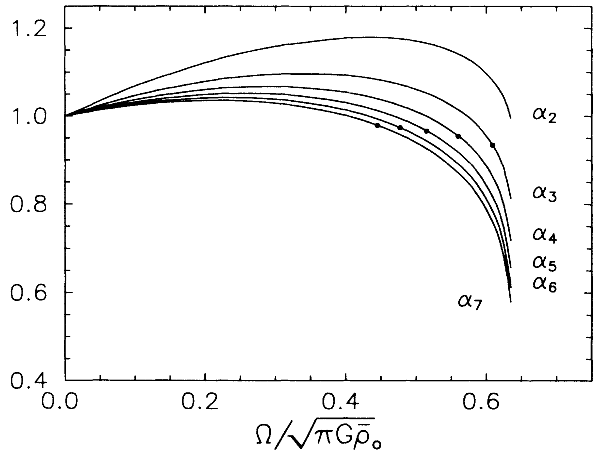

FIG. 1. The angular-velocity dependence of the frequencies of the $l=m$ modes as encoded in the functions α_m. The angular velocities are expressed in units of $(\pi G\bar{\rho}_0)^{1/2}$, where $\bar{\rho}_0$ is the average density of the nonrotating star of the same mass.

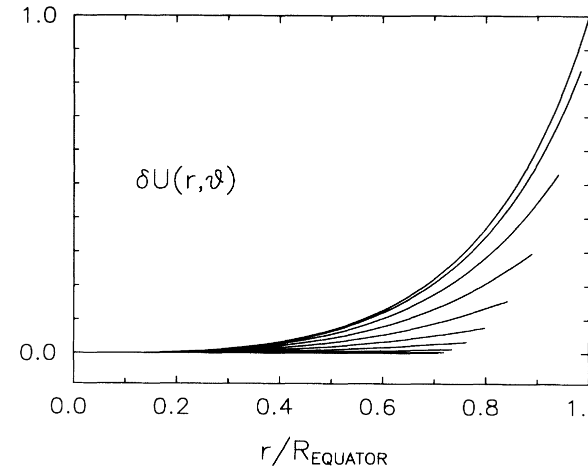

FIG. 2. The eigenfunction $\delta U(r,\theta)$ for the $l=m=3$ mode of a star rotating with angular velocity $\Omega=0.6(\pi G\bar{\rho}_0)^{1/2}$. Each curve in the figure gives the r dependence of $\delta U(r,\theta)$ at one of the fixed values of θ. The curves end at the surface of this highly flattened star.

tion by approximating it as a system of linear algebraic equations on a discreet two-dimensional grid of points. The resulting algebraic eigenvalue problem is solved using standard techniques.

We present here the numerical solutions of Eqs. (2) and (3) for the modes which are the rotating analogs of the $l=m$ f modes of nonrotating stars. These modes are of particular interest in the study of rapidly rotating stars such as the millisecond pulsars because they are the modes driven unstable by GR.[7,8] The equilibrium stars used in this analysis are based on an equation of state of the form $p=K\rho^2$. This equation of state was chosen to have compressibility characteristics similar to realistic neutron-star matter. Figure 1 presents the angular-velocity dependence of the frequencies of these modes for $2\le l=m\le 7$ in the form of the dimensionless function α_m:

$$\alpha_m(\Omega)=\frac{\omega_m(\Omega)-m\Omega}{\omega_m(0)}\,. \qquad (7)$$

These functions, α_m, are independent of the constant K that appears in the equation of state and they are independent of the mass of the star. Dots have been placed on the curves at the angular velocities where the frequency of that mode goes through zero. This is the point at which this mode would become unstable to the emission of GR if viscosity were not present. Figure 2 depicts a typical example of the eigenfunction $\delta U(r,\theta)$ for these modes.

One of the motivations for solving the equations for the frequencies *and* the eigenfunctions of these modes is to allow us to evaluate the effects of GR and viscosity on the oscillations. While GR tends to make these modes unstable in rapidly rotating stars,[9] viscosity tends to damp the modes.[10] The delicate balancing of these two opposing influences will determine whether a given rapidly rotating star is stable. The simplest method for determining the effects of these dissipative effects is to evaluate the rate at which energy is dissipated from the modes. Consider the following energy function:

$$E(t)=\tfrac{1}{4}\int(2\rho\,\delta v^a\,\delta v_a^*+\delta U^*\,\delta\rho+\delta U\,\delta\rho^*)d^3x\,, \qquad (8)$$

where $*$ represents complex conjugation. Since this energy is quadratic in the perturbation variables, its time derivative for a mode (with time dependence $e^{-i\omega t-t/\tau}$) is given by

$$dE/dt=-2E/\tau\,. \qquad (9)$$

Expressions for the time derivatives of this energy due to the effects of dissipation are easily computed and can be used to determine the rate at which the modes are damped (or amplified) from Eq. (9). For viscous dissipation the imaginary part of the frequency is given by

$$\frac{1}{\tau_v}=\frac{1}{E}\int\eta\,\delta\sigma^{ab}\,\delta\sigma_{ab}^*\,d^3x\,, \qquad (10)$$

where η is the viscosity of the star and

$$\delta\sigma_{ab}=\tfrac{1}{2}(\nabla_a\delta v_b+\nabla_b\delta v_a-\tfrac{2}{3}g_{ab}\nabla_c\delta v^c)$$

is the shear.[11] The effect of GR may similarly be evaluated using the post-Newtonian GR potential:[12]

$$\frac{1}{\tau_g}=\frac{1}{2E}(\omega-m\Omega)\sum_{l=l_{\min}}^{\infty}N_l\omega^{2l+1}D_m^lD_m^{*l}\,, \qquad (11)$$

where

$$D_m^l=\int\delta\rho\,r^lY_m^{*l}\,d^3x\,, \qquad (12)$$

$$N_l=\frac{4\pi G(l+1)(l+2)}{c^{2l+1}l(l-1)[(2l+1)!!]^2}\,, \qquad (13)$$

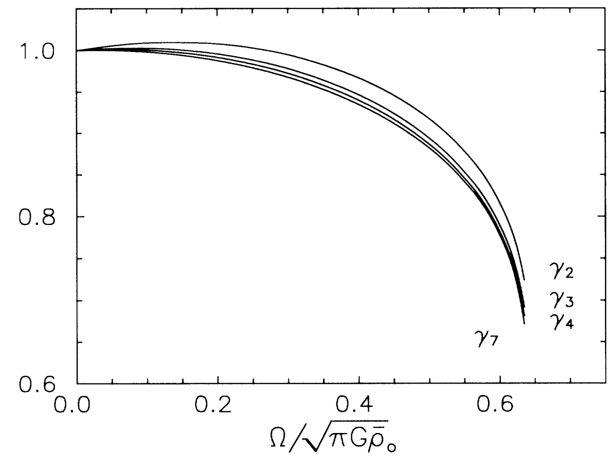

FIG. 3. The functions γ_m representing the angular-velocity dependence of the dissipation time scales.

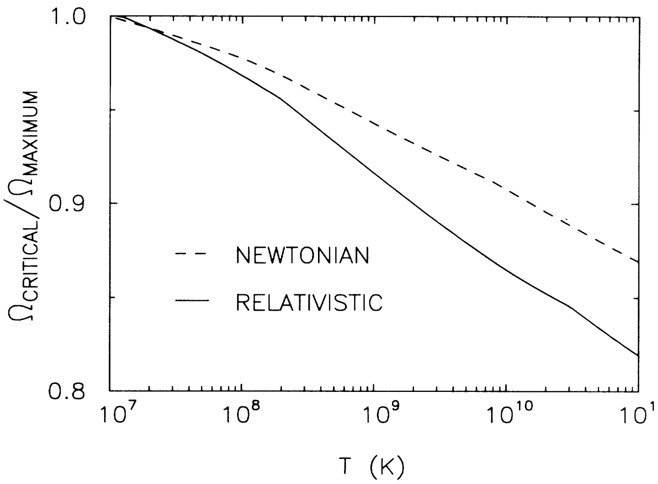

FIG. 4. The temperature dependence of the ratio of the critical angular velocity to the maximum angular velocity of a neutron star.

and l_{min} equals 2 or $|m|$, whichever is larger. Whenever the dissipation is small, the eigenfunctions and frequencies of the adiabatic oscillations from Eqs. (2) and (3) may be used to obtain the lowest-order expressions for the integrals in Eqs. (8), (10), and (12). Both of the time scales, τ_v and τ_g, depend on l and m and the angular velocity of the star. The imaginary part of the frequency of the mode is the sum of the contributions of these individual effects: $1/\tau = 1/\tau_v + 1/\tau_g$. Note that τ_g changes sign whenever the frequency ω goes through zero; this triggers an instability if τ_v is too large.

We have evaluated the effects of viscosity and GR on the imaginary part of the frequencies of the modes of rapidly rotating stellar models using the methods described above. In these computations we use the viscosity for neutron-star matter above the superfluid transition temperature (i.e., T greater than about 10^9 K) given approximately by the formula[13]

$$\eta = 347\frac{\rho^{9/4}}{T^2}, \tag{14}$$

in cgs units. It is also convenient to define dimensionless functions that determine the angular-velocity dependence of the dissipative time scales. We set

$$\gamma_m(\Omega) = \frac{\omega_m(\Omega)}{\omega_m(0)}\left[\frac{\tau_{g,m}(\Omega)}{\tau_{g,m}(0)}\,\frac{\tau_{v,m}(0)}{\tau_{v,m}(\Omega)}\right]^{1/(2|m|+1)}. \tag{15}$$

Figure 3 depicts the functions $\gamma_m(\Omega)$ for the $l=m$ modes. These functions are independent of the constant K in the equation of state, the mass of the star, and the temperature used in the viscosity equation.

To determine when a particular mode becomes unstable to the influence of GR, it is necessary to determine the angular velocity Ω_m at which the imaginary part of the frequency for that mode passes through zero: $0 = 1/\tau_m(\Omega_m)$. When this condition is expressed in terms of the functions defined above, we find the following equation for the critical angular velocities:

$$\Omega_m = -\frac{\omega_m(0)}{m}\left[\alpha_m(\Omega_m) + \gamma_m(\Omega_m)\left[\frac{\tau_{g,m}(0)}{\tau_{v,m}(0)}\right]^{1/(2|m|+1)}\right]. \tag{16}$$

This equation depends on the easily computed frequencies of *nonrotating* stellar models [$\omega_m(0)$, $\tau_{g,m}(0)$, and $\tau_{v,m}(0)$] and the functions $\alpha_m(\Omega)$ and $\gamma_m(\Omega)$ depicted in Figs. 1 and 3. We have solved this equation for two sets of frequencies for nonrotating neutron stars and for a range of neutron-star temperatures. In Fig. 4 we depict the smallest critical angular velocity for these modes as a function of temperature. The two sets of frequencies used in these computations are (1) frequencies for a $1.65 M_\odot$ stellar model based on the Newtonian equations described in this paper, and (2) frequencies based on fully relativistic calculations[14] for the maximum (nonrotating) mass neutron star based on the equation of state of Arponen[15] which admits nonrotating stellar models having masses as large as the binary pulsar PSR 1913+16,[16] and which admits rotating stellar models having periods as short as 0.5 ms.[17] The angular velocities are given in terms of Ω_{max}, the maximum angular velocity for an equilibrium (but unstable) star of the same mass. The slopes of these curves have discontinuities where the mode responsible for the instability changes. At the temperature 10^9 K the mode responsible for determining the critical angular velocity is

$l=m=5$ for the computations based on the Newtonian frequencies and $l=m=4$ for the relativistic frequencies.

The initial temperature of a neutron star is expected to be about 10^{10} K after the initial burst of neutrino emission, cooling to about 10^9 K after two years.[18] Figure 4 indicates that in its hot initial state the neutron star will be unstable if $\Omega \gtrsim (0.86\text{–}0.91)\Omega_{max}$. The growth time for the instability in stars rotating more rapidly than this value depends on the amount by which Ω exceeds Ω_m. This growth time is less than 10^7 s for $\Omega \gtrsim (0.90\text{–}0.94)\Omega_{max}$. For the current temperature of 10^9 K the neutron star will be unstable if $\Omega \gtrsim (0.92\text{–}0.94)\Omega_{max}$ and the growth time for the instability will be short compared to 10^{10} s (the observed lower limit[1] on $P/\dot{P}$ for the pulsar in SN 1987A) if Ω exceeds the critical value by a fraction of a percent. Thus if the 0.508-ms pulsation in SN 1987A is determined to be due to the rotation of a neutron star, there must exist (unstable) equilibrium neutron stars having rotation periods in the range 0.46–0.48 ms. Studies of the structure of rapidly rotating neutron stars[17] have determined that none of the standard equations of state permit equilibrium models with rotation periods this short if they also permit nonrotating models with masses as large as $1.44M_\odot$ (as needed to describe the binary pulsar[16]). The viscosity of neutron-star matter must be significantly larger than that predicted by Eq. (14) or the equation of state must include nonstandard effects (e.g., pion condensation) if there exist neutron stars with 0.5-ms rotation periods.

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