NONLINEAR PATHOLOGIES IN RELATIVISTIC HEAT-CONDUCTING FLUID THEORIES

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Hyperbolicity and stability are analyzed in the nonlinear regimes of two theories of relativistic heat-conducting fluids. Both theories are found to be unstable and non-hyperbolic for sufficiently large deviations from equilibrium. One of these theories (an extended hydrodynamic theory) is well behaved for small (but finite) deviations from equilibrium.

In a number of different physical situations the model of a relativistic heat-conducting fluid is the simplest description of the matter that includes all of the basic phenomena. The collapse of a degenerate stellar core (a supernova) is one such situation involving relativistic matter and very large energy currents (i.e., heat-fluxes) due to the emission of neutrinos. The collision of two heavy ions moving at relativistic velocities and the consequent emission of large numbers of light particles may be another such situation [1]. The purpose of this Letter is to explore some of the fundamental mathematical properties of two theories of relativistic heat-conducting fluids in the nonlinear regime that would be needed to describe such phenomena.

The "standard" theory of a relativistic heat-conducting fluid is the inviscid limit of the relativistic dissipative fluid theory proposed by Eckart [2] (and a similar theory by Landau and Lifshitz [3]). This is the simplest relativistic generalization of a simple single component fluid with thermal conductivity. This theory has been found to be unstable, acausal and ill posed in the linear regime near equilibrium [4,5]. A more complicated extended hydrodynamic theory (in which the heat flux vector is a dynamical field) has been proposed by Israel and Stewart [6–8] which is free of these pathologies in the linear regime [9,10]. In this Letter we extend the analysis of the stability and hyperbolicity of these theories to the nonlinear regime. In order to simplify this analysis we restrict our attention to the case of a fluid having planar symmetry; and we choose the thermodynamic functions to be those appropriate for an extremely high temperature relativistic gas. The assumption of planar symmetry should not be inappropriate to describe, as a first approximation, the collision of two highly Lorentz contracted heavy ions as viewed from the center of mass frame; nor should the assumption be inappropriate to describe locally the regions of a supernova that are far enough away from the center.

The result of our analysis is that both theories have pathologies when states of the fluid that are sufficiently far away from equilibrium are considered. The Eckart theory is unstable (almost any deviation from equilibrium grows without bound) and fails to be hyperbolic for any state of the fluid. The Israel–Stewart theory also exhibits instability and non-hyperbolicity but only in states of the fluid that are far away from equilibrium. The spatially homogeneous mode in our plane symmetric system is found to be unstable only when $|q|/\rho c > 0.50308$ where $|q|$ is the magnitude of the heat-flux vector, ρ is the energy density of the fluid (including rest mass) and c is the speed of light; the system becomes non-hyperbolic when $|q|/\rho c > 0.08898$. The Israel–Stewart theory is well behaved therefore, over the range of fluid states likely to be needed to describe any realistic physical situation.

The evolution equations for the plane symmetric motions of a relativistic heat-conducting fluid (the inviscid limits of the Israel–Stewart theory or the

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Eckart theory) take a particularly simple form if the components of the four-velocity are written in the form $u^a=(\cosh\psi, \sinh\psi, 0, 0)$. (The spacetime metric is taken to be the standard Minkowski metric of special relativity, diag$(-1, 1, 1, 1)$, and cartesian coordinates have been chosen with the x-axis oriented along the direction of spatial variation of the fluid variables.) The heat flow vector, which is defined to be orthogonal to u^a, then has the form $q^a=q(\sinh\psi, \cosh\psi, 0, 0)$. With these definitions, the fluid equations become

$$\cosh\psi\,\partial_t n+\sinh\psi\,\partial_x n+n\sinh\psi\,\partial_t\psi +n\cosh\psi\,\partial_x\psi=0\,, \tag{1}$$

$$\cosh\psi\,\partial_t\rho+\sinh\psi\,\partial_x\rho+[(\rho+p)\sinh\psi +2q\cosh\psi]\partial_t\psi+[(\rho+p)\cosh\psi +2q\sinh\psi]\partial_x\psi+\sinh\psi\,\partial_t q +\cosh\psi\,\partial_x q=0\,, \tag{2}$$

$$\sinh\psi\,\partial_t p+\cosh\psi\,\partial_x p+\cosh\psi\,\partial_t q +\sinh\psi\,\partial_x q+[(\rho+p)\cosh\psi+2q\sinh\psi]\partial_t\psi +[(\rho+p)\sinh\psi+2q\cosh\psi]\partial_x\psi=0\,, \tag{3}$$

and

$$\left(\frac{\sinh\psi}{T}-\frac{\beta q}{2T}\cosh\psi\right)\partial_t T+\left(\frac{\cosh\psi}{T} -\frac{\beta q}{2T}\sinh\psi\right)\partial_x T+\tfrac{1}{2}q(\cosh\psi\,\partial_t\beta+\sinh\psi\,\partial_x\beta) +\beta(\cosh\psi\,\partial_t q+\sinh\psi\,\partial_x q)+(\cosh\psi +\tfrac{1}{2}\beta q\sinh\psi)\partial_t\psi+(\sinh\psi+\tfrac{1}{2}\beta q\cosh\psi)\partial_x\psi +q/\kappa T=0\,, \tag{4}$$

where ρ is the energy density of the fluid, n the number density, p the pressure, T the temperature, κ the thermal conductivity, and β the second-order thermodynamic coefficient (denoted β_1 in refs. [4-10]). The equations for the Eckart theory are obtained by setting $\beta=0$ in these equations.

In order to obtain a closed set of equations, it is necessary to supplement eqs. (1)-(4) by an equation of state for the fluid, as well as thermodynamic expressions for β and κ. We chose in this Letter to examine the high temperature limit of an ideal gas, for which

$$p=\rho/3=nkT \tag{5}$$

is the equation of state; and we take the thermal conductivity to be constant, which is the value given by kinetic theory for the ultrarelativistic limit of a dilute gas with a constant (maxwellian) scattering cross-section [11]. Finally, we take the second-order coefficient, β, to be given by $\beta=5\lambda/4p$. The parameter λ takes the value $\lambda=1$ for an ultrarelativistic gas in the Israel-Stewart theory (based on a kinetic theory analysis [7]), while the value $\lambda=0$ corresponds to the Eckart theory. Using these values for β and κ and the equation of state to eliminate T, and p in favor of ρ and n, eqs. (1)-(4) become a closed set of evolution equations for the four variables (ρ, n, ψ, q).

We first consider the hyperbolicity of these theories. If there exist four distinct real characteristic velocities of this system of equations, then the system is hyperbolic [12]. The fluid equations (1)-(4) may be written symbolically in the form

$$A^\mu{}_\nu{}^a\partial_a Y^\nu+B^\mu\,, \tag{6}$$

where the index ν runs over the set of fluid variables, $Y^\nu=(\rho, n, \psi, q)$, and μ runs over the set of four equations, (1)-(4). The components of $A^\mu{}_\nu{}^a$ and B^μ (which may depend on the Y^ν but not on $\partial_a Y^\nu$) can be read off from the fluid eqs. (1)-(4). The matrices $A^\mu{}_\nu{}^a$ are nonzero only when the spacetime index a takes on the values t and x, since we have restricted attention to the plane symmetric case. The characteristic velocities, v, are defined as the roots of the equation

$$\det(vA^\mu{}_\nu{}^t-A^\mu{}_\nu{}^x)=0\,. \tag{7}$$

These velocities are most easily determined in the rest frame of the fluid, i.e., the frame in which $\psi=0$. Eq. (7) then has the form:

$$[30\lambda-6-45\lambda(q/\rho)^2]v^4-6(5\lambda+2)(q/\rho)v^3 -[2(5\lambda+2)-45\lambda(q/\rho)^2]v^2+12(q/\rho)v+2=0\,, \tag{8}$$

where the dimensionless (in our units, where $c=1$) ratio $|q|/\rho$ is a measure of the deviation from equilibrium. When $|q|/\rho$ is taken to be zero in eq. (8), the resulting velocities are precisely those found for the longitudinal modes in these theories [5,8], specialized to the case of a high-temperature ideal gas.

The fluid equations are hyperbolic only when there exist four distinct real roots of eq. (8). For the case of an Eckart fluid, $\lambda=0$, there are four real roots only when $|q|/\rho>1.523$; however, in the region where the theory is hyperbolic, there is always at least one velocity greater than one (the speed of light in our units). Consequently this theory does not have causal hyperbolic evolution equations.

In theories which have $0.2\leqslant\lambda\leqslant0.4$, there are four real roots to eq. (8) for all values of $|q|/\rho$; an Israel–Stewart theory with λ in this range is thus always hyperbolic, no matter how large the deviation from equilibrium is. These theories with $0.2\leqslant\lambda\leqslant0.4$ are always acausal ($v>1$), however, for all nonzero values of $|q|/\rho$. The upper limit of λ for the set of hyperbolic nonlinear theories corresponds, curiously, to the lower limit of λ consistent with the stability of the linear theory: linear stability requires $\lambda\geqslant0.4$ for a fluid with the thermodynamic functions considered here.

If $\lambda\geqslant0.4$, then there are four real characteristic velocities less than the speed of light for small values of $|q|/\rho$. For the case of the Israel–Stewart fluid with the kinetic theory value of λ, $\lambda=1$, the roots of the characteristic equation are plotted in fig. 1. This graph reveals that this theory is hyperbolic for small

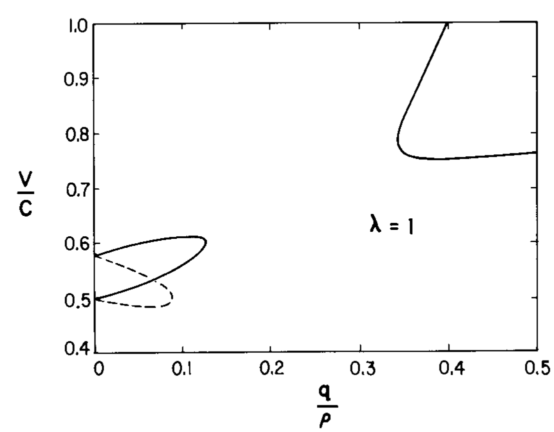

Fig. 1. The characteristic velocities for the nonlinear Israel–Stewart fluid theory with $\lambda=1$. The solid curves represent the characteristic velocities in the same spatial direction as the heat flux ($qv>0$) while the dotted curve represents the characteristic velocities in the opposite direction ($qv<0$). For $|q|/\rho>0.08898$, four real velocities do not exist, and hence the theory is not hyperbolic. For $|q|/\rho>0.4$ one of the real characteristic velocities is greater than the speed of light.

deviations from equilibrium, bounded approximately by $|q|/\rho<0.08898$. For values of $|q|/\rho$ larger than 0.08898, there are either zero or two real characteristic velocities. Since the energy density ρ includes the rest energy of the fluid, bounding the ratio $|q|/\rho$ by 0.08898 is not a severe restriction that is likely to exclude any real physical phenomena.

We turn secondly to the study of the nonlinear stability of these theories. We consider a system to be nonlinearly unstable if the evolution of some state of the system diverges without bound (into the future) in comparison with every equilibrium state of the system. It is already known that the general Israel–Stewart theory is stable (i.e., not unstable) for linear perturbations about equilibrium, if and only if the second-order coefficients are chosen to yield hyperbolic, causal linear-perturbation equations [9,10]. In contrast, the "first-order" theories (such as Eckart [2] and Landau–Lifshitz [3]) are known to be generically unstable and fail to have hyperbolic evolution equations at the level of linear perturbation theory [4,5]. It is natural to ask whether this relationship between stability and hyperbolicity persists in the nonlinear theories as well.

In order to obtain a simple (though nonlinear) problem which can be integrated analytically, we now restrict our attention to spatially homogeneous fluid states. The fluid equations are then given by eqs. (1)–(4) with all x-derivative terms set equal to zero. Eqs. (1)–(4) can then be rewritten as follows:

$$\frac{dN_0}{dt}=\frac{d}{dt}(n\cosh\psi)=0\,, \tag{9}$$

$$\frac{dE_0}{dt}=\frac{d}{dt}(\rho\cosh^2\psi+p\sinh^2\psi+q\sinh 2\psi)=0\,, \tag{10}$$

$$\frac{dP_0}{dt}=\frac{d}{dt}[\tfrac{1}{2}(\rho+p)\sinh 2\psi+q\cosh 2\psi]=0\,, \tag{11}$$

$$\frac{d}{dt}(T\sinh\psi)+\beta T\cosh\psi\frac{dq}{dt}+\tfrac{1}{2}T^2q\frac{d}{dt}\left(\frac{\beta}{T}\cosh\psi\right)+\frac{q}{\kappa}=0\,. \tag{12}$$

Eqs. (9)–(11) can be immediately integrated. The resulting integration constants are, respectively, the particle number density in the rest frame of the fluid (denoted N_0), the total energy density of the fluid (E_0), and the total momentum density of the fluid (P_0). In order to simplify the integration of eq. (12), we consider only fluid motions for which the total momentum density, P_0, vanishes. We use the same equation of state (the relativistic high temperature ideal gas) and the same expressions for the thermodynamic functions β and κ used in the analysis above. With these expressions and the integrals of eqs. (9)–(11) all of the fluid variables can be expressed in terms of the rapidity parameter ψ and the integration constants N_0 and E_0:

$$\rho=3p=\frac{3E_0\cosh 2\psi}{2+\cosh 2\psi}, \tag{13}$$

$$n=\frac{N_0}{\cosh\psi}, \tag{14}$$

$$T=\frac{E_0\cosh\psi\cosh 2\psi}{kN_0(2+\cosh 2\psi)}, \tag{15}$$

$$\beta=\frac{5\lambda(2+\cosh 2\psi)}{4E_0\cosh 2\psi}, \tag{16}$$

$$q=-\frac{2E_0\sinh 2\psi}{2+\cosh 2\psi}. \tag{17}$$

When these values are substituted into eq. (12), a simple differential equation involving only the single variable $w=\cosh 2\psi$ is obtained:

$$\frac{\kappa}{4kN_0}\left(1+\frac{(2w+1)w}{(w^2-1)(2+w)} - \frac{5\lambda}{2}\frac{1}{w(w-1)}\right)\frac{\mathrm{d}w}{\mathrm{d}t}=1\,. \tag{18}$$

This equation can be integrated immediately (by expanding in partial fractions) to obtain the time as a function of the "velocity" variable w. The resulting solution can be interpreted physically more directly if we re-express the answer in terms of the three-velocity, $v=\tanh\psi$; we find

$$t=\tau\left[\frac{1+v^2}{1-v^2}+\log v+2\log\left(\frac{3-v^2}{1-v^2}\right) + \frac{5\lambda}{2}\log\left(\frac{1+v^2}{2v^2}\right)\right]+t_0\,, \tag{19}$$

where the timescale τ is defined by $\tau=\kappa/4kN_0$.

Eq. (19) together with eqs. (13)–(17), give a complete analytic integration of the nonlinear evolution of a spatially homogeneous Eckart or Israel–Stewart fluid. These solutions are graphed for the values $\lambda=0$ (Eckart theory) and $\lambda=1$ (Israel–Stewart theory with kinetic theory value for β) in fig. 2. For small initial values of v the behavior of the curves is, as expected, well described by the results obtained from the linearized theory: the Eckart fluid is exponentially unstable, while in the Israel–Stewart theory the velocity exponentially decays towards the zero-velocity equilibrium state (zero velocity is equivalent to equilibrium here since we chose the total momentum, P_0, to be zero). What is more interesting is the behavior of the evolution curves for initially large values of the velocity. When v is large, the non-logarithmic term in eq. (19) dominates, and the large v evolution is asymptotically the same for the Eckart and the Israel–Stewart fluids (in fact for any value of the parameter λ). Examination of fig. 2 shows that there is a critical velocity above which the Israel–Stewart fluid evolves away from equilibrium, rather than towards it. (As $t\to\infty$ the graph shows that $v\to 1$; and eqs. (13)–(17) imply that $T\to\infty$, $n\to 0$, $\rho=3p\to 3E_0$, and $q\to -2E_0$.) This crit-

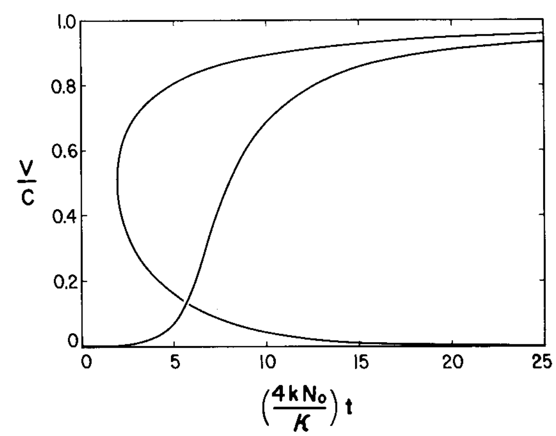

Fig. 2. The evolution of the three-velocity of a spatially homogeneous thermally conducting relativistic fluid, illustrated for the Eckart theory and the Israel–Stewart theory (with the kinetic theory value for β). The evolution for a particular initial value of v is found by finding that value on the graph, and then following the curve to the future. Any nonequilibrium initial state is unstable in the Eckart theory; the Israel–Stewart fluid is stable for initial values $v<0.51188$.

ical velocity is approximately given by $v_c = 0.51188$, which is less than the adiabatic sound speed, $v_s = 3^{-1/2}$. This critical velocity corresponds to a certain deviation of the fluid from equilibrium for which the ratio $|q|/\rho$ (which may be determined from eqs. (13) and (17)) has the approximate value $(|q|/\rho)_c = 0.50308$. This is well outside the domain in which the Israel–Stewart theory is hyperbolic $(|q|/\rho < 0.08898)$.

In summary, we have examined the properties of hyperbolicity and stability in extremely simplified but fully nonlinear versions of Eckart's theory and the Israel–Stewart theory of relativistic dissipative fluids. We have shown that Eckart's theory continues to display the generic instability and acausal, nonhyperbolic behavior that first appeared in the analysis of the linear equations. The Israel–Stewart theory (with β given by its kinetic theory value) fails to be hyperbolic for states of the fluid that are not sufficiently close to equilibrium $|q|/\rho > 0.08898$. In addition the spatially homogeneous solutions in this theory are thermodynamically unstable for all initial values of $|q|/\rho > 0.50308$. Further investigation of the stability of the entire class of solutions will have to be performed before it will be possible to determine whether or not there is as close a relationship between hyperbolicity and stability in the nonlinear theory as there was in the linear regime.

Finally, we should consider briefly how seriously to take these nonlinear pathologies in the Israel–Stewart theory. Since the energy density of the fluid, ρ, includes the rest mass energy of the particles in the fluid, the state of the fluid for which $|q|/\rho = 0.08898$ (the point at which the fluid equations cease to be hyperbolic) represents a spectacularly large heat flux. It is unlikely that there are any potential physical or astrophysical applications in which the deviations from equilibrium become this large. It is also possible to extend the range in which the theory is well behaved by choosing different thermodynamic functions than those considered here. For example by adopting the value $\lambda = 10$ the domain in which the equations are hyperbolic is extended to about $|q|/\rho = 1/3$.

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