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# THE ROLE OF THE VISCOUS SECULAR INSTABILITY IN ROTATING NEUTRON STARS

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## ABSTRACT

For neutron stars colder than $\sim 10^7$ K the viscosity of neutron matter is so large that the well-known gravitational radiation secular instability is completely damped out by the viscosity. That instability probably does not play a role, therefore, in limiting the rotation rate of neutron stars (including possibly the "millisecond pulsar" PSR 1937+214) that were spun up by accretion after they had already cooled. This paper examines the other classic secular instability—the one driven by viscosity. It is shown that the viscosity--driven secular instability can limit the rotation rate of neutron stars colder than $\sim 10^5$ K. Stars with temperatures between $\sim 10^5$ and $10^7$ K appear to be stable to both secular instabilities.

*Subject headings:* dense matter — pulsars — stars: neutron — stars: rotation

## I. INTRODUCTION

The discovery of the pulsar PSR 1937+214 (Backer *et al.* 1982) with period 1.56 ms has stimulated a great deal of interest in determining the minimum possible rotation period for neutron stars. A number of authors (e.g., Friedman 1983; Wagoner 1984) have suggested that the gravitational radiation driven secular instability first discussed by Chandrasekhar (1970*a*, *b*) and Friedman and Schutz (1977) is probably responsible for limiting the angular velocity that neutron stars can achieve. A number of different computations were undertaken, therefore, to determine accurately the minimum rotation period allowed by this instability (Imamura, Friedman, and Durisen 1985; Lindblom 1986; Managan 1986). This paper will suggest that this scenario is probably not correct in sufficiently cold neutron stars ($T \lesssim 10^7$ K). For cold stars it appears that the viscosity of neutron star matter is so large that the gravitational radiation secular instability is completely damped out. In this case another secular instability can occur (in a different set of modes) that is driven by viscosity. For stars that are spun up to high angular velocity by accretion after they have cooled (see, e.g., Ghosh and Lamb 1979), it would appear that the viscous secular instability discussed here, rather than the gravitational radiation secular instability, is probably responsible for providing the upper limit to angular velocity.

The gravitational radiation secular instability tends to make *all* rotating stars unstable (Friedman 1978). Consequently, the maximum angular velocity for a stable, stationary neutron star would be zero except that viscosity blocks the instability in sufficiently slowly rotating stars (see Lindblom and Detweiler 1977; Lindblom and Hiscock 1983). To determine where the actual limit on the angular velocity occurs, therefore, one must evaluate the effects of gravitational radiation *and* viscosity on the relevant oscillation modes of neutron stars. Cutler and Lindblom (1987) recently computed the effects of the appropriate viscosity functions for neutron star matter (see Flowers and Itoh 1979) on the modes of fully relativistic neutron stars. The viscosity of neutron star matter has a temperature dependence of $T^{-2}$ (except when the transition to a superfluid state is taking place). Consequently, for sufficiently low temperatures the viscosity effects the evolution of a given mode more quickly than gravitational radiation. (Superfluidity *increases* the viscosity of neutron star matter by about a factor of 5 over the normal fluid viscosity at the same temperature and density.) Cutler and Lindblom (1987) estimate that in neutron stars having temperatures below $\sim 10^6$ to $10^7$ K (depending on the equation of state) the viscosity will be large enough to suppress the gravitational radiation secular instability completely. It follows that this secular instability cannot be responsible for limiting the angular velocities of stars spun up by accretion after they have already cooled.

The earliest studies of the secular instability of rotating stars by Kelvin in the 19th century (see, e.g., Thomson and Tait 1883) were concerned with a different secular instability: the one driven by viscosity. The viscous secular instability occurs in the uniform density Maclaurin spheroids in the modes having spheroidal harmonic indices $l = -m$ (see Roberts and Stewartson 1963). (In contrast, the gravitational radiation secular instability causes the $l = +m$ modes to grow.) Since gravitational radiation tends to damp out the $l = -m$ modes, the viscous instability will occur only when the viscosity is large enough that it dominates over gravitational radiation in determining the evolution of the mode. In neutron stars this will always occur at sufficiently low temperatures. The purpose of this paper is to develop the techniques needed to estimate the frequencies (including the imaginary parts that determine stability) of these "$l = -m$" modes in realistic neutron stars.

In § II of this paper the $l = -m$ modes of the Maclaurin spheroids are examined. Expressions are found for the angular velocity dependence of the real and the imaginary parts of the frequencies of these modes. The imaginary parts of the frequencies are caused by the viscosity and the gravitational radiation reaction. Section III derives the equations for the critical angular velocities where the secular instabilities in rotating stars first occur. In § IV the frequencies of the $l = -m$ modes of rapidly rotating fully relativistic neutron stars are estimated. These estimates use the real and imaginary parts of the frequencies of fully relativistic but nonrotating stars together with the angular velocity dependence of the modes derived in § II for the Maclaurin spheroids. The critical angular velocities of neutron stars as a function of temperature can be computed from these frequency estimates. Sufficiently rapidly rotating neutron stars with temperatures below $\sim 10^5$ K will be subject to an instability, driven by viscosity, in one of these $l = -m$

modes. The critical angular velocities associated with this instability are probably the ones that limit the rate to which a cold neutron star can be spun up by accretion. These critical angular velocities are computed here for a range of possible neutron star masses and different equations of state for the highest density portion of the neutron star matter.

## II. THE MACLAURIN SPHEROID EXTRAPOLATION

While numerical models of rapidly rotating fully relativistic neutron stars have been computed in recent years (Friedman, Ipser, and Parker 1985, 1986), many of their relevant properties (including oscillations and stability) have yet to be investigated. Such investigations are not trivial and are not likely to be satisfactorily completed in the near future. Until then it is necessary to study the properties of rotating models using approximation techniques. Lindblom (1986) developed a method for extrapolating to large angular velocities the properties of the more easily studied nonrotating stellar models. The value of a quantity in a nonrotating star is extrapolated to nonzero angular velocity using the expression for the angular velocity dependence of this quantity in the Maclaurin spheroids. It was shown by Managan (1986) that this technique predicts the angular velocity dependence of the frequencies of the $l = +m$ modes of rotating Newtonian polytropes to within $\sim$5%–10%. This degree of inaccuracy is acceptably small since the uncertainties caused by the lack of knowledge of the equation of state and other factors in neutron stars are generally much larger. In this section expressions are developed to extrapolate the frequencies (both real and imaginary parts) of the $l = -m$ modes of stars to large angular velocity.

Consider a one-parameter family of rigidly rotating stellar models having the same mass and equation of state but varying angular velocity $\Omega$. Consider the oscillation modes of these stars having angular dependence $e^{im\varphi}$ and time dependence $e^{(i\sigma t - t/\tau)}$. Let $\sigma_l^-(\Omega)$ denote the angular velocity dependent frequency of the $l = -m$ mode for this sequence of stars. (The label $l$ of a continuous family of modes refers to the spherical harmonic index of the zero angular velocity member of that family.) Further, let $1/\tau_l^V$ and $1/\tau_l^G$ represent the contributions to the imaginary part of the frequency of the $l = -m$ mode due respectively to viscosity and gravitational radiation reaction. It is useful to define the dimensionless functions $\alpha_l^-(\Omega)$, $\beta_l^-(\Omega)$, and $\gamma_l^-(\Omega)$ that give the angular velocity dependence of these frequencies for these modes:

$$\alpha_l^-(\Omega) = \frac{\sigma_l^-(\Omega) + (2 - l)\Omega}{\sigma_l^-(0)} , \tag{1}$$

$$\beta_l^-(\Omega) = \frac{\tau_l^V(0)\sigma_l^-(0)}{\tau_l^V(\Omega)[\sigma_l^-(\Omega) - l\Omega]} , \tag{2}$$

$$\gamma_l^-(\Omega) = \frac{\tau_l^G(0)}{\tau_l^G(\Omega)\beta_l^-(\Omega)}\left[\frac{\alpha_l^-(\Omega)\sigma_l^-(0)}{\sigma_l^-(\Omega)}\right]^{2l+1} . \tag{3}$$

These functions are well defined for any sequence of stellar models of this kind; however, in general they are not easy to compute. They have only been computed, to date, for stellar models having the uniform density equation of state—the Maclaurin spheroids.

The uniform density, rigidly rotating Newtonian stellar models (the Maclaurin spheroids) are the only explicit analytic models known that have nonzero angular velocity. The angular velocity, $\Omega$, of a Maclaurin spheroid is related to the geometrical shape of the spheroidal surface of the model by the equation

$$\Omega^2 = 2\pi G\rho\zeta[(1 + 3\zeta^2)\cot^{-1}\zeta - 3\zeta] , \tag{4}$$

where $\rho$ is the density of the star, $G$ is the gravitation constant, $\zeta$ describes the shape of the spheroid: $\zeta = (e^{-2} - 1)^{1/2}$, and $e$ is the eccentricity of the spheroid.

The properties of the oscillations of the Maclaurin spheroids have also been evaluated, and are known in terms of reasonably simple analytic formulae. The oscillations without dissipation were first evaluated by Bryan (1889). The frequencies of the $l = -m$ modes can be written in the form

$$\sigma_l^-[\Omega(\zeta)] = (l - 1)\Omega + \{\Omega^2 - 4\pi G\rho l\zeta \times [R_l(\zeta) + \zeta(\zeta\cot^{-1}\zeta - 1)]\}^{1/2} , \tag{5}$$

where the functions $R_l(\zeta)$ can be generated from the recursion relation

$$R_l(\zeta) = \frac{2l - 1}{2l}(1 + \zeta^2)R_{l-1}(\zeta) - \frac{\zeta}{2l} , \tag{6}$$

and the initial value

$$R_2(\zeta) = \tfrac{1}{8}[3(1 + \zeta^2)^2\cot^{-1}\zeta - \zeta(3\zeta^2 + 5)] . \tag{7}$$

(Eqs. [6]–[7] were derived empirically for $l \leq 5$ using Baumgart and Friedman's 1986 expressions for the associated Legendre functions.) In the limit of zero angular velocity ($\zeta \to +\infty$) these frequencies reduce to Kelvin's expression for the oscillations of an incompressible fluid sphere:

$$\sigma_l^-(0) = \left[\frac{8\pi G\rho l(l - 1)}{3(2l + 1)}\right]^{1/2} \tag{8}$$

Equations (4)–(8) determine the functions $\alpha_l^-(\Omega)$ for the Maclaurin spheroids implicitly in terms of the parameter $\zeta$:

$$\alpha_l^-[\Omega(\zeta)] = \left[\frac{3(2l + 1)}{8\pi G\rho l(l - 1)}\right]^{1/2} [\![\Omega(\zeta) + \{\Omega^2(\zeta) - 4\pi G\rho l\zeta \times [R_l(\zeta) + \zeta(\zeta\cot^{-1}\zeta - 1)]\}^{1/2}]\!] . \tag{9}$$

Figure 1 depicts these functions for $2 \leq l \leq 5$. Note that in the Maclaurin spheroids these functions are identical to the functions $\alpha_l^+(\Omega)$ defined by Lindblom (1986) to describe the angular velocity dependence of the $l = +m$ modes. For more general stellar models this strict equality is not expected to persist.

The oscillations of the $l = \pm m$ modes of the Maclaurin spheroids including the dissipative effects of viscosity and gravitational radiation reaction were evaluated by Comins (1979*a*, *b*). For the $l = -m$ modes the viscous time scale is given by

$$\tau_l^V(\Omega) = \tau_l^V(0)\left(\frac{1 + \zeta^2}{\zeta^2}\right)^{1/3}\frac{\sigma_l^-(\Omega) - (l - 1)\Omega}{\sigma_l^-(\Omega) - l\Omega} , \tag{10}$$

where $\tau_l^V(0)$ is the viscous time scale at zero angular velocity derived by Lamb (1881):

$$\tau_l^V(0) = \frac{\rho R^2}{\eta(l - 1)(2l + 1)} . \tag{11}$$

The shear viscosity is denoted by $\eta$ and the average radius of the star by $R$ in this equation. The gravitational radiation

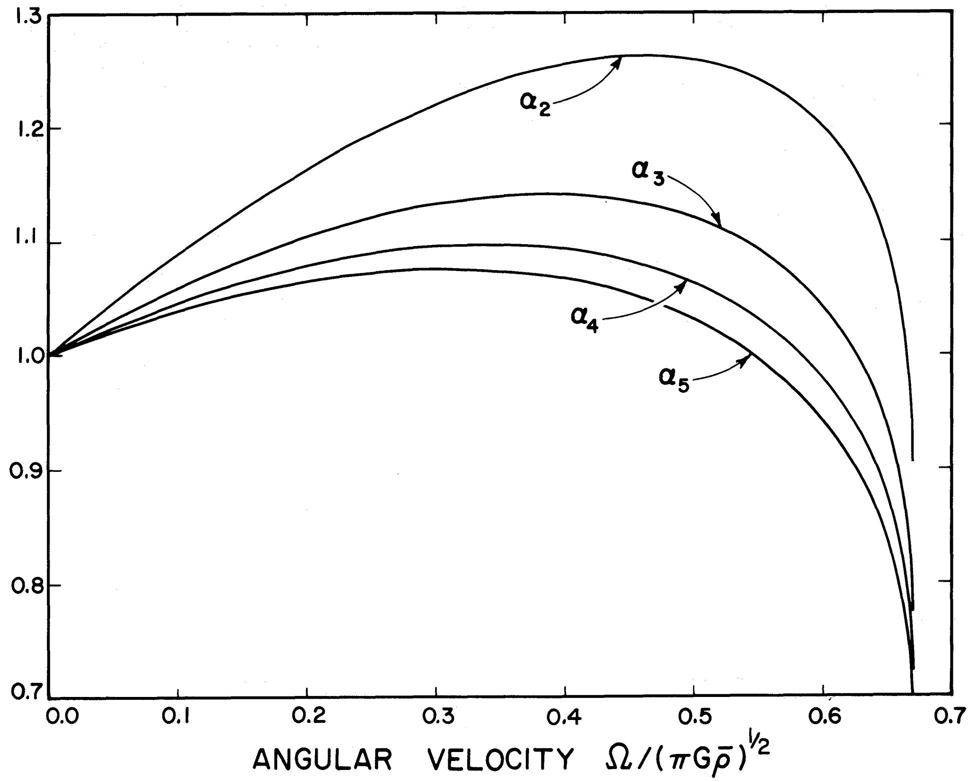


FIG. 1.—Functions, $\alpha_l^-(\Omega)$, relating the frequencies of the $l = -m$ modes of the Maclaurin spheroids to the frequencies of the corresponding nonrotating Kelvin spheres; see eqs. (1) and (9).

reaction time scale is given by

$$\tau_l^G(\Omega) = \tau_l^G(0)\left(\frac{\zeta^2}{1+\zeta^2}\right)^{(l-1)/3} \frac{\sigma_l^-(\Omega) - (l-1)\Omega}{\sigma_l^-(0)}\left[\frac{\sigma_l^-(0)}{\sigma_l^-(\Omega)}\right]^{2l+1}, \tag{12}$$

and $\tau_l^G(0)$ is the gravitational radiation reaction time scale for nonrotating stars derived by Detweiler (1975):

$$\tau_l^G(0) = \frac{2}{3}\frac{(l-1)[(2l+1)!!]^2}{(l+1)(l+2)}\left[\frac{2l+1}{2l(l-1)}\right]^l\left(\frac{c^2R}{GM}\right)^{l+1}\frac{R}{c}. \tag{13}$$

Equations (10)–(13) determine the values of the functions $\beta_l^-(\Omega)$ and $\gamma_l^-(\Omega)$ for the Maclaurin spheroids parametrically in terms of $\zeta$:

$$\beta_l^-[\Omega(\zeta)] = \left(\frac{\zeta^2}{1+\zeta^2}\right)^{1/3}\frac{\sigma_l^-(0)}{\sigma_l^-[\Omega(\zeta)] - (l-1)\Omega(\zeta)}, \tag{14}$$

$$\gamma_l^-[\Omega(\zeta)] = \left(\frac{1+\zeta^2}{\zeta^2}\right)^{l/3}\{\alpha_l^-[\Omega(\zeta)]\}^{2l+1}. \tag{15}$$

The functions $\beta_l^-(\Omega)$ and $\gamma_l^-(\Omega)$ are depicted in Figures 2 and 3 for $2 \leq l \leq 5$.

### III. THE CRITICAL ANGULAR VELOCITIES

The stability of a particular mode of a rotating star is determined by the sign of the imaginary part of the frequency, $1/\tau$, of that mode. For the stellar models considered here, the imaginary parts of the frequencies of the $l = -m$ modes are determined by viscosity and gravitational radiation reaction:

$$\frac{1}{\tau_l} = \frac{1}{\tau_l^V} + \frac{1}{\tau_l^G}. \tag{16}$$

This equation can be written in a more useful form by making use of the definitions of the functions $\alpha_l^-(\Omega)$, $\beta_l^-(\Omega)$, and $\gamma_l^-(\Omega)$ given in eqns. (1)–(3):

$$\frac{1}{\tau_l} = \beta_l^-(\Omega)\left\{\frac{\sigma_l^-(0)\alpha_l^-(\Omega) - 2\Omega}{\tau_l^V(0)\sigma_l^-(0)} + \frac{\gamma_l^-(\Omega)}{\tau_l^G(0)}\left[1 + \frac{(l-2)\Omega}{\alpha_l^-(\Omega)\sigma_l^-(0)}\right]^{2l+1}\right\}. \tag{17}$$

The angular velocities of a star where a mode changes stability are called critical angular velocities. Since the functions $\beta_l^-(\Omega)$ are not expected to vanish anywhere (see Fig. 2), it follows from equation (17) that the condition for the existence of a critical angular velocity is

$$\Omega_l^- = \frac{\sigma_l^-(0)}{2}\left\{\alpha_l^-(\Omega_l^-) + \gamma_l^-(\Omega_l^-)\frac{\tau_l^V(0)}{\tau_l^G(0)} \times \left[1 + \frac{(l-2)\Omega_l^-}{\alpha_l^-(\Omega_l^-)\sigma_l^-(0)}\right]^{2l+1}\right\}. \tag{18}$$

While this equation applies to stellar models having arbitrary equations of state, it is not useful unless the functions $\alpha_l^-(\Omega)$ and $\gamma_l^-(\Omega)$ are known. These functions are not easy to compute, and have not been evaluated to date except for the Maclaurin

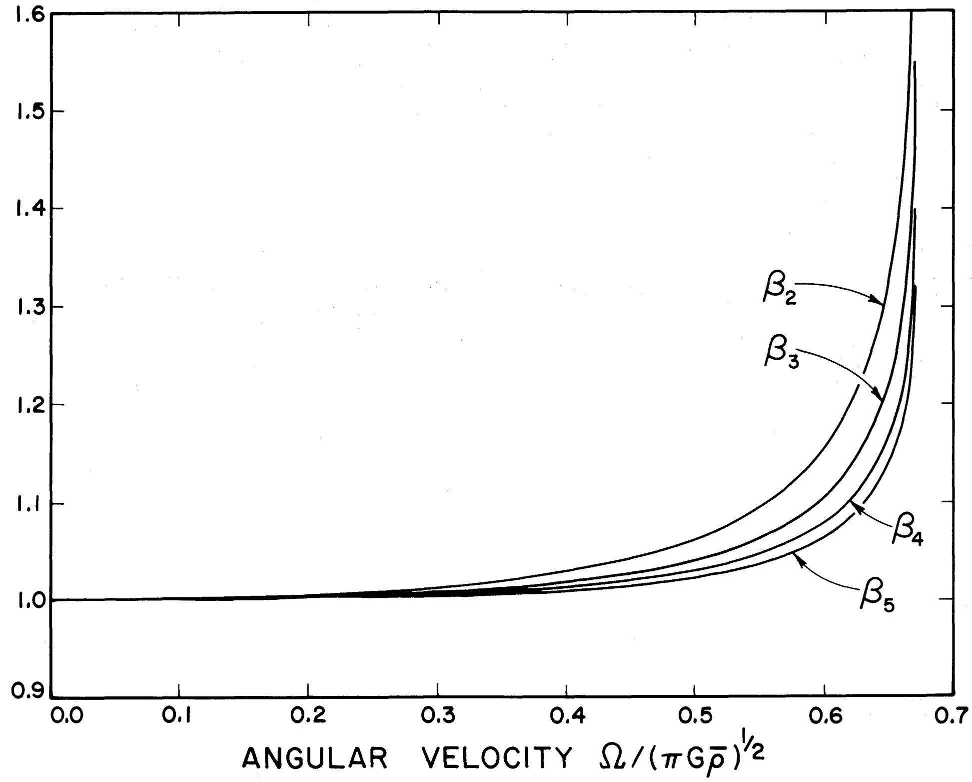


FIG. 2.—Functions, $\beta_l^-(\Omega)$, giving the angular velocity dependence of the viscous time scales for the $l = -m$ modes in the Maclaurin spheroids; see eqs. (2) and (14).

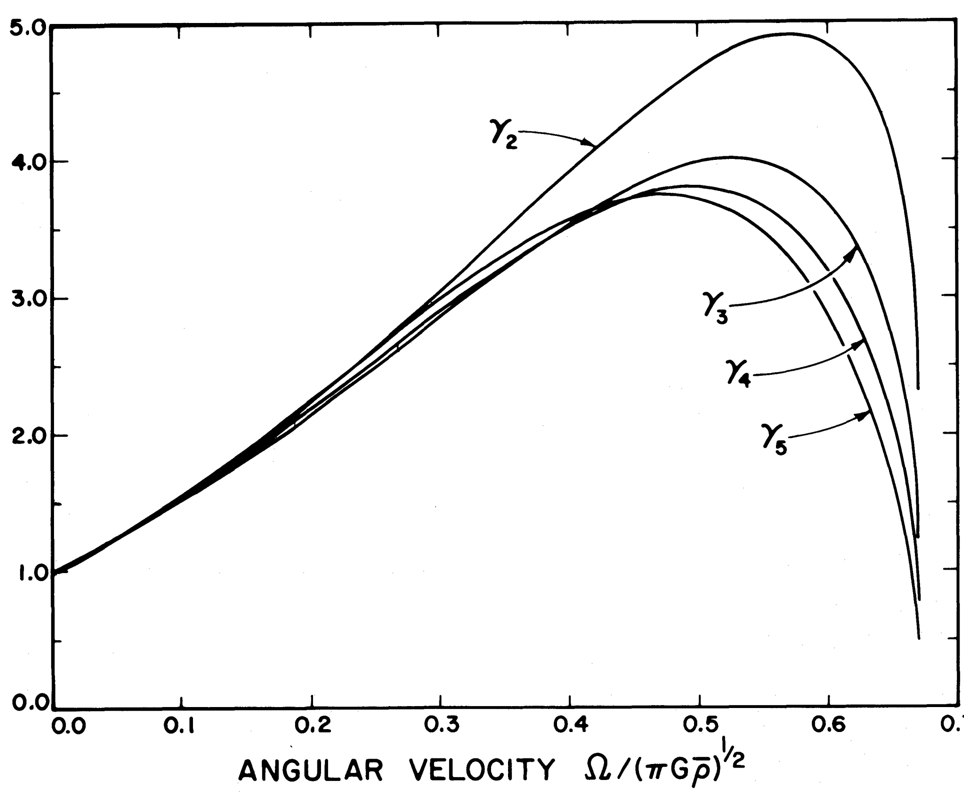


FIG. 3.—Functions, $\gamma_l^-(\Omega)$, giving the angular velocity dependence of the gravitational radiation reaction time scales for the $l = -m$ modes in the Maclaurin spheroids; see eqs. (3) and (15).

spheroids. Since the analogous functions for the $l = +m$ modes do not vary significantly between equations of state (Managan 1986), it seems likely that these functions for the $l = -m$ modes will not either. Thus it is expected that reasonably accurate estimates for the critical angular velocities of arbitrary stellar models can be obtained by solving equation (18) using the functions $\alpha_l^-(\Omega)$ and $\gamma_l^-(\Omega)$ given in equations (9) and (15) for the Maclaurin spheroids.

In the limit that the viscous time scale is much shorter than the gravitational radiation time scale, $\tau_l^V \ll \tau_l^G$, equation (18) reduces to the simpler form:

$$\Omega_l^- = \tfrac{1}{2}\sigma_l^-(0)\alpha_l^-(\Omega_l^-)\,. \tag{19}$$

The first two solutions of this equation for the Maclaurin spheroids are given by

$$\begin{aligned}\Omega_2^- &= 0.611743(\pi G\rho)^{1/2}\,,\\ \Omega_3^- &= 0.663434(\pi G\rho)^{1/2}\,.\end{aligned} \tag{20}$$

Since the viscosity of neutron star matter varies with temperature as $T^{-2}$, it follows that the critical angular velocities that occur in the limit of zero temperature stars will be the solutions of equations (19). The smallest angular velocity solution to equation (19) is the one associated with the $l = 2$ mode, and so in the limit of small temperature this mode will limit the angular velocity of rotating neutron stars.

As the temperature of the neutron star becomes sufficiently large, the second term (proportional to $\gamma_l^-$) in equation (18) dominates, and the critical angular velocities are forced to approach the maximum angular velocity for that type of stellar model. For the Maclaurin spheroids the maximum angular velocity is given by

$$\max(\Omega) = 0.670322(\pi G\rho)^{1/2} \tag{21}$$

(see, e.g., Chandrasekhar 1969). This value agrees to within 5% with the maximum angular velocities for the 1.4 $N_\odot$ neutron star models computed with a variety of realistic equations of state by Friedman, Ipser, and Parker (1984).

## IV. ESTIMATES FOR REALISTIC NEUTRON STARS

The critical angular velocities of realistic neutron stars can be evaluated using equation (18) as soon as the frequencies $\sigma_l^-(0)$, $\tau_l^V(0)$, and $\tau_l^G(0)$ for nonrotating neutron stars, and the functions $\alpha_l^-(\Omega)$ and $\gamma_l^-(\Omega)$ are known. The frequencies $\sigma_l^-(0)$ and the gravitational radiation damping times $\tau_l^G(0)$ were evaluated by Lindblom (1986) for a variety of neutron star models based on a number of different "realistic" equations of state and for a range of possible neutron star masses. The viscous time scales $\tau_l^V(0)$ were computed more recently for realistic neutron stars by Cutler and Lindblom (1987). They found that the viscous time scale for a neutron star with central temperature $T$ is given by

$$\tau_l^V(0) = \frac{\Upsilon_l}{347}\frac{T^2R^2}{\bar{\rho}^{5/4}(l-1)(2l+1)}\,. \tag{22}$$

The parameters $\Upsilon_l$ are tabulated by Cutler and Lindblom (1986) for a variety of neutron star models. The values of these parameters were found to be confined to the range $0.1 \lesssim \Upsilon_l \lesssim 0.6$ for neutron star matter assumed to be in the normal state, and to the range $0.03 \lesssim \Upsilon_l \lesssim 0.1$ for neutron star matter cold enough to be completely superfluid. Given these frequencies for realistic neutron stars, and the angular velocity dependent functions $\alpha_l^-$ and $\gamma_l^-$ from equations (9) and (15), it is straightforward to solve equation (18) numerically to estimate the critical angular velocities for realistic neutron stars.

An argument was given in § III that the critical angular velocities of neutron stars would depend on the temperature of the star. For the Maclaurin spheroids these critical angular velocities varied from the values given in equation (20) for zero temperature stars to the value given in equation (21) for very high temperature stars. In Table 1 the analogous minimum and maximum critical rotation periods ($P_c = 2\pi/\Omega_c$) allowed by the viscosity-driven secular instability are presented for a variety of neutron star models. The different equations of state are identified by a letter (A, B, C, etc.) and are defined in Lindblom (1986). The maximum period, max ($P_c$), given in Table 1 was computed by solving equation (19) using the actual pulsation frequencies, $\sigma_l^-(0)$, for these stellar models together with the functions $\alpha_l^-$ from equation (9). The minimum periods, min ($P_c$), were estimated from equation (21) for these models.

Table 1 also gives estimates for the temperature, $T_{1/2}$, where the angular velocity is halfway between these extreme values. The temperature at which a given angular velocity will become critical in these stellar models can be determined by combining equation (18) for the critical angular velocities with equation (22) for the viscous time scales for neutron star matter:

$$T^2 = \frac{347(l-1)(2l+1)}{\Upsilon_l R^2}\,\bar{\rho}^{5/4}\tau_l^G(0)\,\frac{2\Omega - \sigma_l^-(0)\alpha_l^-(\Omega)}{\sigma_l^-(0)\gamma_l^-(\Omega)} \times \left[1 + \frac{(l-2)\Omega}{\alpha_l^-(\Omega)\sigma_l^-(0)}\right]^{-(2l+1)}\,. \tag{23}$$

The temperatures, $T_{1/2}$, were obtained by evaluating equation (23) at the angular velocity halfway between the minimum and maximum allowed by the viscous instability. The two entries in Table 1 for this quantity correspond to the assumptions that

TABLE 1

PROPERTIES OF VISCOUS INSTABILITY IN NEUTRON STARS[a]

| Equation of State | $M/M_\odot$ | $R$ (km) | min ($P_c$) | max ($P_c$) | Normal $T_{1/2}$ | Superfluid $T_{1/2}$ |
|---|---|---|---|---|---|---|
| M ......... | 1.277 | 16.057 | 1.692 | 1.693 | $1.97 \times 10^3$ | $4.12 \times 10^3$ |
| | 1.752 | 11.903 | 0.920 | 0.922 | $5.13 \times 10^3$ | $9.09 \times 10^3$ |
| L ......... | 1.311 | 14.944 | 1.499 | 1.507 | $2.94 \times 10^3$ | $6.06 \times 10^3$ |
| | 2.661 | 13.619 | 0.915 | 0.954 | $8.30 \times 10^3$ | $1.56 \times 10^4$ |
| N ......... | 1.385 | 13.784 | 1.292 | 1.305 | $3.62 \times 10^3$ | $7.24 \times 10^3$ |
| | 2.563 | 12.270 | 0.798 | 0.842 | $1.32 \times 10^4$ | $2.42 \times 10^4$ |
| O ......... | 1.282 | 12.798 | 1.201 | 1.212 | $4.08 \times 10^3$ | $8.01 \times 10^3$ |
| | 2.380 | 11.581 | 0.759 | 0.800 | $1.32 \times 10^4$ | $2.39 \times 10^4$ |
| C ......... | 1.317 | 12.027 | 1.080 | 1.084 | $4.03 \times 10^3$ | $7.62 \times 10^3$ |
| | 1.852 | 9.952 | 0.685 | 0.701 | $1.33 \times 10^4$ | $2.18 \times 10^4$ |
| F ......... | 1.262 | 10.325 | 0.877 | 0.881 | $5.68 \times 10^3$ | $1.02 \times 10^4$ |
| | 1.463 | 7.966 | 0.552 | 0.560 | $2.01 \times 10^4$ | $2.70 \times 10^4$ |
| A ......... | 1.246 | 9.783 | 0.814 | 0.823 | $6.97 \times 10^3$ | $1.24 \times 10^4$ |
| | 1.653 | 8.427 | 0.565 | 0.585 | $2.06 \times 10^4$ | $3.05 \times 10^4$ |
| B ......... | 1.223 | 8.209 | 0.632 | 0.638 | $1.08 \times 10^4$ | $1.63 \times 10^4$ |
| | 1.412 | 7.000 | 0.463 | 0.479 | $3.45 \times 10^4$ | $4.35 \times 10^4$ |

[a] The quantities min ($P_c$) and max ($P_c$) are the minimum and maximum critical rotation periods (given in milliseconds) allowed by the viscosity-driven secular instability. The maximum critical rotation period occurs when the temperature of the star is $T = 0$ K, while the minimum occurs at $T = \infty$. The quantities $T_{1/2}$ are the temperatures (given in kelvins) for which the critical angular velocity of the star is the average of the maximum and minimum values. These quantities are given for both normal and superfluid models of the viscosity of neutron star matter.

the neutron star matter is either normal or superfluid respectively.

For each $l = -m$ mode equation (23) can be evaluated for the temperatures corresponding to different critical angular velocities. Since there will typically be several angular velocities that result in the same temperature, only the lowest will correspond to the actual critical angular velocity for that temperature. Figure 4 shows the temperature dependence of the critical angular velocities for the two stellar models computed from equation of state $N$. At low temperature all of these curves (for any of the equations of state studied) are essentially indistinguishable. The discontinuity in the slope of the solid curve in Figure 4 occurs at the point where the mode responsible for the instability changes from $l = 2$ at low temperatures to $l = 3$ at higher temperatures. Figure 4 illustrates that the critical angular velocity changes from the minimum to the maximum value over a relatively narrow temperature range centered around the temperature $T_{1/2}$. Thus, qualitatively, for temperatures above $T_{1/2}$ the viscous instability will not be effective in limiting the rotation rate of the neutron star, while below $T_{1/2}$ the rotation period will be limited to the value max ($P_c$). These characteristic temperatures lie in the range $2 \times 10^3 \lesssim T_{1/2} \lesssim 4 \times 10^4$ K for the equations of state studied.

To give a more global picture of the effects of the secular instabilities on rotating neutron stars, the temperature dependence of the critical angular velocities have been computed for both the viscosity driven secular instability in the $l = -m$ modes and the gravitational radiation driven secular instability in the $l = +m$ modes. Figure 5 depicts these critical angular velocities for the stellar models computed from equation of state $N$. At low temperature ($T \lesssim 10^5$ K) the critical angular velocity is determined by the viscous instability in the $l = -m$ modes. Several different modes can determine the maximum angular velocity of the neutron star depending on the temperature. At higher temperatures ($T \gtrsim 10^7$ K) the critical angular velocities are determined by the gravitational radiation driven instability in the $l = +m$ modes. Again, the particular mode responsible for limiting the angular velocity of the neutron star changes with the temperature. Figure 5 also reveals that the viscosity-driven secular instability is only capable of reducing the maximum angular velocity of a rotating neutron star by $\sim 5\%$ below the maximum angular velocity (up to 9% for the Maclaurin spheroids). The gravitational radiation secular instability is somewhat more effective in limiting the rotation rate; however, at realistic neutron star temperatures even this instability only reduces the maximum angular velocity by $\sim 20\%$.

The cooling times of neutron stars are sufficiently rapid (see, e.g., Tsuruta 1979, 1986; Nomoto and Tsuruta 1986, 1987) that within a few weeks after creation their surface temperatures are expected to fall to $\sim 10^7$ K, and within $10^5$ yr to $\sim 10^5$ K if no reheating mechanism is operating. The most significant reheating mechanism in neutron stars probably occurs in stars undergoing accretion from a companion. In the pulsating X-ray sources, where accretion onto the surface of a rotating neutron star is believed to be responsible for the observed

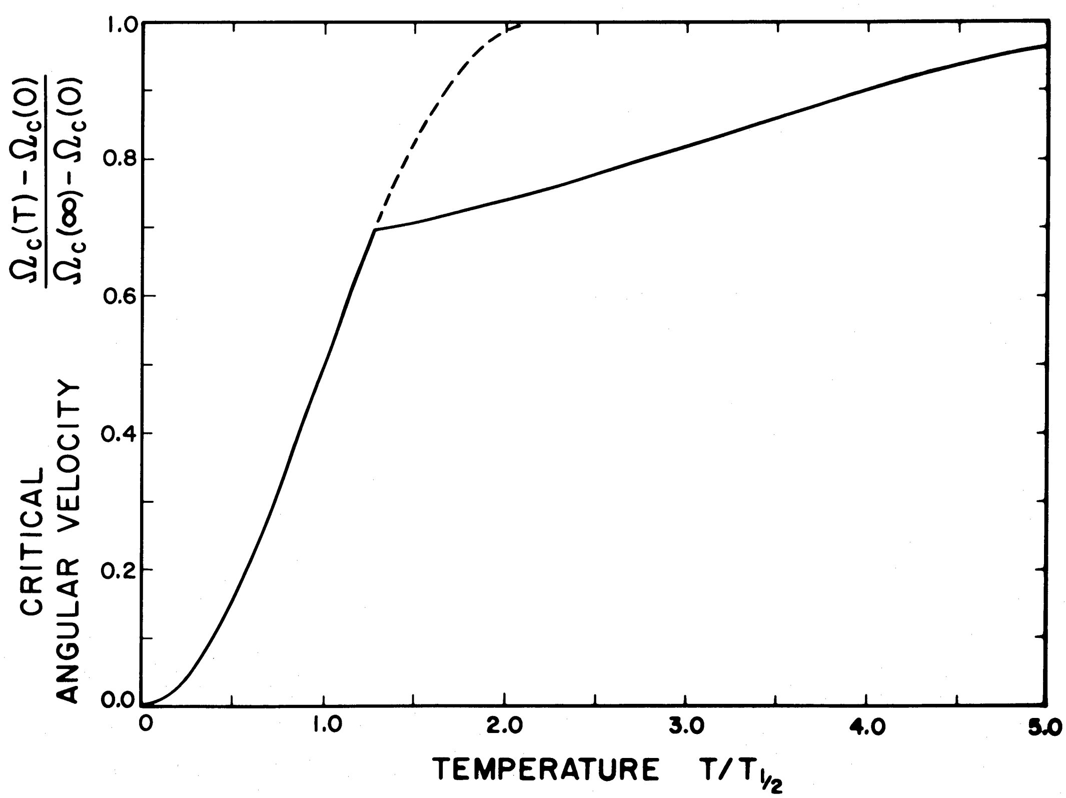


FIG. 4.—Curves depicting the critical angular velocity due to the viscous secular instability as a function of temperature in rotating neutron stars. The temperature scale used, $T_{1/2}$, is given in Table 1 for each equation of state. While these particular curves were computed for the two stellar models constructed from equation of state $N$ described in Table 1, the curves are very representative of the other equations of state. All of the computed curves are essentially indistinguishable for temperatures below $\sim 1.3T_{1/2}$. Solid curve is the maximum mass stellar model, while dashed curve is the 1.4 $N_\odot$ model. The viscosity used in these models includes the effects of superfluidity on the neutron star matter.

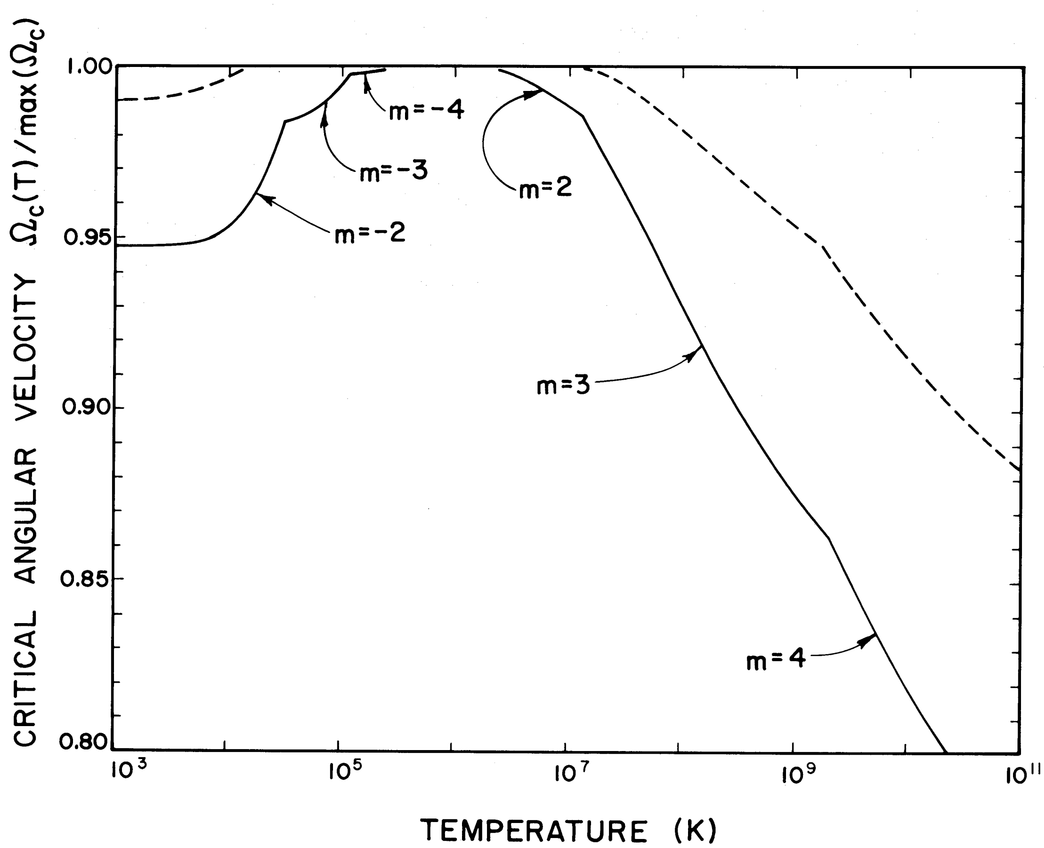


FIG. 5.—Critical angular velocity as a function of temperature for the maximum mass (*solid curve*) and the 1.4 $N_\odot$ (*dashed curve*) stellar models using equation of state *N* and the viscosity function appropriate for superfluid neutron star matter. Both the effects of the viscous secular instability in the $l = -m$ modes that limit the angular velocity at low temperatures and the gravitational radiation instability in the $l = +m$ modes that limit the angular velocity at high temperatures are depicted. The particular mode responsible for the instability is identified in each smooth portion of the curve for the maximum mass star.

luminosity, the surface temperatures of observed sources must be less than $\sim 10^7$ K to be compatable with the observed X-ray flux. The central temperatures of neutron stars that have relaxed to a quasi-equilibrium state are typically only about a factor of 2 higher than the surface temperatures. It follows that the gravitational radiation driven secular instability is only expected to play a role in limiting the angular velocities of newly created neutron stars.

Neutron stars can be spun up by accretion on time scales of $\sim 10^4$ yr (see, e.g., Rappaport and Joss 1977; Ghosh and Lamb 1979). Consequently, it is possible that the angular velocity of a rather old neutron star could be increased to the point where an instability sets in to prevent any further increase in angular momentum. The X-ray luminosities in sources where such an accretion process is occurring are generally in the neighborhood of $10^{37}$ ergs s$^{-1}$. This is equivalent to a surface temperature of $\sim 10^7$ K for a neutron star. This temperature appears to be in the range where neither of the secular instabilities is capable of operating. Thus, a neutron star whose quasi-equilibrium temperature during the accretion process lies in the range $10^5$–$10^7$ K may be spun up to the point where mass shedding or some dynamical instability sets in. After the accretion from the companion star turns off, the neutron star will begin to cool. If no other reheating mechanism becomes significant, the temperature of the neutron star will drop to the point where the viscosity-driven secular instabilities may become operative. A sufficiently rapidly rotating neutron star will then become unstable and quickly radiate away its excess angular momentum via gravitational radiation. Since the cooling times for neutron stars (see, e.g., Tsuruta 1986) are typically much longer than the dissipation time scales for gravitational radiation, the angular velocity of the star would simply follow the critical angular velocity curve for that stellar model, analogous to the one in Figure 5. It is possible, however, that perpetual heating of the neutron star by accretion from the interstellar medium may prevent the temperature of neutron stars from ever falling below $\sim 10^5$ K (see Tsuruta 1979). In that case the viscous secular instability discussed here may never play a role in neutron stars. Better calculations of these secular instabilities in fully relativistic rapidly rotating neutron star models, and better estimates of possible reheating mechanisms are needed before it will be possible to confidently predict which scenario is relevant to real neutron stars.

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