Static uniform-density stars must be spherical in general relativity

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In this paper the uniqueness of the static solutions of Einstein's equation that represent isolated uniform-density perfect-fluid stellar models is demonstrated: any static asymptotically flat space-time containing only a uniform-density perfect fluid confined to a spatially compact world tube is necessarily spherically symmetric. This result generalizes to relativistic uniform-density models the well known Newtonian theorem of Carleman and Lichtenstein.

I. INTRODUCTION

The inevitability of spherical symmetry in isolated static (i.e., time independent and nonrotating) fluid stellar models was first demonstrated in the Newtonian theory by Carleman[1] and Lichtenstein.[2,3] To date the analogous result has not been established in general relativity theory. A number of studies of the properties of static relativistic stellar models have been published, however. Some of the more interesting results of these investigations are as follows. Masood-ul-Alam[4] has shown that the topology of the space-times containing these static stellar models must be diffeomorphic to $\mathbf{R}^3\times\mathbf{R}$. Avez,[5,6] Künzle,[7] and Lindblom[8,9] have studied the geometry of this general class of space-times. They established the equivalence of spherical symmetry and a number of other geometrical properties (e.g., spatial conformal flatness) in these space-times. Künzle and Savage[10] showed that the spherical static space-times are isolated in the sense that no continuous family of static fluid space-times exists which contains both spherical and nonspherical space-times.

Recently Masood-ul-Alam[11] explored the implications of the positive mass theorem[12–14] on the geometry of static fluid space-times. He demonstrated that the positive mass theorem could be used to prove the necessity of spherical symmetry in a subset of these space-times that satisfies certain special properties. He assumed that the fluid obeyed a particular equation of state, which having $d\rho/dp<0$ is unfortunately extremely unphysical. He also limited his attention to a subset of the stellar models based on this equation of state which have $\rho\geqslant 0$. Since the spherical models in this subset all have $\rho=0$ at the center of the star, he has implicitly assumed that the central pressure in these (potentially nonspherical) models is never greater than that achieved in the corresponding spherical model. When stated in this way the additional assumption, $\rho\geqslant 0$, seems to me to be an unnatural auxiliary assumption in the context of the particular equation of state considered by him.

In this paper the necessity of spherical symmetry in isolated static uniform-density stellar models is demonstrated. These stellar models have a somewhat more physically acceptable equation of state than the one considered by Masood-ul-Alam. Furthermore, no unnatural auxiliary assumption is necessary in this case. Thus Masood-ul-Alam's[11] recognition of the importance of the positive mass theorem in the study of static space-times is further supported. This work also supersedes portions of Ref. 8 which erroneously claimed to prove the necessity of spherical symmetry in static uniform-density stellar models. (This error has been noted previously in Ref. 9.) Section II of this paper reviews the established properties of static stellar models that are needed in this analysis. Section III presents the proof that spherical symmetry is a necessary property of isolated static uniform-density stellar models in general relativity theory. The method of proof is to perform a particular conformal transformation on the spatial metric which sets the mass to zero and leaves the scalar curvature non-negative. The demonstration that the scalar curvature resulting from this transformation is non-negative requires the use of the divergence identities for static stellar models found in Ref. 8. The positive mass theorem implies that this conformally transformed metric is flat. The desired result follows from the already established equivalence of spatial conformal flatness and spherical symmetry in static fluid space-times.[8]

II. STATIC STELLAR MODELS

In this section some of the basic properties of static perfect-fluid space-times are reviewed. Careful derivations of these results can be found in the literature. The statements in this section are valid for any static perfect-fluid space-time while those in the next section are valid only for uniform-density stellar models.

A static space-time must admit a hypersurface orthogonal timelike Killing vector field, t^a. Let t be a function whose level surfaces are orthogonal to t^a, and let $t^a\,\partial_a t=1$. The space-time metric can then be represented in the form

$$ds^2=-V^2\,dt^2+g_{ab}\,dx^a\,dx^b, \qquad (1)$$

where g_{ab} is the positive definite three-metric of the constant-t surfaces and $0=\partial_t V=\partial_t g_{ab}$. Einstein's equation for such a space-time with a perfect-fluid stress-energy tensor is equivalent to the system of equations

$$D^aD_aV=4\pi V(\rho+3p), \qquad (2)$$

$$R_{ab}=V^{-1}D_aD_bV+4\pi(\rho-p)g_{ab}, \qquad (3)$$

where D_a and R_{ab} are the three-dimensional covariant derivative and the Ricci curvature tensor associated with g_{ab}, ρ is the total energy density (including rest-mass energy), and p is the pressure of the fluid. To these equations must be added an equation of state: a function $\rho=\rho(p)$ that summarizes the microscopic properties of the particular fluid. This function must be positive and monotonically increasing to be

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physically relevant. Associated with Eq. (3) is a Bianchi identity, which is equivalent to Euler's equation for these static fluids,

$$D_a p = -V^{-1}(\rho + p) D_a V. \quad (4)$$

The solutions of Eqs. (2) and (3) that are of interest in this paper are the physically isolated solutions. Thus we only consider solutions in which the support of the pressure is spatially compact, and in which the space-time metric is asymptotically flat in an appropriate sense. We assume that V and g_{ab} are given asymptotically by expressions of the form,

$$V = 1 - m/r + v, \quad (5)$$

$$g_{ab} = (1 + 2m/r)\delta_{ab} + h_{ab}, \quad (6)$$

where δ_{ab} is the standard flat metric on a constant-t surface, the function r is the asymptotic spherical coordinate given by $r^2 = \delta_{ab} x^a x^b$, and the x^a are the Cartesian coordinates associated with δ_{ab} on each constant-t surface. The quantities v and h_{ab} must vanish like r^{-2} as $r \to \infty$, and their first and second derivatives must vanish with successively higher powers of r^{-1}. The constant m is the mass of the star. These asymptotic conditions on the solutions of Eqs. (2) and (3) can be deduced from rather mild asymptotic falloff assumptions.[15]

To avoid the possibility of surface stresses and surface energy densities and thereby violate the assumption that the stress energy in these space-times is purely that of a perfect fluid, some care must be taken to ensure the proper matching conditions at the boundary between the exterior vacuum region and the interior fluid portion of the stellar model (and in addition at any interior surface on which the equation of state is not continuous). From Eq. (4) it follows that p (and consequently ρ) must have level surfaces that coincide with the level surfaces of V. It also follows from Eq. (4) that p must be a continuous function (since V must be continuous) to avoid the existence of surface stresses on the boundary between the interior and exterior of the star. The pressure must vanish, therefore, on this boundary. Let $V = V_s$ be the level surface of V that corresponds to this boundary. We must also impose an appropriate discontinuity in $D_a D_b V$ at this surface if the equation of state is one for which $\rho(0) \neq 0$ [see, e.g., Eq. (2)]. The needed condition is most easily expressed in terms of the function $W \equiv D^a V D_a V$. This function must satisfy the following discontinuity condition[8] on the surface $V = V_s$:

$$[W^{-1} D_a V D^a W] = -8\pi V_s \rho(0), \quad (7)$$

where $[Q]$ represents the discontinuity (exterior minus interior) in the quantity Q on the surface $V = V_s$.

The conformal properties of a three-geometry are expressed in terms of a certain third-rank tensor field R_{abc} defined by

$$R_{abc} = D_c R_{ab} - D_b R_{ac} + \tfrac{1}{4}(g_{ac} D_b R - g_{ab} D_c R), \quad (8)$$

where $R = R_{ab} g^{ab}$. This tensor vanishes if and only if the geometry is conformally flat.[16] Two different expressions for R_{abc} will be useful in the analysis that follows. The first relates R_{abc} to the geometrical properties of the constant-V two-surfaces in static perfect-fluid space-times:

$$R_{abc} R^{abc} = 8V^{-4} W^2 (\psi_{ab} \psi^{ab} + \tfrac{1}{8} W^{-2} \beta^{ab} D_a W D_b W), \quad (9)$$

where ψ_{ab} is the trace-free part of the extrinsic curvature and β_{ab} is the intrinsic metric of the constant-V two-surfaces. If the metric g_{ab} were conformally flat then the left-hand side of Eq. (9) would vanish. Since the metric g_{ab} is positive definite it would follow that

$$\psi_{ab} = \beta^{ab} D_b W = 0, \quad (10)$$

in this case. Avez[5,6] and Künzle[7] have shown that these conditions, Eq. (10), are equivalent to spherical symmetry. Therefore, Eq. (9) establishes the equivalence of spatial conformal flatness and spherical symmetry for static perfect-fluid space-times.[8] Using Eqs. (2) and (3) R_{abc} can also be expressed completely in terms of V, the fluid variables, and their derivatives. An expression of this type that will be useful in the analysis that follows is given by

$$\begin{aligned}\tfrac{1}{4} V^4 W^{-1} R_{abc} R^{abc} &= D^a D_a W - V^{-1} D^a V D_a W \\ &\quad - \tfrac{3}{4} W^{-1} D^a W D_a W + 8\pi W(\rho + p) \\ &\quad + 4\pi V W^{-1}(\rho + 3p) D^a V D_a W \\ &\quad - 16\pi^2 V^2 (\rho + 3p)^2 - 8\pi V D^a V D_a \rho. \end{aligned} \quad (11)$$

One further property of the conformal transformation of three-geometries will be useful. Consider the conformal metric $\bar{g}_{ab} = \psi^4 g_{ab}$. The conformally transformed scalar curvature $\bar{R}$ is related to R by the equation[16]

$$\bar{R} = \psi^{-4}(R - 8\psi^{-1} D^a D_a \psi), \quad (12)$$

where R and D_a are the scalar curvature and covariant derivative associated with g_{ab}.

III. STATIC UNIFORM-DENSITY STARS MUST BE SPHERICAL

The necessity of spherical symmetry in isolated static uniform-density stellar models will be demonstrated by showing that any such model must be spatially conformally flat. To accomplish this an explicit conformal transformation is performed on the metric. The scalar curvature of the conformally transformed metric is shown to be non-negative. The positive mass theorem is then used to demonstrate that the transformed metric is in fact flat.

Consider the conformal transformation $\bar{g}_{ab} = \psi^4 g_{ab}$, where ψ is the following function of V:

$$\psi(V) = \begin{cases} \tfrac{1}{2}(1 + V), & V_s < V < 1, \\ \tfrac{1}{2}(1 + V_s)^{3/2}(1 + 3V_s - 2V)^{-1/2}, & 0 < V < V_s. \end{cases} \quad (13)$$

Note that $\psi(V)$ and its first derivative are continuous at the surface $V = V_s$. Also note that $\psi''(V)$, the second derivative of $\psi(V)$, vanishes for $V_s < V < 1$ and is positive for $0 < V < V_s$ since

$$\psi''(V) = \tfrac{3}{2}(1 + V_s)^{3/2}(1 + 3V_s - 2V)^{-5/2} > 0. \quad (14)$$

The scalar curvature associated with the metric $\bar{g}_{ab}$ can now be computed using Eq. (12) with the result

$$\overline{R} = 8\psi^{-5}\psi''\{W_0(V) - W\}. \quad (15)$$

The function $W_0(V)$ used in Eq. (15) is defined by

$$W_0(V) = \begin{cases} \frac{2}{3}\pi\rho(1 - V^2)^4(1 - V_s^2)^{-3}, & V_s \leqslant V \leqslant 1, \\ \frac{8}{3}\pi\rho V(3V_s - V) + \frac{2}{3}\pi\rho(1 - 9V_s^2), & 0 < V \leqslant V_s. \end{cases} \quad (16)$$

To establish Eq. (15) it is necessary to use Eqs. (2) and (3) and the fact that the integral of Eq. (4) for the pressure can be written in the form

$$p = \rho V^{-1}(V_s - V) \quad (17)$$

for uniform-density fluids in the domain $0 < V \leqslant V_s$.

The next step is to demonstrate that the scalar curvature $\overline{R}$ given in Eq. (15) is non-negative. Since ψ and ψ'' are non-negative from Eqs. (13) and (14) it remains only to determine the sign of $W_0(V) - W$. The function $W_0(V)$ is continuous at the surface $V = V_s$, while its first derivative satisfies the following discontinuity condition:

$$[W^{-1}D_a VD^aW_0] = -8\pi V_s \rho. \quad (18)$$

This is precisely the same, for uniform-density stellar models, as the discontinuity condition, Eq. (7), satisfied by the first derivative of W. Consequently the function $W_0(V) - W$ and its first derivative are continuous everywhere including the boundary surface $V = V_s$.

The sign of $W_0(V) - W$ will be determined using two identities and the maximum principle for elliptic differential operators. Using Eqs. (11), (16), and (17) it is straightforward to show that in the interior of the star (i.e., the region $0 < V \leqslant V_s$) the following identity must be satisfied[8]:

$$\begin{aligned} D^a\{V^{-1}D_a(W - W_0)\} &= \tfrac{1}{4}V^3W^{-1}R_{abc}R^{abc} \\ &\quad + \tfrac{3}{4}V^{-1}W^{-1}D_a(W - W_0)D^a(W - W_0). \end{aligned} \quad (19)$$

The right-hand side of Eq. (19) is non-negative. The left-hand side is an elliptic differential operator acting on the function $W - W_0$. The maximum principle (see, e.g., Ref. 17) states that $W - W_0$ must achieve its maximum value at a boundary point of the domain on which Eq. (19) is valid (i.e., on the surface $V = V_s$ in this case). Furthermore the gradient $D_a(W - W_0)$ must be nonvanishing and directed out of the domain (the interior of the star in this case) at this maximum point unless the function $W - W_0$ is in fact constant.

A similar identity exists in the exterior of the star[11,18] (i.e., the region where $V_s \leqslant V \leqslant 1$):

$$D_a\{V^{-1}D^aY\} = \frac{V^4R_{abc}R^{abc} + 3X_aX^a}{4VW(1 - V^2)^3}, \quad (20)$$

where X_a and Y are defined by

$$X_a = D_aW + 8VW(1 - V^2)^{-1}D_aV, \quad (21)$$

$$Y = (W - W_0)/(1 - V^2)^3. \quad (22)$$

The right-hand side of Eq. (20) is also non-negative while the left-hand side is an elliptic differential operator on the function Y. The maximum principle implies that the maximum of Y must occur either on the surface of the star where $V = V_s$ or at infinity where $V = 1$.

Consider first the case where the maximum Y occurs at infinity. The asymptotic falloff conditions [i.e., Eqs. (5) and (6)] imply that W and W_0 go to zero like r^{-4} while $1 - V^2$ vanishes like r^{-1}. Therefore Y vanishes at infinity. If the maximum of Y occurs at infinity then $W - W_0$ is necessarily nonpositive in the exterior of the star from Eq. (22). By continuity and the argument given above for the location of the maximum of $W - W_0$ in the interior of the star, it follows that $W \leqslant W_0$ everywhere in the space-time in this case. (This case was inadvertently overlooked in Ref. 8.) Finally, it follows from Eq. (15) that the conformally transformed scalar curvature is non-negative in this case: $\overline{R} \geqslant 0$.

Consider next the case where the maximum of Y (with respect to the exterior of the star) occurs on the surface of the star, $V = V_s$. In this case $Y \geqslant 0$ at this maximum since $Y = 0$ at infinity. It follows that the maximum of the function $Y(1 - V^2)^3$ must occur at the same location as the maximum of Y in this case, since Y is non-negative and the maximum of $(1 - V^2)^3$ occurs on the surface of the star $V = V_s$. Therefore the maximum of $W - W_0 = Y(1 - V^2)^3$ with respect to both the interior and the exterior regions must occur on the surface of the star in this case. Since $D_a(W - W_0)$ is continuous it must vanish at this maximum point. The gradient of Y at this maximum point is given therefore by $D_aY = 6VY(1 - V^2)^{-1}D_aV$. Since V is larger in the exterior of the star than the interior, this gradient points *into* the exterior region. The maximum principle demands that this gradient points *out of* the exterior region unless Y is constant. Since $Y = 0$ at infinity it follows that Y must vanish everywhere in this case, and consequently $W = W_0$ everywhere as well. Thus the conformally transformed scalar curvature would vanish identically in this case: $\overline{R} = 0$.

To summarize, the conformally transformed scalar curvature $\overline{R}$ is necessarily non-negative in a static asymptotically flat uniform-density fluid stellar model.

To complete the proof of the necessity of spherical symmetry the asymptotic behavior of the conformally transformed metric $\bar{g}_{ab}$ must be determined. The asymptotic expansion of the conformal factor defined in Eq. (13) can be determined by the asymptotic form of V given in Eq. (6): $\psi = 1 - m/2r + \phi$, where ϕ vanishes like r^{-2} as $r \to \infty$. It follows that the conformal metric is given in this limit by $\bar{g}_{ab} = \delta_{ab} + \bar{h}_{ab}$, where $\bar{h}_{ab}$ vanishes like r^{-2}. Thus the mass associated with the metric $\bar{g}_{ab}$ vanishes. The positive mass theorem[12–14] states that any three-geometry having non-negative scalar curvature and zero mass is in fact flat. Therefore the metric $\bar{g}_{ab}$ is flat. The physical spatial metric g_{ab} is consequently conformally flat, and the stellar model is therefore spherical.

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