PHYSICAL REVIEW D VOLUME 35, NUMBER 12 15 JUNE 1987

Linear plane waves in dissipative relativistic fluids

William A. Hiscock and Lee Lindblom
Department of Physics, Montana State University, Bozeman, Montana 59717
(Received 17 February 1987)

This paper analyzes the dispersion relations for linear plane waves in the Eckart and the Israel-Stewart theories of dissipative relativistic hydrodynamics. We show that in the long-wavelength (compared to a typical mean-free-path-length) limit the complicated dynamical structure of the Israel-Stewart theory reduces to the familiar dynamics of classical fluids: 9 of the 14 modes of an Israel-Stewart fluid are strongly damped in this limit, two propagate at the adiabatic sound speed (with appropriate viscous and thermal damping), two transverse shear modes decay at the classical viscous damping rate, and the final longitudinal mode is damped at the classical thermal diffusion rate. The short-wavelength limit of these dispersion relations is also examined. We demonstrate that the phase and group velocities of these waves must approach the characteristic velocities in the short-wavelength limit. Finally, we show how some of the perturbations of an Eckart fluid violate causality.

I. INTRODUCTION

In this paper we investigate the properties of linear plane waves in the Eckart and the Israel-Stewart theories of dissipative relativistic hydrodynamics. The purpose of this investigation is to explore the dynamical structure of these fluid theories in a context that is appropriate for making comparisons with experimental studies. We are interested in determining, for example, whether the complicated dynamical structure of the "second-order" Israel-Stewart theory (with its 14 dynamical degrees of freedom) reduces in some appropriate limit to the familiar dynamical structure of relativistic ideal fluids (with only sound modes propagating). To accomplish this we derive expressions for the dispersion relations for the various types of waves contained in these theories. These dispersion relations show that the dynamics of these complicated theories do reduce in appropriate limits to the dynamics of a relativistic ideal fluid or to the dynamics of a nonrelativistic Navier-Stokes-Fourier fluid. Furthermore, these dispersion relations could, in principle, be used to determine the values of the various thermodynamic quantities (e.g., viscosities, thermal conductivity, and second-order coefficients) directly from experimental data. Such an analysis should make it possible to distinguish experimentally between the Eckart and the Israel-Stewart theories.

The simplest covariant generalization of the Navier-Stokes-Fourier theory of dissipative fluids is the first-order theory developed by Eckart.[1] This theory is referred to as "first order" because the expression used for the entropy current contains only first-order terms in the deviations away from equilibrium. This theory is known to have generic instabilities[2] (no stable equilibrium states exist); also, it is not hyperbolic. In this paper we examine a number of interesting features of the linear plane-wave solutions in the Eckart theory. We show that the transverse components of the velocity perturbation satisfy *elliptic* differential equations in the Eckart theory. The solutions to these equations violate any reasonable definition of causality. We show, in fact, that for an initial δ-function perturbation, the solution to the nonrelativistic diffusion equation is a good approximation everywhere inside a future cone whose boundary is defined by the velocity $(\eta c^4/\kappa T)^{1/2}$. For water at 300 K, this velocity is about 10^6 times the speed of light. The Eckart theory, although written in covariant form, is thus not truly a "relativistic" theory, since information can be transmitted faster than the speed of light. We also find some interesting features of the short-wavelength limits of the dispersion relations for an Eckart fluid. The phase and group velocities of the propagating longitudinal modes diverge as $k^{1/3}$ in the Eckart theory, while in the Navier-Stokes-Fourier theory they diverge like k. These diverging velocities probably signal causality violation in these modes as well, although it is very difficult to analyze the evolution of a wave packet of finite spatial extent in a dispersive dissipative theory such as this.

We also consider in this paper the dissipative relativistic fluid theory proposed by Israel[3] and Stewart.[3–6] This is referred to as a second-order theory because the expression used for the entropy current is second order in the deviations away from an equilibrium state. The first-order Eckart theory may be viewed as a (singular) limit of the second-order Israel-Stewart theory. This theory is known to admit stable equilibrium states, and fluctuations about equilibrium are known to propagate causally via hyperbolic differential equations.[7] The theory is therefore an attractive alternative to the simpler but pathological first-order theories.

There are, however, at least two fundamental questions that have yet to be satisfactorily resolved for the second-order theories: (a) How does the complicated dynamical structure of the second-order theories (with their 14 degrees of freedom) reduce to the familiar dynamics of classical fluids (with only sound modes propagating)? (b) Are the second-order theories capable of describing strong shock waves? In this paper we address the first question

 ©1987 The American Physical Society

by evaluating the general dispersion relations for linear plane waves in the Israel-Stewart theory. We show that, in the long-wavelength limit, all of the modes are strongly damped except for those which are the relativistic analogues of the familiar modes of a Navier-Stokes-Fourier fluid. These remaining modes have dispersion relations which are (in the long-wavelength limit) equivalent to the dispersion relations for an Eckart fluid: simple relativistic generalizations of their Navier-Stokes-Fourier counterparts. Thus the Israel-Stewart theory does have an appropriate classical fluid limit. We do not address the second question in the present work. It has been suggested[8,9] that the second-order theories fail to describe adequately the structure of strong shock waves. Those suggestions, however, are based on an analysis of the kinetic theory limit of the Israel-Stewart theory. It is unclear to us to what extent this pathology will persist in the general Israel-Stewart theory.

In Sec. II of this paper we derive the general dispersion relations for longitudinal and transverse plane waves which are solutions to the Israel-Stewart fluid equations linearized about a homogeneous static background equilibrium state. Kranyš[10] has previously studied the analogous linear wave solutions in the kinetic theory limit of the Israel-Stewart theory. We work entirely in the context of the phenomenological fluid theory, however. We do not appeal to kinetic theory except to offer opinions about the anticipated magnitudes of some quantities in laboratory fluids. In Sec. III we examine the long-wavelength (small-wave-number) limit of these dispersion relations. This is the limit (when the fluid fluctuation length scale is much longer than the interparticle separation or the mean free path of the underlying microscopic theory) that we expect the behavior of the theory to closely mimic the predictions of the relativistic theory of perfect fluids, with small dissipative corrections which should be given by the relativistic generalizations of well-known results in the Navier-Stokes-Fourier theory. We expand the dispersion relations for each of the modes in powers of the wave number k up to the first order in which the second-order coefficients α_i and β_i appear. In Sec. IV we examine the large-wave-number limit of the dispersion relations, now expanding the expression for each mode in powers of k^{-1} up to the first term in which the second-order coefficients appear. In Sec. V we examine the limiting case of the dispersion relations for the first-order Eckart theory, and address the issue of whether the Eckart theory violates causality by allowing signal propagation outside the light cone. Finally, in an Appendix, we show that the small-wavelength limits of the phase and group velocities of linear plane waves are equal to the characteristic velocities for systems of differential equations such as those which determine the evolution of the perturbations to an Israel-Stewart fluid.

II. LINEAR PLANE-WAVE PERTURBATIONS

In this section we derive the dispersion relations for linear plane-wave perturbations about a homogeneous and isotropic background equilibrium state of an Israel-Stewart fluid. We use the same notation and conventions as those used in Ref. 7. We treat the perturbations within the Eulerian framework in order to avoid the difficulties associated with gauge ambiguities in the Lagrangian approach.[11,12]

The difference between the actual nonequilibrium value of a field Q at a given point of spacetime and the value which Q has in the fiducial background equilibrium state at the same spacetime point will be denoted by δQ. The quantities $\delta\rho$, δn, $\delta\tau$, δu^a, δq^a, and $\delta\tau^{ab}$ are the fields which describe the perturbations of an Israel-Stewart fluid about its equilibrium state. Any fields which do not include the prefix δ (e.g., ρ, n, u^a) refer to the background equilibrium configuration, and are assumed to satisfy the usual equilibrium constraints (see Ref. 7, Sec. II B), in addition to being homogeneous and isotropic.

The equations of motion for the perturbation fields δQ are obtained by linearizing the general equations for an Israel-Stewart fluid[6,7] about the fiducial homogeneous and isotropic equilibrium state. The equations then become

$$0=\nabla_a\delta T^{ab}\ , \tag{1}$$

$$0=u^a\nabla_a\delta n+n\nabla_a\delta u^a\ , \tag{2}$$

$$0=\zeta^{-1}\delta\tau+\nabla_a\delta u^a+\beta_0 u^a\nabla_a\delta\tau-\alpha_0\nabla_a\delta q^a\ , \tag{3}$$

$$\begin{aligned}0=(\kappa T)^{-1}\delta q^a+q^{ab}(T^{-1}\nabla_b\delta T+u^c\nabla_c\delta u_b\\ +\beta_1 u^c\nabla_c\delta q_b-\alpha_0\nabla_b\delta\tau\\ -\alpha_1\nabla_c\delta\tau^c{}_b)\ ,\end{aligned} \tag{4}$$

$$0=(2\eta)^{-1}\delta\tau^{ab}+\langle\nabla^a\delta u^b+\beta_2 u^c\nabla_c\delta\tau^{ab}-\alpha_1\nabla^a\delta q^b\rangle\ , \tag{5}$$

where $q^{ab}=g^{ab}+u^a u^b$,

$$\begin{aligned}\delta T^{ab}=(\rho+p)(\delta u^a u^b+u^a\delta u^b)+\delta\rho u^a u^b\\ +(\delta p+\delta\tau)q^{ab}+\delta\tau^{ab}+u^a\delta q^b+u^b\delta q^a\ ,\end{aligned} \tag{6}$$

and

$$\langle A^{ab}\rangle=\tfrac{1}{2}q^a{}_c q^b{}_d(A^{cd}+A^{dc}-\tfrac{2}{3}q^{cd}q_{ef}A^{ef})\ , \tag{7}$$

for any second-rank tensor field A^{ab}. The perturbation variables also satisfy the constraints[7]

$$0=u^a\delta q_a=u^a\delta u_a=\delta\tau^{ab}-\langle\delta\tau^{ab}\rangle\ . \tag{8}$$

In order to simplify the analysis, we consider only exponential plane-wave perturbations about a homogeneous and isotropic background equilibrium state in Minkowski space. These assumptions imply that there exists a set of orthonormal Cartesian coordinates (t,x,y,z) such that

$$u^a\partial_a=\partial_t\ . \tag{9}$$

The x axis of the Cartesian coordinate system is chosen to coincide with the direction of spatial variation of the plane waves, so that the perturbation fields have the form

$$\delta Q=\delta Q_0\exp(\Gamma t+ikx)\ , \tag{10}$$

where δQ_0 is a constant. We have chosen to write Eq. (10) with a "frequency" Γ, which is defined so that its real part gives the growth (or decay) time scale of the mode, and its imaginary part gives the oscillation frequency of the mode. We choose this convention because the dispersion relations are simpler when written in terms of Γ, and most of the modes will turn out to be strongly damped. We will hereafter refer to Γ as the frequency of a mode, even when the mode is purely damped, and there is no oscillation in time. The phase velocity and group velocity of such a wave will then be defined as $i\Gamma/k$ and $id\Gamma/dk$, respectively.

Under these conditions the perturbation equations become a set of simple algebraic equations:

$$U^A{}_B \delta Y^B = 0 \,, \tag{11}$$

where $U^A{}_B$ is a 14×14 complex-valued matrix, and δY^B represents the list of the 14 perturbation fields. The index B runs over these 14 fields, while the index A runs over the 14 equations governing the perturbation variables. The columns of $\mathbf{U}$ are defined by choosing a set of perturbation variables, δY^B. We use the set

$$\delta Y^B = \{\delta\rho, \delta n, \delta\tau, \delta u^x, \delta q^x, \delta\tau^{xx}, \delta u^y, \delta q^y, \delta\tau^{xy}, \delta u^z, \delta q^z, \delta\tau^{xz}, \delta\tau^{yz}, \delta\tau^{yy}-\delta\tau^{zz}\} \,. \tag{12}$$

The matrix $U^A{}_B$ may then be put into block-diagonal form, as in the case of the first-order theories:[2]

$$\mathbf{U} = \begin{bmatrix} \mathbf{Q} & 0 & 0 & 0 \\ 0 & \mathbf{R} & 0 & 0 \\ 0 & 0 & \mathbf{R} & 0 \\ 0 & 0 & 0 & \mathbf{S} \end{bmatrix} , \tag{13}$$

where

$$\mathbf{Q} = \begin{bmatrix} 0 & \Gamma & 0 & ink & 0 & 0 \\ \Gamma & 0 & 0 & i(\rho+p)k & ik & 0 \\ 0 & 0 & \zeta^{-1}+\beta_0\Gamma & ik & -ik\alpha_0 & 0 \\ ik\left[\frac{\partial p}{\partial \rho}\right]_n & ik\left[\frac{\partial p}{\partial n}\right]_\rho & ik & (\rho+p)\Gamma & \Gamma & ik \\ \frac{ik}{T}\left[\frac{\partial T}{\partial \rho}\right]_n & \frac{ik}{T}\left[\frac{\partial T}{\partial n}\right]_\rho & -ik\alpha_0 & \Gamma & (\kappa T)^{-1}+\beta_1\Gamma & -ik\alpha_1 \\ 0 & 0 & 0 & \frac{2}{3}ik & -\frac{2}{3}ik\alpha_1 & (2\eta)^{-1}+\beta_2\Gamma \end{bmatrix} , \tag{14}$$

$$\mathbf{R} = \begin{bmatrix} (\rho+p)\Gamma & \Gamma & ik \\ \Gamma & (\kappa T)^{-1}+\beta_1\Gamma & -ik\alpha_1 \\ ik & -ik\alpha_1 & \eta^{-1}+2\beta_2\Gamma \end{bmatrix} , \tag{15}$$

and

$$\mathbf{S} = [(2\eta)^{-1}+\beta_2\Gamma] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} . \tag{16}$$

There exist exponential plane-wave solutions of Eq. (11) whenever Γ and k have values which satisfy the dispersion relation

$$\det(\mathbf{U}) = 0 \,. \tag{17}$$

The determinant of $\mathbf{U}$ will be zero whenever the determinant of one of its diagonal blocks is zero, since

$$\det(\mathbf{U}) = [\det(\mathbf{Q})][\det(\mathbf{R})]^2[\det(\mathbf{S})] \,. \tag{18}$$

The set of all exponential plane-wave solutions is characterized then by the collection of roots obtained by setting the determinants of $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{S}$ separately equal to zero.

The dispersion relations obtained by setting $\det(\mathbf{R})=0$ or $\det(\mathbf{S})=0$ correspond to the solutions of the perturbation equations which are referred to as transverse modes, since the matrices $\mathbf{R}$ and $\mathbf{S}$ operate on the components of the perturbation variables which are orthogonal to the direction of spatial variation (x). There are three distinct sets of transverse modes. First, there are the modes associated with the perturbation variables $\delta\tau^{yz}$ and $\delta\tau^{yy}-\delta\tau^{zz}$, which are governed by the matrix $\mathbf{S}$. Second, there are the modes associated with the variables $(\delta u^y, \delta q^y, \delta\tau^{xy})$, determined by the matrix $\mathbf{R}$. Third, there is a set of modes associated with the variables $(\delta u^z, \delta q^z, \delta\tau^{xz})$, which are also controlled by the matrix $\mathbf{R}$. Since we have assumed that the background equilibrium state is isotropic, the second and third sets of transverse modes obey the same equations of motion, defined by $\mathbf{R}$, and the same dispersion relation, $\det(\mathbf{R})=0$.

The modes associated with the matrix $\mathbf{S}$ (excitations of $\delta\tau^{yy}-\delta\tau^{zz}$ or $\delta\tau^{yz}$) are very simple. Their dispersion relation is given by

$$0 = \det(\mathbf{S}) = [(2\eta)^{-1}+\beta_2\Gamma]^2 \,. \tag{19}$$

These components of the shear stress simply decay exponentially with the time constant,

$$\Gamma_{T1} = -1/(2\beta_2\eta) , \qquad (20)$$

without propagating spatially. Assuming the kinetic theory values of the second-order coefficients[5] are at least approximately correct, these modes will be very strongly damped in laboratory fluids, and are probably unobservable.

The transverse modes associated with the matrix **R** are more interesting. The determinant of **R** is given by

$$0=\eta\kappa T \det(\mathbf{R})=2\eta\kappa T\beta_2[\beta_1(\rho+p)-1]\Gamma^3+\{2\eta\beta_2(\rho+p)+\kappa T[\beta_1(\rho+p)-1]\}\Gamma^2 + \{\rho+p+\eta\kappa T[\beta_1+2\alpha_1+(\alpha_1)^2(\rho+p)]k^2\}\Gamma+\eta k^2 . \qquad (21)$$

In an Israel-Stewart fluid which possesses stable equilibrium states (i.e., which satisfies the stability conditions given in Ref. 7, Sec. III B), the coefficients of all of the terms in Eq. (21) are positive for real wave numbers k. This implies the existence of at least one nonpositive real root for Γ, a nonpropagating decaying mode.

The perturbations which are governed by the matrix **Q** are referred to as longitudinal modes, since the perturbation variables controlled by **Q** are either scalars or the components of tensors in the direction of spatial variation (x) of the exponential plane waves. The frequencies of the longitudinal modes are given by the roots of the dispersion relation, $\det(\mathbf{Q})=0$. The determinant of **Q** may be put into the form

$$0=-\zeta\eta\kappa T\det(\mathbf{Q})=A\Gamma^6+B\Gamma^5+(C+Dk^2)\Gamma^4+(E+Fk^2)\Gamma^3+(Gk^2+Hk^4)\Gamma^2+(Ik^2+Jk^4)\Gamma+Kk^4 , \qquad (22)$$

where $A,B,C,\ldots$ are the following functions of the thermodynamic variables:

$$A=\beta_0\beta_2[\beta_1(\rho+p)-1]\zeta\eta\kappa T , \qquad (23)$$

$$B=\beta_0\beta_2(\rho+p)\zeta\eta+\tfrac{1}{2}(\beta_0\zeta+2\beta_2\eta)[\beta_1(\rho+p)-1]\kappa T , \qquad (24)$$

$$C=\tfrac{1}{2}(\beta_0\zeta+2\beta_2\eta)(\rho+p)+\tfrac{1}{2}[\beta_1(\rho+p)-1]\kappa T , \qquad (25)$$

$$D=\frac{\zeta\eta\kappa T}{\rho+p}\left\{-\beta_0\beta_2(\rho+p)\left[\frac{\partial\Theta}{\partial s}\right]_p+\left[\tfrac{2}{3}\beta_0+\beta_2+\beta_0\beta_2(\rho+p)\left[\frac{\partial p}{\partial\rho}\right]_s\right][\beta_1(\rho+p)-1] + \tfrac{2}{3}\beta_0[\alpha_1(\rho+p)+1]^2+\beta_2[\alpha_0(\rho+p)+1]^2\right\} , \qquad (26)$$

$$E=\tfrac{1}{2}(\rho+p) , \qquad (27)$$

$$F=-\tfrac{1}{2}(\beta_0\zeta+2\beta_2\eta)\left[\frac{\partial\Theta}{\partial s}\right]_p+\frac{2\eta}{3(\rho+p)}\{\beta_0\zeta(\rho+p)+[\alpha_1(\rho+p)+1]^2\} +[\beta_1(\rho+p)-1]\left[\tfrac{1}{2}(\beta_0\zeta+2\beta_2\eta)\left[\frac{\partial p}{\partial\rho}\right]_s+(\tfrac{2}{3}\eta+\tfrac{1}{2}\zeta)\frac{\kappa T}{\rho+p}\right] +\frac{\zeta}{2(\rho+p)}\{2\beta_2\eta(\rho+p)+[\alpha_0(\rho+p)+1]^2\}+\beta_0\beta_2\eta\zeta(\rho+p)\left[\frac{\partial p}{\partial\rho}\right]_s , \qquad (28)$$

$$G=\tfrac{2}{3}\eta+\tfrac{1}{2}\zeta+\tfrac{1}{2}(\rho+p)\left[\frac{\partial p}{\partial\rho}\right]_s(\beta_0\zeta+2\beta_2\eta)-\tfrac{1}{2}\kappa T\left[\left[\frac{\partial\Theta}{\partial s}\right]_p+\left[\frac{\partial p}{\partial\rho}\right]_s[1-\beta_1(\rho+p)]\right] , \qquad (29)$$

$$H=\zeta\eta\kappa T\left\{\tfrac{2}{3}(\alpha_1-\alpha_0)^2-\beta_0\beta_2\left[\frac{\partial p}{\partial\rho}\right]_s\left[\frac{\partial\Theta}{\partial s}\right]_p+\tfrac{2}{3}\beta_0 n\left[\frac{\partial p}{\partial n}\right]_s\left[\alpha_1+\frac{1}{T}\left[\frac{\partial T}{\partial p}\right]_s\right]^2 +\beta_2 n\left[\frac{\partial p}{\partial n}\right]_s\left[\alpha_0+\frac{1}{T}\left[\frac{\partial T}{\partial p}\right]_s\right]^2+(\tfrac{2}{3}\beta_0+\beta_2)(nT^2)^{-1}\left[\frac{\partial T}{\partial s}\right]_p\right\} , \qquad (30)$$

$$I=\tfrac{1}{2}(\rho+p)\left[\frac{\partial p}{\partial\rho}\right]_s , \qquad (31)$$

$$J=\tfrac{1}{2}\kappa T\left\{(\tfrac{4}{3}\eta+\zeta)(nT^2)^{-1}\left[\frac{\partial T}{\partial s}\right]_p-(\beta_0\zeta+2\beta_2\eta)\left[\frac{\partial p}{\partial\rho}\right]_s\left[\frac{\partial\Theta}{\partial s}\right]_p +\tfrac{4}{3}\eta n\left[\frac{\partial p}{\partial n}\right]_s\left[\alpha_1+\frac{1}{T}\left[\frac{\partial T}{\partial p}\right]_s\right]^2+\zeta n\left[\frac{\partial p}{\partial n}\right]_s\left[\alpha_0+\frac{1}{T}\left[\frac{\partial T}{\partial p}\right]_s\right]^2\right\} , \qquad (32)$$

and

$$K=-\tfrac{1}{2}\kappa T\left[\frac{\partial p}{\partial\rho}\right]_s\left[\frac{\partial\Theta}{\partial s}\right]_p . \tag{33}$$

For a stable Israel-Stewart fluid all of these coefficients, $A,\ldots,K$, are non-negative.

The dispersion relations [Eqs. (19), (21), and (22)] give the relationships between the frequency Γ and the wave number k for plane-wave perturbations. These relationships can be used in two different ways. In the first way the wave number k is taken to be a given real number and the dispersion relation is then used to determine the frequency with which such a spatially sinusoidal perturbation evolves in time. This way of using the dispersion relations could be used to evolve arbitrary initial data for the perturbation fields by Fourier transforming the initial data and evolving each Fourier component with the frequency determined from the dispersion relation. When looked at in this way the dispersion relation [Eq. (17)] has 14 roots (not all distinct) for Γ for each value of k. These correspond to the 14 modes of the fluid. The dispersion relation $\det(\mathbf{R})=0$ is a cubic polynomial in Γ, which could, in principle, be solved explicitly. We have not found it enlightening to do so. The dispersion relation $\det(\mathbf{Q})=0$ is a sixth-order polynomial in Γ. We have not been successful in factoring it, and it may not be possible to do so explicitly for general fluids.

The second way of using the dispersion relation takes the frequencies Γ to be given imaginary numbers. This might correspond to an experimental situation in which perturbations in the fluid are driven at a given frequency by some external agent. The dispersion relations could then be used to determine the wave numbers k and the resulting spatial variation of the fluid perturbations. In this case one can "solve" the dispersion relations explicitly in general. Equation (21) is a simple linear equation for k^2 while Eq. (22) is a quadratic equation for k^2 in terms of Γ. Note that there exist only eight roots (not all distinct) of these equations for k in terms of Γ. It is not possible to excite all of the dynamical degrees of freedom of these fluids by driving them externally at a given frequency. Some of the modes (e.g., those associated with $\mathbf{S}$) cannot be made to oscillate at a given frequency.

III. THE SMALL-WAVE-NUMBER LIMIT

Since the dispersion relations are so complicated, it is informative to examine their limiting forms in order to obtain some insight into these modes. We first consider the $k\rightarrow 0$ limit of the dispersion relations. In this limit the perturbations are assumed to vary slowly in space (i.e., they must have large wavelengths compared to the characteristic length scales of the fluid such as the interparticle separation or the mean free path). This limit corresponds to the regime in which classical fluid mechanics is known to represent real fluids well. This is the appropriate context, therefore, in which to look for a correspondence between the Israel-Stewart theories, with their 14 dynamical degrees of freedom, and the much simpler dynamical structure of classical fluid dynamics. We first consider the transverse modes and then discuss the longitudinal modes in this limit.

A. Transverse modes

Expanding the dispersion relation for the transverse modes, Eq. (21), about $k=0$ in a Taylor series in k, we find the three transverse-mode frequencies in this limit have the values

$$\Gamma_{T2}=-\frac{1}{2\beta_2\eta}+O(k^2)\ , \tag{34}$$

$$\Gamma_{T3}=-\frac{\rho+p}{\kappa T[\beta_1(\rho+p)-1]}+O(k^2)\ , \tag{35}$$

$$\Gamma_{T4}=-\frac{\eta}{\rho+p}k^2-\left[\frac{\eta}{\rho+p}\right]^2\left[2\beta_2\eta-\frac{\kappa T}{\rho+p}[1+\alpha_1(\rho+p)]^2\right]k^4+O(k^6)\ . \tag{36}$$

The two transverse modes with frequencies Γ_{T2} and Γ_{T3} are very strongly damped (if kinetic theory is even approximately correct), and are probably not observable in laboratory fluids. The third mode with frequency Γ_{T4} represents, at order k^2, the well-known viscous damping of transverse stresses; we also calculated the k^4 term in this expansion to show the first second-order corrections to the dispersion relation for this mode. Higher-order effects of this type should be observable, e.g., in drag measurements on Couette flow between rotating cylinders.[13]

These limiting forms of the dispersion relations [Eqs. (20) and (34)–(36)] show that the 8 transverse degrees of freedom of the Israel-Stewart fluid dynamics do reduce to the appropriate classical behavior in this limit. The six modes having frequencies Γ_{T1}, Γ_{T2}, and Γ_{T3} are all severely damped in this limit and are probably unobservable. The remaining two modes with frequencies Γ_{T4} have the same diffusive dispersion relation in this limit as the transverse modes in an Eckart fluid. These expressions are simply the relativistic generalizations of those for the transverse modes of a Navier-Stokes-Fourier fluid.

B. Longitudinal modes

Expanding the dispersion relation for the longitudinal modes in powers of k about $k=0$, we find that there are three longitudinal modes whose frequencies do not go to zero as k goes to zero:

$$\Gamma_{L1}=-\frac{1}{\beta_0\zeta}+O(k^2)\ , \tag{37}$$

$$\Gamma_{L2}=-\frac{\rho+p}{\kappa T[\beta_1(\rho+p)-1]}+O(k^2)\ , \tag{38}$$

and

$$\Gamma_{L3}=-\frac{1}{2\beta_2\eta}+O(k^2)\ . \tag{39}$$

These modes are all nonpropagating (up to order k^2) and are probably very strongly damped in laboratory fluids.

The next two longitudinal modes vary linearly with k in the small-k limit. Their dispersion relation is given by

$$\Gamma_{L4}^{\pm} = \pm i \left[\frac{\partial p}{\partial \rho} \right]_s^{1/2} k - \frac{k^2}{2(\rho+p)} \left[\tfrac{4}{3}\eta + \zeta + \kappa \left[\frac{\partial p}{\partial \rho} \right]_s \left[\frac{\partial \rho}{\partial s} \right]_p^2 (n^2 T)^{-1} \right] + O(k^3) , \quad (40)$$

up to order k^3. To lowest order in k these two longitudinal modes propagate with phase and group velocities equal to $(\partial p/\partial \rho)_s{}^{1/2}$, the adiabatic sound speed. The next-order (k^2) contribution to the frequency of these modes is purely real and determines the damping rate of these sound waves. The Israel-Stewart theory predicts exactly the same velocities and damping coefficient (to lowest orders in k) as the first-order Eckart theory. Furthermore, these expressions are simply the relativistic generalizations of the usual results for the velocities and damping of nonrelativistic sound waves in the Navier-Stokes-Fourier theory.[14] The second-order coefficients α_i and β_i enter the dispersion relation for this mode in the k^3 term. Even though it is straightforward to evaluate, we were unable to find a simple expression for the k^3 term in the dispersion relation for this mode; consequently we chose not to reproduce our lengthy expression here. There exists[13] a great deal of experimental data on the velocity and damping of extremely high-frequency sound waves. These data could be used to determine empirically the values of the second-order coefficients that appear in this dispersion relation. We are not aware that these data have ever been analyzed in this way, however.

Finally, the sixth longitudinal mode has the dispersion relation

$$\Gamma_{L5} = -\frac{\kappa}{nT} \left[\frac{\partial T}{\partial s} \right]_p k^2 + O(k^4) , \quad (41)$$

through order k^2. This mode is another nonpropagating decaying mode; it is, however, not as strongly damped in laboratory fluids as the modes with frequencies Γ_{L1}, Γ_{L2}, and Γ_{L3}. The first term (order k^2) in this dispersion relation [Eq. (41)] represents the classic thermal diffusion of temperature fluctuations. This term is the same in the first-order (Eckart) and second-order (Israel-Stewart) theories. The second-order coefficients first appear in the k^4 term in this dispersion relation. We do not include that expression here because it is unenlightening and somewhat lengthy.

We note that the dynamical behavior of the longitudinal modes in a second-order (Israel-Stewart) fluid reduce to the familiar behavior of a classic fluid in the long-wavelength limit considered here. Three of the six longitudinal modes (with frequencies Γ_{L1}, Γ_{L2}, and Γ_{L3}) are severely damped in this limit and are probably unobservable in laboratory fluids. Two of the longitudinal modes have dispersion relations [Eq. (40)] which yield group and phase velocities identical to perfect-fluid adiabatic sound waves through order k^2, and have damping coefficients which are the simple relativistic generalizations of the classic Navier-Stokes-Fourier results[14] at order k^2. And the final longitudinal mode is identical to classic thermal diffusion to lowest order. Thus we conclude that the dynamics of the second-order theories do indeed reduce to the simpler dynamics of classical fluid dynamics for normal laboratory fluids, and (to lowest orders) to the dynamics of the simpler adiabatic relativistic fluids where appropriate.

Note that none of the phase velocities or the group velocities of these modes in this limit are equal to any of the characteristic velocities for these equations.[7] The characteristic velocities determine the rate at which discontinuities in the perturbations propagate in the fluid. Any discontinuity in the initial data will necessarily involve contributions (in the Fourier transform of the perturbation) from the large-k (short-wavelength) structure of the theory. Consequently it is not surprising that the long-wavelength velocities derived in this section have little relation to the characteristic velocities.

IV. THE LARGE-WAVE-NUMBER LIMIT

Next, we consider the short-wavelength ($k \to \infty$) limit of the dispersion relations. One could argue that this limit is not physically relevant for a fluid theory because the derivations of the fluid equations from microphysics are generally not valid when the characteristic length scales become smaller than a typical mean-free-path length in the underlying microscopic theory. However, we do not believe that this argument limits the range of applicability of a phenomenological fluid theory *a priori*. Experience with nonrelativistic fluids has demonstrated that a phenomenological fluid theory (e.g., Navier-Stokes-Fourier) may be quite useful in situations where the derivation of the theory from microphysics is inadequate.

A. General results

In the Appendix we show that the short-wavelength (large-k) limits of the phase and group velocities for linear plane waves are necessarily equal, and that these limits are also equal to the characteristic velocities of the associated system of differential equations. This result, which applies to a very general class of first-order linear differential equations, guarantees that the leading term in the dispersion relation for these waves will have the form

$$\Gamma = -ivk + O((k^{-1})^0) , \quad (42)$$

where v is one of the characteristic velocities. These modes will consequently be propagating modes at lowest order (in an expansion in powers of k^{-1}) if the characteristic velocities are real (as they must be in a hyperbolic system of equations) and if $v \neq 0$.

B. Transverse modes

The transverse modes have two characteristic velocities given by

$$(v_T)^2=[(\rho+p)(\alpha_1)^2+2\alpha_1+\beta_1]\{2\beta_2[\beta_1(\rho+p)-1]\}^{-1}\ , \tag{43}$$

and one characteristic velocity which vanishes. We have shown[7] that these characteristic velocities [Eq. (43)] are real and bounded above by the speed of light for stable Israel-Stewart fluids. The leading term in the dispersion relations for these two modes are consequently given by Eq. (42) with $v=v_T$. The leading term in the expansion of the dispersion relation for the third transverse mode is given by

$$\Gamma=-\{\kappa T[(\alpha_1)^2(\rho+p)+2\alpha_1+\beta_1]\}^{-1}+O(k^{-1})\ . \tag{44}$$

This mode is strongly damped in this large-k limit.

C. Longitudinal modes

The longitudinal modes have two characteristic velocities which vanish and four characteristic velocities which are the roots of the quartic polynomial:

$$A(v_L)^4-D(v_L)^2+H=0\ , \tag{45}$$

where A, D, and H are given by Eqs. (23), (26), and (30). The roots of this equation are known to be real and bounded above by the speed of light for stable Israel-Stewart fluids.[7] The leading terms in the dispersion relations for these four modes are consequently given by Eq. (42) with $v=v_L$. The remaining two longitudinal modes have vanishing characteristic velocities. The leading terms in the expansion of the dispersion relations for these two modes are given by

$$\Gamma=-[J\pm(J^2-4HK)^{1/2}]/(2H)+O(k^{-1})\ , \tag{46}$$

where H, J, and K are given by Eqs. (30), (32), and (33). These coefficients are all positive for a stable Israel-Stewart fluid, so that the modes are damped at highest order in k in this case.

V. THE ECKART LIMIT

The second-order Israel-Stewart fluid equations become the first-order Eckart equations[1,15,16] in the limit that the coefficients α_i and β_i are set equal to zero. The dispersion relations for this limiting theory have been studied previously in the context of determining the stability of the equilibrium states of an Eckart fluid.[2] In this final section we describe a few interesting, previously unnoticed, features of the linear plane-wave solutions of an Eckart fluid. We focus principally on the question of whether or not linear perturbations in the Eckart theory violate causality. There have been frequent assertions in the literature that the Eckart theory does violate causality, and hence cannot be an acceptable relativistic theory, but to the best of our knowledge no proof of the supposed noncausal behavior has previously appeared.

We first reexamine the transverse perturbations of an Eckart fluid. The differential equations for the perturbations in the transverse variables (δu^i, δq^i, and $\delta\tau^{xi}$, where $i=y$ or z) can be decoupled into a single second-order equation for the transverse components of the perturbed velocity:

$$\kappa T\partial_t{}^2\delta u^i-(\rho+p)\partial_t\delta u^i+\eta\partial_x{}^2\delta u^i=0\ . \tag{47}$$

This is an *elliptic* equation for the δu^i. Clearly the solutions to this equation violate any reasonable definition of causality. To illustrate this noncausal behavior, we have integrated this equation to determine the "evolution" of a perturbation which at time $t=0$ is a simple δ function: $\delta u^i(x,0)=\delta(x)$. The evolution of these "initial data" can be determined by the usual Fourier transform techniques. The solution of Eq. (47) with this initial condition is given by

$$\delta u^i(x,t)=\frac{(\rho+p)t}{2\pi\kappa t}\left[\frac{\eta}{\kappa T}t^2+x^2\right]^{-1/2}K_1\left[\frac{\rho+p}{2\sqrt{(\kappa T\eta)}}\left[\frac{\eta}{\kappa T}t^2+x^2\right]^{1/2}\right]\exp\left[\frac{(\rho+p)t}{2\kappa T}\right]\ , \tag{48}$$

where $K_1(z)$ is a modified Bessel function. This equation reduces to the standard classical expression for the diffusion of shear stresses:

$$\delta u^i(x,t)=\left[\frac{\rho+p}{4\pi\eta t}\right]^{1/2}\exp\left[-\frac{x^2(\rho+p)}{4\eta t}\right]\ , \tag{49}$$

in the limit that $t^2\gg(\kappa T/\eta)x^2$ and $t\gg\kappa T/(\rho+p)$. Thus the classical expression is valid inside the future cone determined by the velocity $(\eta c^4/\kappa T)^{1/2}$. For a normal laboratory fluid this velocity is very large; for water at $T=300$ K, for example, the velocity is about 10^6 times the speed of light [simple kinetic theory indicates that this velocity should generally be of order $(mc^2/kT)^{1/2}$ times the speed of light]. The characteristic time $\kappa T/[(\rho c^2+p)c^2]$ is very short for normal fluids (about 10^{-35} sec for water at 300 K). Therefore the classical expression for the diffusion of shear stresses is valid in a region which includes and extends outside the future light cone of the plane $(x,t)=(0,0)$ where the initial disturbance in δu^i occurs. Consequently, Eq. (48), the Green's function for the evolution of transverse perturbations in the fully relativistic Eckart theory, violates causality just as badly as the analogous expression for the perturbations of a Navier-Stokes-Fourier fluid. The Eckart theory thus cannot be considered to be an acceptable relativistic theory, even if one were willing to overlook its other serious problems (e.g., lack of stable equilibria).

The longitudinal modes of an Eckart fluid also have some interesting properties. In the long-wavelength limit the dispersion relations for these modes are the obvious limits of the expressions for the dispersion relations for the modes $L2$, $L4^{\pm}$, and $L5$ given in Eqs. (38), (40), and (41). In the short-wavelength limit, however, the dispersion relations for the modes of an Eckart fluid behave rather differently than their Israel-Stewart counterparts. In particular, three of the modes have the dispersion relation

$$\Gamma = \mu k^{4/3} + O(k^{2/3}) , \qquad (50)$$

where μ is any one of the three cube roots of

$$\mu^3 = (\tfrac{4}{3}\eta + \zeta)\frac{1}{T}\left[\frac{\partial T}{\partial \rho}\right]_n . \qquad (51)$$

Assuming that the fluid has positive specific heats, the quantity μ^3, defined in Eq. (51), is a postitive real number. Its cube roots, which give the frequencies of these modes, consist of a positive real number (which corresponds to a growing nonpropagating mode), and a pair of complex conjugate values for μ (which represent two decaying propagating modes). The interesting feature of these dispersion relations is their dependence on fractional powers of the wave number, a behavior which is substantially different than in the nonrelativistic Navier-Stokes-Fourier theory. The relativistic modifications of the fluid equations (to render them covariant) have changed the small-length-scale behavior of the theory somewhat. The phase velocity and group velocity of these propagating waves still diverge (as $k^{1/3}$) in the Eckart theory, but not as rapidly as in the nonrelativistic Navier-Stokes-Fourier theory where they are proportional to k.

Finally, it is interesting to examine one more special case: the first-order Eckart theory in the limit when the viscosity coefficients are set to zero (a first-order relativistic version of the Fourier theory of heat flow). The resulting dispersion relations in this theory have a quite different behavior than the Newtonian theory in the large-k limit:

$$\Gamma^2 = -\frac{k^2}{2}\left[\frac{\partial p}{\partial \rho}\right]_s \left\{1 + \left[\frac{\partial \Theta}{\partial s}\right]_p \left[\frac{\partial \rho}{\partial p}\right]_\Theta \pm \left\{\left[1 + \left[\frac{\partial \Theta}{\partial s}\right]_p \left[\frac{\partial \rho}{\partial p}\right]_\Theta\right]^2 - 4\left[\frac{\partial \Theta}{\partial s}\right]_p \left[\frac{\partial \rho}{\partial p}\right]_s\right\}^{1/2}\right\} + O(k^4) . \qquad (52)$$

The two modes corresponding to the minus sign in Eq. (52) are nonpropagating (one growing and one decaying), the other two modes (with the + sign) are propagating nondecaying modes. These modes are the large-wave-number limit of sound waves; the coefficient of the k^2 term in Eq. (52) is the square of the phase and group velocity of these waves. Note that it is not simply $(\partial p/\partial \rho)_s$; the nonzero thermal conductivity changes the dynamics of these waves in this limit in a way which does not depend on the value of the thermal conductivity [except that it must be nonzero for Eq. (52) to hold]. It is also interesting to note that the only propagating modes in this case have *finite* phase and group velocities; there is no mode in this limited theory (Eckart with viscosities set to zero) which satisfies a parabolic-type dispersion relation in the large-k limit. The large-k behavior of this relativistic Fourier theory thus seems distinctly improved relative to the nonrelativistic Fourier theory of heat flow, in which thermal perturbations obey a parabolic dispersion relation. Unfortunately, this theory possesses no stable equilibrium states, unlike its nonrelativistic counterpart.

ACKNOWLEDGMENTS

We thank R. Geroch, K. Nordtvedt, and R. Robiscoe for helpful discussions during the course of this work, which was supported in part by National Science Foundation Grants Nos. PHY 85-05484 and PHY 85-18490.

APPENDIX

In this appendix we prove the following proposition which establishes the equality between the values of the phase and group velocities and the characteristic velocity for general linear plane waves in the large-wave-number limit ($k \to \infty$).

Proposition. Consider a dispersion relation of the form

$$\det[\omega(k)\mathbf{A} + k\mathbf{B} + \mathbf{C}] = 0 , \qquad (A1)$$

where $\mathbf{A}$, $\mathbf{B}$, *and* $\mathbf{C}$ *are constant* $n \times n$ *matrices with* $\det(\mathbf{A}) \neq 0$. *Assume that* $v \equiv \lim(\omega/k)$ *is finite as* $k \to \infty$. *Then the short-wavelength* ($k \to \infty$) *limits of the phase and group velocities are equal for this system:*

$$\lim(d\omega/dk) = \lim(\omega/k) = v ; \qquad (A2)$$

and v *satisfies*

$$\det(v\mathbf{A} + \mathbf{B}) = 0 , \qquad (A3)$$

the characteristic equation for the associated system of differential equations.

Proof. It follows that v satisfies Eq. (A3) by dividing Eq. (A1) by k^n and taking the limit as $k \to \infty$.

The proof of the equality of the phase and group velocities is more straightforward when there are no asymptotically degenerate roots of the dispersion relation, Eq. (A1). We proceed to give the proof in some detail in this case and then outline the proof in the more general degenerate case. First define the matrices

$$E^a{}_b = \omega A^a{}_b + kB^a{}_b + C^a{}_b , \qquad (A4)$$

$$M^a{}_b = \frac{1}{n!}k^{-n+1}\epsilon^{bb_2\cdots b_n}\epsilon_{aa_2\cdots a_n}E^{a_2}{}_{b_2}\cdots E^{a_n}{}_{b_n} , \qquad (A5)$$

where $\epsilon^{abc\cdots n}$ and $\epsilon_{abc\cdots n}$ are the totally antisymmetric tensors and summation over repeated indices is implied.

The dispersion relation, Eq. (A1), for these waves is equivalent to det(**E**)=0, which may be written in the form

$$0=\det(\mathbf{E})=k^{n-1}E^a{}_b M^b{}_a \ . \quad \text{(A6)}$$

Equation (A6) holds for all values of k, therefore the derivative of this expression also vanishes for all k. This derivative may be written in the form

$$0=d[\det(\mathbf{E})]/dk=nk^{n-1}M^b{}_a dE^a{}_b/dk \ . \quad \text{(A7)}$$

We combine Eqs. (A6) and (A7) to obtain the desired relation between the phase and group velocities:

$$0=(dE^a{}_b/dk-E^a{}_b/k)M^b{}_a = \left[\frac{d\omega}{dk}-\frac{\omega}{k}\right]A^a{}_b M^b{}_a-\frac{1}{k}C^a{}_b M^b{}_a \ . \quad \text{(A8)}$$

The final step is to take the limit of this expression as $k\to\infty$. We note that $k^{-1}E^a{}_b$ and consequently $M^a{}_b$ are well behaved in this limit since $\lim(\omega/k)$ was assumed to be well behaved. It follows that

$$0=\lim_{k\to\infty}\left[\frac{d\omega}{dk}-\frac{\omega}{k}\right]\lim_{k\to\infty}A^a{}_b M^b{}_a \ . \quad \text{(A9)}$$

The desired equality between the group and phase velocities follows unless $\lim A^a{}_b M^b{}_a=0$. This coefficient will unfortunately always vanish when the dispersion relation admits roots that are asymptotically degenerate.

To extend this proof to the asymptotically degenerate case we will differentiate the dispersion relation, Eq. (A1), m times [where m is the degeneracy of the root of Eq. (A1) in question] to remove the degeneracy. To accomplish this it is helpful to define the quantities

$$Q_{i,j}=\lim_{k\to\infty}(k^{i+j-n}\epsilon_{a_1\cdots a_i a_{i+1}\cdots a_{i+j}a_{i+j+1}\cdots a_n}\epsilon^{b_1\cdots b_i b_{i+1}\cdots b_{i+j}b_{i+j+1}\cdots b_n} \times A^{a_1}{}_{b_1}\cdots A^{a_i}{}_{b_i}B^{a_{i+1}}{}_{b_{i+1}}\cdots B^{a_{i+j}}{}_{b_{i+j}}E^{a_{i+j+1}}{}_{b_{i+j+1}}\cdots E^{a_n}{}_{b_n}) \ . \quad \text{(A10)}$$

If the system is degenerate then some of the quantities $Q_{i,0}$ will vanish. In particular assume that $Q_{i,0}=0$ for $1\le i\le m-1$ but that $Q_{m,0}\neq 0$ for some m. [Since $Q_{n,0}=n!\det(\mathbf{A})$ and since $\det(\mathbf{A})$ is assumed to be nonzero, we know that there will always be some m for which $Q_{m,0}\neq 0$.] These $Q_{i,j}$ satisfy the recursion relation

$$Q_{i,j}=vQ_{i+1,j}+Q_{i,j+1} \ , \quad \text{(A11)}$$

from which it follows by induction that $Q_{i,j}=0$ for $1\le i+j\le m-1$.

We next compute the mth derivative of det(**E**). This quantity vanishes for all k, so in the limit $k\to\infty$ we have

$$0=\lim_{k\to\infty}\{k^{m-n}d^m[\det(\mathbf{E})]/dk^m\}=\lim_{k\to\infty}\left[\frac{1}{m!}k^{m-n}\epsilon_{a_1\cdots a_m a_{m+1}\cdots a_n}\epsilon^{b_1\cdots b_m b_{m+1}\cdots b_n} \times(dE/dk)^{a_1}{}_{b_1}\cdots(dE/dk)^{a_m}{}_{b_m}E^{a_{m+1}}{}_{b_{m+1}}\cdots E^{a_n}{}_{b_n}\right] \ . \quad \text{(A12)}$$

The terms involving higher derivatives of ω vanish in this limit because they always multiply some vanishing $Q_{i,j}$. Finally, Eq. (A12) can be brought into the following form by adding to it appropriate linear combinations of the vanishing $Q_{i,j}$:

$$0=\lim_{k\to\infty}\left[k^{m-n}\epsilon_{a_1\cdots a_m a_{m+1}\cdots a_n}\epsilon^{b_1\cdots b_m b_{m+1}\cdots b_n}\left[\frac{dE}{dk}-\frac{E}{k}\right]^{a_1}{}_{b_1}\cdots\left[\frac{dE}{dk}-\frac{E}{k}\right]^{a_m}{}_{b_m}E^{a_{m+1}}{}_{b_{m+1}}\cdots E^{a_n}{}_{b_n}\right] \ . \quad \text{(A13)}$$

When the explicit forms for the $dE^a{}_b/dk$ and $E^a{}_b/k$ are used in this expression we find

$$0=Q_{m,0}\lim_{k\to\infty}\left[\frac{d\omega}{dk}-\frac{\omega}{k}\right]^m \ . \quad \text{(A14)}$$

Since $Q_{m,0}\neq 0$ by assumption, the desired equality between the group and phase velocities follows. Note that the assumption, $\det(\mathbf{A})\neq 0$, was made only to assure the inequality $Q_{m,0}\neq 0$ for some m. This assumption can clearly be considerably weakened.

[1] C. Eckart, Phys. Rev. **58**, 919 (1940).
[2] W. A. Hiscock and L. Lindblom, Phys. Rev. D **31**, 725 (1985).
[3] W. Israel, Ann. Phys. (N.Y.) **100**, 310 (1976).
[4] J. M. Stewart, Proc. R. Soc. London **A357**, 59 (1977).
[5] W. Israel and J. M. Stewart, Proc. R. Soc. London **A365**, 43 (1979).
[6] W. Israel and J. M. Stewart, Ann. Phys. (N.Y.) **118**, 341 (1979).
[7] W. A. Hiscock and L. Lindblom, Ann. Phys. (N.Y.) **151**, 466 (1983).
[8] H. Grad, Commun. Pure Appl. Math. **5**, 257 (1952).
[9] A. M. Anile and A. Majorana, Meccanica **16**, 149 (1981); A. Majorana and S. Motta, J. Non-Equilib. Thermodyn. **10**, 1

(1985).
[10]M. Kranyš, Ann. Inst. Henri Poincaré **25**, 197 (1976).
[11]B. F. Schutz and R. Sorkin, Ann. Phys. (N.Y.) **107**, 1 (1977).
[12]J. L. Friedman and B. F. Schutz, Astrophys. J. **221**, 937 (1078).
[13]F. S. Sherman and L. Talbot, in *Rarified Gas Dynamics*, edited by F. M. Devienne (Academic, New York, 1960), p. 161.
[14]L. Landau and E. M. Lifshitz, *Fluid Mechanics* (Addison Wesley, Reading, MA, 1958), Sec. 77.
[15]S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972), pp. 53–57.
[16]C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), pp. 567 and 568.