A ONE-FLUID MODEL OF SUPERFLUIDS

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A phenomenological macroscopic one-fluid description of a classical superfluid is presented which is completely equivalent to Landau's two-fluid model in the limit of small departures from equilibrium. The one-fluid model described here is a special case of the "extended hydrodynamic" theories in which the heat-flux vector and stress tensor become dynamical variables. The dynamics of the heat-flux in our model replaces the dynamics of the superfluid component in the two-fluid model. Like Landau's two-fluid model, our model contains undetermined equations of state that must be fixed by experiment or by calculations based in microphysics. The one-fluid model presented here generalizes the one-fluid model of Greco and Müller by allowing an equation of state that is compatible with the observed temperature dependence of the sound velocities in superfluid ^{4}He, and by including the effects of viscosity.

The Landau [1] two-fluid model of superfluid ^{4}He is based on the recognition that the observed dynamics of this material require a second dynamical vector field in addition to the velocity and thermodynamic variables needed to describe a normal classical fluid. Based on a suggestion of Tisza [2], Landau treated superfluid ^{4}He as a particular non-interacting mixture of "normal" and "superfluid" components. The velocity of the superfluid component became the needed second dynamical vector field. His equations to describe this material are a reasonably straightforward expression of conservation of mass, entropy and momentum, and the requirement that the superfluid velocity evolves in a curl-free manner.

A long recognized [3] deficiency of the two-fluid model is that the two components cannot in principle be separated. In this Letter we describe a one-fluid model of superfluids based on the "extended hydrodynamic" (EH) theory of dissipative fluids. In EH the heat-flux vector and the stress tensor are dynamical fields in addition to the velocity and thermodynamic variables that describe the dynamics of a normal classical fluid. Therefore, there exists ample dynamical structure in EH to accommodate the needs of a theory of superfluids. The EH theory which is analyzed here is given by the newtonian limit of Israel and Stewart's [4,5] relativistic equations, and is a generalization of Müller's [6] earlier theory. The theory contains a number of undetermined thermodynamic functions which govern the evolution of the additional dynamical fields. In our one-fluid model of superfluids, we make phenomenological arguments based on the general observed properties of superfluid ^{4}He to fix several of these additional thermodynamic functions. The resulting theory is completely equivalent to the Landau [1] theory in the limit of small departures from a non-rotating equilibrium state, the only regime in which the two-fluid model is an adequate description of superfluid ^{4}He. We also comment on the differences between our one-fluid model and a related one-fluid model proposed by Greco and Müller [7].

Any single-component classical fluid can be described in terms of a number of scalar, vector and tensor fields: ρ, the mass density; ϵ, the internal energy (per unit mass); T, the temperature; s, the entropy (per unit mass); p, the thermodynamic pressure; τ, the non-thermodynamic (i.e., viscous) pressure; v^i, the velocity; q^i, the heat-flux vector; and τ^{ij}, the trace-free (viscous) stress tensor. In terms of the variables described above the conservation laws for mass, energy and momentum are given by [8]

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$$0=\partial_t\rho+\nabla_i(\rho v^i)\ , \tag{1}$$

$$0=\partial_t(\rho\epsilon+\tfrac{1}{2}\rho v^i v_i)+\nabla_j[(\tfrac{1}{2}\rho v^i v_i+\rho\epsilon+p+\tau)v^j+\tau^{ji}v_i+q^j]\ , \tag{2}$$

$$0=\partial_t(\rho v^i)+\nabla_j[(p+\tau)\delta^{ij}+\rho v^i v^j+\tau^{ij}]\ . \tag{3}$$

In these equations ∂_t is the partial derivative with respect to time t, ∇_i is the partial derivative with respect to the cartesian coordinate x^i, with $i=1$, 2, or 3 (or the covariant derivative in curvilinear coordinates), δ^{ij} is the unit matrix in cartesian coordinates (or the components of the inverse metric tensor in curvilinear coordinates), and summation over repeated indices is implied. These equations are supplemented by the first law of thermodynamics,

$$d\epsilon=T\,ds+p\,d\rho/\rho^2\ , \tag{4}$$

which defines the temperature T and thermodynamic pressure p once an equation of state, say $s=s(\rho,\epsilon)$, is given for a particular material.

To complete the system of equations, the evolution of the variables τ, q^i, and τ^{ij} must be specified. The simplest model sets these field equal to zero: $\tau=q^i=\tau^{ij}=0$. Eqs. (1)–(4) are then the laws for a perfect fluid in terms of the usual dynamical variables ϵ, ρ, and v^i. The standard Navier–Stokes–Fourier theory which includes the effects of dissipation is obtained by defining the quantities τ, q^i, and τ^{ij} in terms of spatial gradients of the dynamical variables of the perfect fluid theory: $\tau=-\varsigma\nabla_i v^i$, $q^i=-\kappa\nabla^i T$, and $\tau^{ij}=-\eta(\nabla^i v^j+\nabla^j v^i-\frac{2}{3}\delta^{ij}\nabla_k v^k)$. The quantities ς, η and κ are the viscosity coefficients and thermal conductivity which are assumed to be positive functions of the thermodynamic variables. This theory has the same set of dynamical variables (ϵ, ρ and v^i) as perfect fluid dynamics, since τ, q^i and τ^{ij} are determined by the dynamical variables at each instant of time.

In EH the set of dynamical variables is extended to include the fields τ, q^i, and τ^{ij}. The evolution equations for these quantities are given by

$$0=\tau+\varsigma\{\nabla_i v^i+\beta_0 D_t\tau-\alpha_0\nabla_i q^i-\gamma_0 Tq^i\nabla_i(\alpha_0/T)+\tfrac{1}{2}\tau T[D_t(\beta_0/T)+(\beta_0/T)\nabla_i v^i]\}\ , \tag{5}$$

$$0=q^i+\kappa T\{T^{-1}\nabla^i T+\beta_1 D_t q^i-\alpha_0\nabla^i\tau-\alpha_1\nabla_j\tau^{ij}+\gamma_2 q_j\nabla^{[i}v^{j]}-(1-\gamma_0)\tau T\nabla^i(\alpha_0/T)-(1-\gamma_1)T\tau^{ij}\nabla_j(\alpha_1/T)+\tfrac{1}{2}Tq^i[D_t(\beta_1/T)+(\beta_1/T)\nabla_j v^j]\}\ , \tag{6}$$

$$0=\tau^{ij}+2\eta\langle\nabla^i v^j-\alpha_1\nabla^i q^j+\beta_2 D_t\tau^{ij}-\gamma_1 Tq^i\nabla^j(\alpha_1/T)+\gamma_3\tau^i{}_k\nabla^{[j}v^{k]}+\tfrac{1}{2}T\tau^{ij}[D_t(\beta_2/T)+(\beta_2/T)\nabla_k v^k]\rangle\ . \tag{7}$$

In these equations the differential opeator D_t is defined to be the co-moving time derivative given by $D_t=\partial_t+v^i\nabla_i$; the functions α_A, β_A and γ_A are functions of the thermodynamic variables; square brackets, [ ], surrounding a pair of coordinate indices indicate anti-symmetrization; and the bracket operation, $\langle\ \rangle$, is defined by $\langle A^{ij}\rangle=\frac{1}{2}(A^{ij}+A^{ji}-\frac{2}{3}\delta^{ij}A^k{}_k)$ for any tensor A^{ij}. Eqs. (1)–(7) form a complete system of equations for the evolution of the fourteen dynamical variables ϵ, ρ, τ, v^i, q^i, and τ^{ij} of EH once they are supplemented with "equations of state" for s, ς, η, κ, α_A, β_A and γ_A. The EH equations were first derived with specific values of the α's, β's and γ's appropriate for a dilute gas of point particles by Grad [9]. Müller [6] derived these equations for a somewhat more general set of α's and β's using an argument based on a second-order implementation of the second law of thermodynamics. The equations presented here (allowing non-uniform α's, β's and γ's) are the newtonian limit of equations first given by Israel and Stewart [4,5] for relativistic fluids. The EH theory has received considerable attention in the context of relativistic fluids [4,5,10,11] since it appears to be the *only* viable (causal, stable) theory of a relativistic fluid which includes the effects of viscosity and thermal conductivity [11,12]. Note that eqs. (1)–(7) reduce to the Navier–Stokes–Fourier theory when $\alpha_A=\beta_A=0$.

The form of the evolution equations for τ, q^i, and τ^{ij} was explicitly chosen so that the second law of thermodynamics would take the simple form

$$\partial_t(\rho\bar{s})+\nabla_i\{\rho\bar{s}v^i+T^{-1}[(1+\alpha_0\tau)\delta^{ij}+\alpha_1\tau^{ij}]q_j\}=T^{-1}(\tau^2/\varsigma+q_i q^i/\kappa T+\tau_{ij}\tau^{ij}/2\eta)\ , \tag{8}$$

where the generalized specific entropy function, $\bar{s}$, is defined by

$$\rho\bar{s}=\rho s-\tfrac{1}{2}T^{-1}(\beta_0\tau^2+\beta_1 q^i q_i+\beta_2\tau^{ij}\tau_{ij}) . \qquad (9)$$

In equilibrium this function reduces to the thermodynamic specific entropy s, but for systems out of equilibrium $\bar{s}$ reflects the fact that the system is not in a state of maximal entropy. Eq. (8) implies that the total entropy, defined as the volume integral of $\rho\bar{s}$, is strictly increasing for isolated systems.

We now wish to choose the functional form of the α's, β's, κ, etc., in such a way that the resulting system is an appropriate description of a superfluid. We begin by noting that the standard Landau two-fluid theory is an adequate description of superfluid ^{4}He only when the velocity of the fluid is a small fraction of the characteristic velocities (e.g., the first and second sound velocities) of the system [3]. Since the simple macroscopic description of superfluid ^{4}He appears to be adequate only near equilibrium, we confine our attention hereafter to the dynamics of small departures from an equilibrium state. The equilibrium states of eqs. (1)–(7) must have $\tau=q^i=\tau^{ij}=0$ to avoid generating entropy in eq. (8) and must have $v^i=0$ (up to a galilean transformation) unless the fluid is rotating. These conditions and eqs. (1)–(7) imply in turn that the thermodynamic variables are time independent and spatially uniform. The equations that describe the evolution of small departures away from such an equilibrium state are found by linearizing eqs. (1)–(7). The difference between the nearly equilibrium value of a quantity Q and the value that it takes in the fiducial equilibrium state is denoted by δQ; quantities Q without the prefix δ hereafter refer to that quantity's value in the background equilibrium state. The linearized evolution equations are:

$$0=\partial_t\delta\rho+\rho\nabla_i\delta v^i , \qquad (10)$$

$$0=\partial_t(\rho\delta\epsilon+\epsilon\delta\rho)+(\rho\epsilon+p)\nabla_i\delta v^i+\nabla_i\delta q^i , \qquad (11)$$

$$0=\rho\partial_t\delta v^i+\nabla_j[(\delta p+\delta\tau)\delta^{ij}+\delta\tau^{ij}] , \qquad (12)$$

$$0=\delta\tau+\varsigma(\nabla_i\delta v^i+\beta_0\partial_t\delta\tau-\alpha_0\nabla_i\delta q^i) , \qquad (13)$$

$$0=\delta q^i+\kappa T(T^{-1}\nabla^i\delta T+\beta_1\partial_t\delta q^i -\alpha_0\nabla^i\delta\tau-\alpha_1\nabla_j\delta\tau^{ij}) , \qquad (14)$$

$$0=\delta\tau^{ij}+2\eta\langle\nabla^i(\delta v^j-\alpha_1\delta q^j)+\beta_2\partial_t\delta\tau^{ij}\rangle . \qquad (15)$$

Eqs. (10)–(15) imply the first-order form of the entropy generation law:

$$0=\partial_t(\rho\delta s+s\delta\rho)+\nabla_i(\rho s\delta v^i+T^{-1}\delta q^i) . \qquad (16)$$

Eqs. (10)–(16) describe the evolution of small departures from an arbitrary non-rotating equilibrium state in the general EH theory. The functional form of the α's and β's which appear in these equations must now be chosen in such a way that the resulting system describes a superfield. We first note that although the phenomenology of superfluid ^{4}He demands the existence of an additional dynamical vector field, it does not suggest the existence of additional scalar or tensor degrees of freedom. We thus set $\beta_0=\beta_2=0$ to remove the dynamics from the fields $\delta\tau$ and $\delta\tau^{ij}$ in eqs. (13) and (15). Second, since superfluid ^{4}He appears to be extremely efficient at conducting heat, we set $1/\kappa T=0$ in eq. (14). Finally third, we hypothesize that the vector field which transports entropy, $\delta v^i+\delta q^i/\rho sT$, in eq. (16) (i.e., on a microscopic level the current of atoms not condensed into the zero entropy ground state) should be identified with the vector field whose shear determines the viscous stress, $\delta v^i-\alpha_1\delta q^i$, in eq. (15) (i.e., the current of atoms capable of transporting momentum through scattering). Thus, we choose $\alpha_1=-1/\rho sT$. A similar consideration might lead one to choose the same value for α_0 to limit the coupling of the bulk viscosity in eq. (13); however, following Khalatnikov [13] we choose to allow a more general bulk viscosity coupling. With these choices, eqs. (13)–(15) reduce to the following:

$$0=\delta\tau+\varsigma(\nabla_i\delta v^i-\alpha_0\nabla_i\delta q^i) , \qquad (17)$$

$$0=\beta_1\partial_t\delta q^i+T^{-1}\nabla^i\delta T-\alpha_0\nabla^i\delta\tau+(1/\rho sT)\nabla_j\delta\tau^{ij} , \qquad (18)$$

$$0=\delta\tau^{ij}+2\eta\langle\nabla^i(\delta v^j+\delta q^j/\rho sT)\rangle . \qquad (19)$$

Eqs. (10)–(12) and (17)–(19) represent the linearized version of an EH description of a superfluid. In these equations we have not fixed the thermodynamic functions α_0 and β_1, the viscosity coefficients ς and η, nor the equation of state $s=s(\rho,\epsilon)$ of the fluid itself.

The one-fluid model of a superfield described above may be compared to the standard Landau two-fluid model by introducing new variables ρ_s, ρ_n, ς_1, ς_2, v_s^i, and v_n^i which are related to the EH variables ρ, β_1, ς, α_0, v^i, and q^i by

$$\rho=\rho_s+\rho_n\,, \tag{20}$$

$$\rho v^i=\rho_s v_s^i+\rho_n v_n^i\,, \tag{21}$$

$$q^i=\rho_s sT(v_n^i-v_s^i)\,, \tag{22}$$

$$\beta_1=\rho_n/\rho\rho_s s^2T^2\,. \tag{23}$$

$$\alpha_0=(\rho\varsigma_1-\varsigma_2)/\varsigma_2\rho sT\,, \tag{24}$$

$$\varsigma=\varsigma_2\,. \tag{25}$$

Using these new variables eqs. (10), (12), (16)–(19) can be written in the following form:

$$0=\partial_t(\delta\rho_s+\delta\rho_n)+\nabla_i(\rho_s\delta v_s^i+\rho_n\delta v_n^i)\,, \tag{26}$$

$$0=\partial_t(\rho_s\delta v_s^i+\rho_n\delta v_n^i)+\nabla_j[(\delta p+\delta\tau)\delta^{ij}+\delta\tau^{ij}], \tag{27}$$

$$0=\partial_t(\rho\delta s+s\delta\rho)+\nabla_i(\rho s\delta v_n^i)\,, \tag{28}$$

$$0=\partial_t\delta v_s^i+\nabla^i(\delta\mu+\varsigma_1\delta\tau/\varsigma_2)\,, \tag{29}$$

$$0=\delta\tau+\varsigma_1\nabla_i[\rho_s(\delta v_s^i-\delta v_n^i)]+\varsigma_2\nabla_i\delta v_n^i\,, \tag{30}$$

$$0=\delta\tau^{ij}+2\eta\langle\nabla^i\delta v_n^j\rangle\,. \tag{31}$$

The thermodynamic variable μ in eq. (29) is the chemical potential defined by $\mu=\epsilon+p/\rho-sT$. Eqs. (26)–(31) are precisely the two-fluid equations of Landau [1], with viscous terms added by Khalatnikov [13], when written in a form that describes linear perturbations about a non-rotating equilibrium state. At this linear order, our equations precisely agree with Khalatnikov's shear viscosity coupling. The two bulk-viscosity coefficients in our model, ς_1 and ς_2, also correspond exactly to Khalatnikov's coefficients of the same names. Khalatnikov's theory also admits a third and fourth type of "bulk viscosity" which in our equations take particular values for the associated coefficients given by $\varsigma_3=\varsigma_1^2/\varsigma_2$, and $\varsigma_4=\varsigma_1$ (this last condition is also a constraint in Khalatnikov's equations). The one-fluid model proposed here is therefore precisely equivalent to Landau's two-fluid model in the limit of small perturbations about an equilibrium state, with viscosity coupling that is only slightly less general than that proposed by Khalatnikov. The one-fluid model will describe, therefore, all of the properties of superfluid ^{4}He that can be described within the two-fluid model including: normal sound waves, second sound, and even transverse sound [14] waves.

While we are unaware of appropriate experimental data at this time, the one- and two-fluids models are in principle distinguishable experimentally. The one-fluid model with the transformed variables given in eqs. (20)–(25) differs from the two-fluid model at non-linear order. Also, the effects of finite thermal conductivity could naturally be introduced into the one-fluid model either by keeping κT finite in eq. (14) or by changing the relationship between the one- and two-fluid variables in eq. (22) to $q^i=\rho_s sT(v_n^i-v_s^i)-\kappa'\nabla^i T$. Neither of these possibilities is identical to Khalatnikov's [13] model of finite thermal conductivity.

Greco and Müller [6] have also proposed a one-fluid model of superfluidity based on EH. Their model differs from ours in two ways. First, it is based on a special case of the EH equations proposed by Liu and Müller [15] instead of the general theory, eqs. (1)–(17), used here. As a result, their model makes specific predictions about the equations of state that relate the various functions that appear in the theory (e.g. ρ, β_1, T, etc.) which do not agree with the observed temperature dependence of the first and second sound velocities. Our model leaves these equations of state undetermined a priori so that it can describe the body of experimental data precisely as well as the two-fluid model. The second difference is that we have included the effects of viscosity in our equations while they have not.

The equivalence of the two-fluid model of superfluidity and a certain single-component EH theory is interesting for three reasons: (a) Since superfluids cannot be physically separated into constituent normal and superfluid components, it is more attractive to have a mathematical description of this material that is fundamentally a single-component fluid. (b) The description of superfluid ^{4}He within EH gives an interesting, *extreme*, example of the dynamical behavior that is possible within the EH framework. (c) The relativistic EH theories are well behaved causal theories; therefore the relativistic analogy of the one-fluid superfluid model discussed here may well provide the appropriate model for relativistic superfluids such as the interiors of neutron stars.

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