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THE EFFECT OF VISCOSITY ON NEUTRON STAR OSCILLATIONS

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ABSTRACT

In this paper we compute the rate at which neutron star oscillations are damped by the presence of viscosity and thermal conductivity in the neutron matter. In our computation we use fully relativistic equations to describe the neutron star oscillations, we use the best available expressions for the dissipation coefficients in neutron matter, and we approximate the effects of superfluidity on these dissipation mechanisms. We compute damping times for a range of neutron star masses and temperatures, and we use a variety of equations of state to describe the highest density portion of the neutron matter. The time scales computed here are used to improve previous estimates of the maximum rotation rate of neutron stars.

Subject headings: equation of state — hydrodynamics — relativity — stars: neutron — stars: pulsation — stars: rotation

I. INTRODUCTION

The effects of viscosity play a crucial role in determining the maximum rotation rate that a neutron star of given number of baryons may attain. This in turn directly determines the minimum periods of pulsars that we can expect to observe. The instability that limits the rotation rate of these stars is driven by gravitational radiation. The viscosity of the fluid has a damping influence on these modes. If viscosity were absent, all rotating stars would be unstable. To determine which modes are stable, and which unstable, it is necessary to determine in detail the relative time scales of the gravitational radiation and the viscosity influences on the appropriate oscillation modes of neutron stars.

The purpose of this paper is to determine as accurately as possible the rate at which viscosity damps the appropriate nonradial modes of neutron star oscillations. Even though there does not exist at present a completely satisfactory theory of relativistic hydrodynamics that includes the effects of viscosity, we present in § II an approximate method for computing the rate at which viscosity (and thermal conductivity) dissipate energy from fluctuations in the fluid away from an equilibrium state. This approximation is expected to be valid in the large Reynolds number (small viscosity) limit. The approximation does not require that the gravitational field of the neutron star be weak or that the sound speed be a small fraction of the speed of light.

The viscosity and thermal conductivity of neutron star matter have been computed in the important density range (above 10^{14} g cm^{-3}) by Flowers and Itoh (1979). In § III we present simple analytic formulae that reproduce their viscosities and thermal conductivities to within a few percent. We use these simple analytic expressions for the viscosity in all of our numerical work.

Neutron star matter is expected to be superfluid at sufficiently low temperatures. Superfluidity should influence the effects of viscosity on the star's oscillations in two ways: (*a*) the viscosity due to neutron-neutron scattering should effectively vanish in the superfluid state, and (*b*) the superfluidity should affect the dynamical equations of motion of the oscillating star. In the present work we correctly modify the viscosity coefficients when the temperature of the fluid falls below the superfluid critical temperature, but we do not attempt to modify appropriately the dynamics of the fluid.

The detailed results of our computations of the damping times of neutron star oscillations due to viscous dissipation are presented in § IV. We present damping times for the nonradial modes of static spherical fluid neutron stars. The modes that we examine have spherical harmonic index, l, in the range: $2 \leq l \leq 5$. We evaluate these damping times for a range of different temperatures, neutron star masses, and different equations of state. We find that these damping times do agree fairly well (within an order of magnitude) with the well-known Lamb formula for the viscous damping time of incompressible Newtonian stars. In an Appendix we derive analogous analytic formulae for the bulk viscosity and thermal conductivity time scales as well. We find that our numerical time scales agree with these new analytic formulae with comparable accuracy.

In the last section of our paper, § V, we use our viscous time scales to improve the estimates of the maximum rotation rates of neutron stars. For sufficiently cold neutron stars, the viscosity is so large that the gravitational radiation instability is completely suppressed. We find that the minimum temperature that a neutron star may have before these instabilities are completely suppressed is in the range 10^6–10^7 K, depending on the equation of state.

II. THE FLUID EQUATIONS

While there is at present no completely satisfactory theory to describe the effects of viscosity on the dynamics of relativistic fluids (see, e.g., Hiscock and Lindblom 1983, 1985), there are a number of partially successful theories: Eckart (1940), Landau and Lifshitz (1959), and Israel (1976). Even though the details of the equations of motion for a viscous fluid differ among these theories, the theories do agree on the equation for the evolution for the energy contained in small fluctuations about an equilibrium state. This equation is

$$\frac{dE}{dt} = -\int \left[\frac{\delta\tau^{ab}\delta\tau_{ab}}{2\eta} + \frac{(\delta\tau)^2}{\zeta} + \frac{\delta q^a \delta q_a}{\kappa T}\right] d^3x - F_{\mathrm{GR}} , \quad (1)$$

where η, ζ, and κ are the coefficients of viscosity and thermal conductivity; $\delta\tau^{ab}$, $\delta\tau$, and δq^a are the (traceless) spatial stress tensor of the perturbed fluid, the trace, and heat flow vector; and F_{GR} is the energy flux in gravitational radiation. The energy E is a functional of the perturbed fluid variables (see, e.g., Lindblom and Hiscock 1983; Hiscock and Lindblom 1983). Since this equation is the obvious generalization of the analogous equation for Navier-Stokes fluids, and since it is also a feature of all the proposed relativistic theories, we feel confident in using it as the basis of our calculation of the damping of viscous neutron star oscillations.

The rate at which the normal modes of neutron star oscillations are damped by the dissipative forces can be determined from equation (1). For a normal mode the fluid perturbations will have time dependence $e^{i\omega t}$, and consequently the energy functional E (being bilinear in the fluid perturbations) will have time dependence $e^{-2\,\mathrm{Im}\,(\omega)t}$. The time derivative of the energy can therefore be determined directly for normal modes:

$$\frac{dE}{dt} = -2\,\mathrm{Im}\,(\omega)E . \quad (2)$$

The characteristic damping time of the fluid perturbations, τ, therefore is given by

$$\frac{1}{\tau} \equiv \mathrm{Im}\,(\omega) = -\frac{1}{2E}\frac{dE}{dt} . \quad (3)$$

Therefore, if we can evaluate the integrals in equation (1) to determine the rate of energy dissipation in the fluid, and if we can evaluate the energy of the perturbations, E, then the rate at which the normal mode is damped is easily determined by equation (3).

We consider first the problem of evaluating the dissipation integrals in equation (1). Each of the proposed relativistic fluid theories has a different equation to determine the spatial stress tensor $\delta\tau^{ab}$, $\delta\tau$, and the heat flow δq^a that appear in equation (1). All of the different expressions for these quantities reduce, however, to the same equations under the conditions (*a*) that the dissipation coefficients are sufficiently small, and (*b*) that the time derivatives of the perturbed quantities remain sufficiently small. In this limit the stresses and heat flow are given by

$$\delta\tau^{ab} = -2\eta\delta\sigma^{ab} , \quad (4)$$

$$\delta\tau = -\zeta\delta\theta , \quad (5)$$

$$\delta q^a = -\frac{\kappa n T^2}{\rho + p} q^{ab}\nabla_b \delta\Theta , \quad (6)$$

where $\delta\sigma^{ab}$ is the shear of the perturbed fluid motion; $\delta\theta$ is the expansion; $q^{ab} = g^{ab} + u^a u^b$ projects quantities into the three-space orthogonal to the four-velocity, u^a, of the unperturbed fluid; n, T, s, ρ, and p are the number density, temperature, entropy per particle, energy density (including rest mass), and pressure of the unperturbed fluid; and $\delta\Theta$ is the perturbation in the thermodynamic variable

$$\Theta = \frac{\rho + p}{nT} - s . \quad (7)$$

Once the evolution of the perturbed fluid has been determined, it is straightforward to evaluate the integrals in equation (1) using these expressions. We turn next, therefore, to the evolution of the fluid itself.

We assume that the evolution of a relativistic viscous fluid will smoothly approach the evolution of a perfect fluid as the magnitude of the dissipation coefficients are taken to zero. (The violation of this condition is the fatal flaw in some of the proposed relativistic theories (Hiscock and Lindblom 1985). Equations (4)–(6) then give the lowest order contributions to the perturbed stresses by using the perfect fluid evolutions to determine $\delta\theta$, $\delta\sigma^{ab}$ and $\delta\Theta$. Similarly, to determine the time scale τ from equation (3), to first order in the dissipation coefficients, one need only evaluate E using the perfect fluid evolution of the perturbed fluid.

The perfect fluid evolution of neutron star oscillations has, of course, received a great deal of attention in the literature, starting with Thorne and Campolattaro (1967*a*, *b*). Here, we use the notation of Detweiler and Lindblom (1985). We describe the deviations of the fluid in a stellar model from its static, spherical equilibrium state by the Lagrangian fluid displacement, ξ_a, which we expand in spherical harmonics. One component of this expansion is represented by

$$\xi_a = [W(r)e^{\lambda/2}r^{-1}Y^l_m\nabla_a r - V(r)\nabla_a Y^l_m]r^l e^{i\omega t} , \quad (8)$$

where ω is the frequency of the stellar pulsation, Y^l_m is the standard spherical harmonic function, e^{λ} is the radial component of the metric tensor of the equilibrium star, and W and V are functions (to be determined) that characterize the stellar pulsations. The geometry of the spacetime of the perturbed stellar model is described by the metric tensor, whose perturbations can also be expanded in spherical harmonics. One component of this expansion is given by

$$\begin{aligned} ds^2 = &-e^{\nu}(1 + r^l H_0 Y^l_m e^{i\omega t})dt^2 \\ &- 2i\omega r^{l+1} H_1 Y^l_m e^{i\omega t} dt dr \\ &+ e^{\lambda}(1 - r^l H_0 Y^l_m e^{i\omega t})dr^2 \\ &+ r^2(1 - r^l K Y^l_m e^{i\omega t})(d\theta^2 + \sin^2\theta d\varphi^2) , \end{aligned} \quad (9)$$

where e^{ν} and e^{λ} are the metric components of the unperturbed stellar model, and H_0, H_1, and K are the functions that characterize the perturbed spacetime. The equations that determine the perturbation functions H_0, H_1, K, V, and W and the algorithm used to solve those equations are described in Detweiler and Lindblom (1985) and Lindblom and Detweiler (1983) and will not be repeated here.

The perturbed thermodynamic variables, the shear, and the expansion needed to evaluate the dissipation integrals in equation (1) are simple functions of the Lagrangian displacement ξ_a and the perturbed metric h_{ab}. These expressions can be found in Lindblom and Hiscock (1983). Here we write out only the dissipation integral due to the shear viscosity η. (In § IV we argue that the other dissipation integrals are much smaller for neutron star matter.) Using the expressions for ξ_a and h_{ab} in equations (8) and (9), and performing the angular integrals involving the Y^l_m, we find that the dissipation integral due to shear viscosity is given by

$$\begin{aligned} \frac{1}{\tau_\eta} = \frac{(\mathrm{Re}\,\omega)^2}{E}\int_0^R \eta r^{2(l-1)} e^{-\nu} e^{\lambda/2} \Big\{ &\frac{3}{2}(\alpha_1)^2 + l(l+1)(\alpha_2)^2 \\ &+ l(l+1)\left[\frac{1}{2}l(l+1) - 1\right]V^2 \Big\} dr , \end{aligned} \quad (10)$$

where

$$\alpha_1 = \frac{r^2}{3}\left\{\frac{2}{r}\, e^{-\lambda/2}\left[\frac{dW}{dr} + (l-2)\frac{W}{r}\right] + K - H_0 - l(l+1)\frac{V}{r^2}\right\}, \tag{11}$$

and

$$\alpha_2 = \frac{r}{2}\left[\frac{dV}{dr} + (l-2)\frac{V}{r} - e^{\lambda/2}\frac{W}{r}\right]e^{-\lambda/2}. \tag{12}$$

All that remains to be done before equation (10) can be used to evaluate the viscous time scales of oscillating neutron stars is to give an expression for the energy, E, contained in those oscillations. Integral expressions for this energy exist in the literature (Detweiler and Ipser 1973; Detweiler 1975), but we use a simpler method to obtain it. Once the eigenfunctions H_0, H_1, K, etc., are known for the oscillations of the star, it is straightfoward to determine the flux of gravitational radiation, $F_{\rm GR}$, by examining their asymptotic forms (see, e.g., Thorne 1969). Furthermore the time scale, τ_G, for gravitational radiation damping of the oscillation is computed as part of the solution of the evolution of the perfect fluid equations. Thus, equation (3) can be used to determine the energy contained in the perfect fluid oscillations of the star.

$$E = \tfrac{1}{2}\tau_G F_{\rm GR}\,. \tag{13}$$

We find that this energy agrees with the energy integral of Detweiler and Ipser (1973) (with the correction to the boundary integral pointed out by Finn 1986) to within a few percent for $2 \le l \le 4$.

III. THE DISSIPATION COEFFICIENTS

To evaluate the integral in equation (1) for the dissipation of the energy, we must know the values of the dissipation coefficients η, ζ, and κ. These coefficients are generally functions of the variables that characterize the thermodynamic state of the fluid: e.g., the energy density, ρ, and the temperature, T. For neutron stars, the bulk of the fluid has densities above 10^{14} g cm^{-3}, and temperatures in the range 10^6–10^{10} K are typical for young stars. Thus, for neutron stars the dissipation coefficients must be determined for matter in this range of thermodynamic states.

The only calculations of dissipation coefficients that have been done (to date) for matter in this extreme density range are by Flowers and Itoh (1979). The dissipation coefficients in this regime are determined by the transport of momentum and energy due to the neutrons, protons, and electrons that are present in neutron star matter. Flowers and Itoh (1979) compute each of these contributions for neutron matter in the density range $10^{14} \lesssim \rho \lesssim 4 \times 10^{14}$ g cm^{-3}. In this range they find that the viscosity is dominated by neutron transport, and the thermal conductivity is dominated by electron transport. We find that the dissipation coefficients as calculated by Flowers and Itoh (1979) are given, to an excellent approximation, by simple power-law formulae. The dissipation coefficients that will be needed in the present work are given by

$$\eta = 347\rho^{9/4}T^{-2}\,, \tag{14}$$

$$\kappa = 2.6 \times 10^7 \rho^{5/3} T^{-1} \tag{15}$$

where η has units g cm^{-1} s^{-1}, κ has units ergs cm^{-1} s^{-1}, ρ is given in g cm^{-3}, and T has units K. These formulae reproduce the Flowers and Itoh computations of viscosity to within $\sim$3%, and their thermal conductivity to within $\sim$5%.

These formulae, equations (14)–(15), for the dissipation coefficients are only known to represent accurately the transport properties of neutron matter in the density range computed by Flowers and Itoh (1979): $10^{14} \lesssim \rho \lesssim 4 \times 10^{14}$ g cm^{-3}. Using the above formula for η, we find that the contribution to the dissipation integral, equation (10), from densities below 10^{14} g cm^{-3} is always less than 3% of the total integral for neutron stars. Furthermore, this simple expression for η is greater than the more realistic values for the low-density viscosity given by Flowers and Itoh (1976) and by Nandkumar and Pethick (1984), at least down to densities of 10^{12} g cm^{-3} and $T \lesssim 10^9$ K. Therefore, the low-density contributions to the dissipation integrals are insignificant, and we do not bother to use more accurate values for the dissipation coefficients for densities below 10^{14} g cm^{-3}.

For densities above 4×10^{14} g cm^{-3} the situation is more serious. In extreme cases as much as 99.8% of the contribution to the dissipation integral comes from densities above 4×10^{14} g cm^{-3}. Since no one has published dissipation coefficients in this ultra–high-density range, we have simply extrapolated equations (14)–(15) to the needed densities. In the extreme case this extrapolation must be made up to 6×10^{15} g cm^{-3}, a factor of 15 above the range where the dissipation coefficients are known. An extension of the transport computations clearly needs to be undertaken in this extreme density range. A crude "back-of-the-envelope" estimate of the neutron contribution to the viscosity gives $\eta_n = 10^{11}\rho^{5/3}T^{-2}$. (The viscosity is approximately the momentum density of the neutrons times their mean free path. The mean free path can be estimated using the "bare" neutron-neutron cross section of 25 mbarns for neutrons in the appropriate energy range. The effective cross section must be reduced from this bare number to account for the fact that only neutrons within $\sim kT$ of the Fermi surface are capable of scattering at all. This factor gives the characteristic $1/T^2$ dependence.) This estimate agrees with the Flowers and Itoh (1979) result in equation (14) to within a factor of 2 over the density range $10^{14} \lesssim \rho \lesssim 4 \times 10^{14}$ g cm^{-3}, and still agrees with the extrapolated values to within a factor of 4 up to densities of 6×10^{15} g cm^{-3} We are reasonably confident, therefore, that our extrapolated viscosities are not wildly unrealistic.

The discussion thus far has ignored the fact that neutrons in neutron star matter are likely to be in a superfluid state over much of the relevant density and temperature range (see, e.g., Pines and Alpar 1985). The superfluidity of neutron star matter will, of course, have drastic effects both on the dissipation coefficients and on the dynamics of this material. In this paper we will approximate the effects of superfluidity on the dissipation time scales by correcting the dissipation coefficients, while ignoring the effects of superfluidity on the dynamics of the fluid.

When the temperature of the neutron star matter falls below the superfluid critical temperature, the effective scattering cross section of the neutrons with other particles quickly falls to zero. The neutrons, consequently, cease to contribute to the dissipation coefficients. The protons are also expected to make the transition to a superfluid state at about the same temperatures and densities as the neutron component of the neutron star material.

Consequently, only the normal electrons will be able to transport momentum and energy by scattering, and only electron-electron scattering will occur. The expression for the electron-electron scattering viscosity has been derived by

Flowers and Itoh (1976). The extreme relativistic limit of their expression, relevant to the high-density neutron star matter of interest here, is given by

$$\eta_e = \frac{\hbar}{3\pi^2}\left(\frac{p_e}{\hbar}\right)^3\left(\frac{2p_e c}{5\pi\alpha kT}\right)^2\left[\frac{\alpha}{\pi}\left(1+\frac{p_p m_p c}{p_e^2}\right)\right]^{1/2}, \quad (16)$$

where p_e and p_p are the Fermi momenta of the electrons and protons, and m_p is the proton rest mass. We have used the tables of these parameters in neutron star matter computed by Baym, Bethe, and Pethick (1971) to evaluate this expression. We find that this electron-electron scattering viscosity can be approximated in the density range $10^{14} \lesssim \rho \lesssim 4 \times 10^{14}$ g cm^{-3} covered by the Baym, Bethe, and Pethick calculations by the following simple power-law formula:

$$\eta_e = 6.0 \times 10^6 \rho^2 T^{-2} . \quad (17)$$

Equation (17) reproduces equation (16) to within ~5% in this density range. It is interesting to note that contrary to our experience with other superfluids like He_4, neutron star matter becomes more viscous in the superfluid state than it was in the normal state.

The viscosity in a neutron star containing some material in the superfluid state will therefore have the form:

$$\eta_s = (1-\theta)\eta + \theta\eta_e , \quad (18)$$

where $\theta(\rho, T)$ is defined by

$$\theta(\rho, T) = \begin{cases} 0 , & T > T_c(\rho) , \\ 1 , & T < T_c(\rho) . \end{cases} \quad (19)$$

The superfluid critical temperatures for neutron star matter have been computed by Takatsuka (1972) and by Amundsen and Ostgaard (1985). For our computations we use the values of $T_c(\rho)$ computed by Amundsen and Ostgaard (1985) for an effective neutron mass of $m^* = 0.8$. This function is depicted in Figure 1. We chose to use the Amundsen and Ostgaard critical temperatures, rather than Takatsuka's, because $T_c(\rho) > 0$ over a wider range of densities in their work. Consequently, the Amundsen and Ostgaard critical temperatures will have a larger effect on the dissipation time scales. Our results, therefore give some indication of the maximum effect that superfluidity is likely to have on the dissipation time scales.

IV. THE RESULTS

We are now prepared to evaluate the dissipation integrals, and the energy integrals needed to determine the dissipation time scale via equation (1). It is always informative to compare the results of complicated numerical computations with simple "back-of-the-envelope" analytic formulae. In the Appendix we derive simple expressions for the dissipative time scales based on a quasi-uniform density Newtonian model for the neutron stars. This formula for each of the dissipative time scales is

$$\frac{1}{\tau_\eta} = (l-1)(2l+1)\frac{\eta}{\rho R^2} , \quad (20)$$

$$\frac{1}{\tau_\zeta} = \frac{1}{2}\left(\frac{3}{5}\right)^4 \frac{l^3}{2l+3}\frac{\zeta}{\rho R^2} , \quad (21)$$

$$\frac{1}{\tau_\kappa} = \frac{3}{4\pi}\left(\frac{3}{25}\right)^2 \frac{l^3(2l+1)}{l-1}\frac{\kappa T}{G\rho^2 R^4} . \quad (22)$$

We find that these simple expressions correctly predict the results of our detailed numerical computations to within about a factor of 10.

These formulae reveal that the shear viscosity time scale is expected to be shorter (and therefore more important) than either the bulk viscosity or the thermal conductivity time scale. The ratio of the bulk to the shear viscosity time scale is easily estimated from the expressions in equations (20) and (21) to be

$$\frac{\tau_\zeta}{\tau_\eta} = 2\left(\frac{5}{3}\right)^4(2l+3)(2l+1)(l-1)\frac{1}{l^3}\frac{\eta}{\zeta} > 61\frac{\eta}{\zeta} . \quad (23)$$

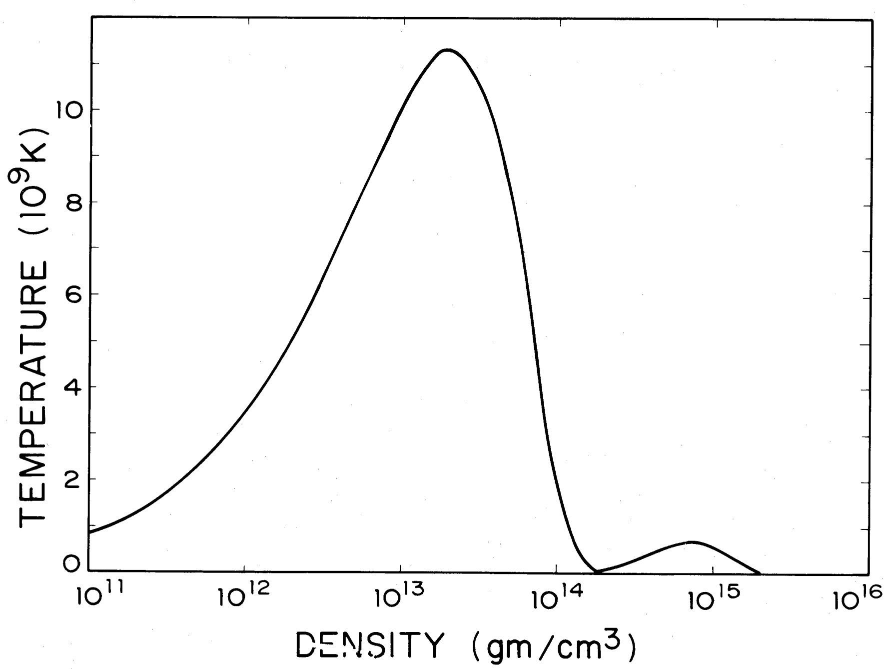

FIG. 1.—This is the superfluid critical temperature as a function of density for neutron star matter, as calculated by Amundsen and Ostgaard (1985)

The bulk viscosity coefficient, ζ, has yet to be computed for neutron star matter; however, it is generally comparable in size to the shear viscosity (see, e.g., Landau and Lifshitz 1959). Thus we conclude that the bulk viscosity time scale is expected to be about two orders of magnitude longer than the shear viscosity time scale. The ratio of the thermal conductivity time scale to the shear viscosity time scale can be determined from equations (14), (15), (20), and (22) to be given approximately by

$$\frac{\tau_\kappa}{\tau_\eta} = 3.8 \times 10^6 \, \frac{(l-1)^2}{l^3} \left(\frac{\rho}{10^{14}}\right)^{19/12} \left(\frac{R}{10^6}\right)^2 \left(\frac{10^9}{T}\right)^2 . \quad (24)$$

Thus the thermal conductivity can be neglected compared to the shear viscosity as the dominant energy dissipation mechanism in neutron stars for small values of l. (It is unclear to us if this is still true if a superfluid convective heat transport mechanism exists in neutron star matter analogous to that in superfluid He_4).

The dominant energy dissipation mechanism in neutron star matter is therefore shear viscosity. The time scale for this mechanism to damp neutron star oscillations is (from eqs. [14] and [20]) given approximately by

$$\tau_\eta = \frac{9.1 \times 10^9}{(l-1)(2l+1)} \left(\frac{10^{14}}{\rho}\right)^{5/4} \left(\frac{T}{10^9}\right)^2 \left(\frac{R}{10^6}\right)^2 \text{ s} ; \quad (25)$$

that is, less (sometimes much less) than 100 yr for typical neutron star parameters.

These formulae (eqs. [20]–[22]) give qualitative estimates of the dissipation time scales for neutron star oscillations. The purpose of this paper, however, is to compute more accurate time scales for more realistic neutron star models. We solved the neutron star pulsation equations for the eigenfunctions H_0, H_1, K, W, and V (as described in § II). These were used together with the viscosities described in § III to evaluate the dissipation integral in equation (10). We computed this time scale for the lowest few values of the spherical harmonic index l. We used eight different equations of state (labeled "A," "B," "C," "F," "L," "M," "N," "O;" see, e.g., Lindblom and Detweiler 1983 for references) to describe the structure of neutron star matter. For each equation of state we evaluate the dissipation time scales for two different models: one having the maximum mass possible for that equation of state, the other having 1.4 $N_\odot$ baryons, the minimum mass that can be formed astrophysically. Our results are presented in Table 1. There we give the ratio of this "exact" numerically determined dissipation time scale to the value estimated by equation (25). Note that this ratio is independent of temperature for nonsuperfluid matter. We also give in Table 1 the time scales corresponding to neutron stars with central temperature $T = 10^6$ K and superfluid neutrons. For temperatures below 10^6 K the ratios given in Table 1 will also be independent of temperature even for superfluid neutrons. This is because at 10^6 K essentially all of the fluid in the star capable of making the superfluid transition will have done so. For temperatures between 10^6 and 2×10^9 K the superfluid time scales will be intermediate between those given in Table 1 for normal fluid and those for superfluid neutrons at 10^6 K. Above $\sim 2 \times 10^9$ K essentially all of the material in the star will be too hot to be in the superfluid state, so the normal fluid time scales given in Table 1 will be the appropriate ones.

We note that the shear viscosity time scales for the normal neutron fluid are systematically shorter than the estimates from equation (25). This comes about because of the strong density dependence of the viscosity. The viscosity varies like $\rho^{9/4}$ (see eq. [14]). Since the largest portion of the dissipation integral comes from densities that are above the average value, the effective average viscosity is larger than that used to derive equation (25). As a consequence, the real dissipation integral is larger than that given in equation (A7) and the resulting realistic time scale is shorter than the estimate in equation (25).

V. DISCUSSION

Perhaps the most interesting application of these dissipative time scales is the role they play in limiting the angular velocity of rapidly rotating neutron stars. All rotating stars are subject to instabilities driven by gravitational radiation emission (see Friedman 1978). All of these instabilities are generally suppressed by the dissipative processes (viscosity and thermal conductivity) in the star (see Lindblom and Hiscock 1983). In slowly rotating stars, the viscous time scales are shorter than the gravitational radiation time scales, and all of the modes are in fact stable. More rapidly rotating stars are unstable to a

TABLE 1

VISCOUS DAMPING TIMES OF NEUTRON STAR OSCILLATIONS[a]

Equation of State	$M/M_\odot$	R (km)	Normal Neutrons Υ_2	Υ_3	Υ_4	Υ_5	Superfluid Neutrons ($T = 10^6$ K) Υ_2	Υ_3	Υ_4	Υ_5
M	1.277	16.057	0.222	0.344	0.465	0.55	0.051	0.074	0.096	0.11
	1.759	11.903	0.100	0.244	0.413	0.57	0.032	0.066	0.102	0.13
L	1.311	14.944	0.259	0.361	0.451	0.51	0.061	0.082	0.099	0.11
	2.661	13.619	0.107	0.184	0.248	0.29	0.030	0.048	0.062	0.07
N	1.385	13.784	0.264	0.360	0.441	0.49	0.066	0.087	0.104	0.11
	2.563	12.270	0.083	0.143	0.192	0.22	0.025	0.039	0.051	0.06
O	1.282	12.798	0.258	0.355	0.431	0.47	0.067	0.089	0.105	0.11
	2.380	11.581	0.094	0.158	0.210	0.24	0.029	0.045	0.057	0.06
C	1.317	12.027	0.198	0.324	0.448	0.56	0.056	0.085	0.112	0.13
	1.852	9.952	0.091	0.176	0.254	0.32	0.034	0.055	0.073	0.09
F	1.262	10.325	0.177	0.310	0.440	0.53	0.055	0.089	0.120	0.14
	1.463	7.966	0.075	0.164	0.263	0.35	0.042	0.066	0.088	0.11
A	1.246	9.783	0.203	0.308	0.399	0.45	0.064	0.092	0.115	0.13
	1.653	8.427	0.090	0.163	0.226	0.27	0.041	0.058	0.073	0.08
B	1.223	8.209	0.147	0.254	0.360	0.45	0.064	0.090	0.116	0.14
	1.412	7.000	0.069	0.129	0.187	0.24	0.043	0.061	0.074	0.08

[a] $\Upsilon_l \equiv 347\tau_{\eta,l}(2l+1)(l-1)\bar{\rho}^{5/4}R^{-2}T^{-2}$.

wider range of these instabilities, however. The modes that become unstable at higher angular velocities have smaller values of the spherical harmonic indicies l and m. These modes have shorter gravitational radiation time scales and longer viscous time scales. At sufficiently high angular velocities, a mode can become unstable in neutron stars when the viscous time scale is not sufficiently short to suppress the gravitational radiation instability. When this occurs, the star deforms itself into a nonaxisymmetric configuration and radiates away its excess angular momentum as gravitational radiation.

To determine the critical angular velocity, where the instability first sets in, it is necessary to compute the effects of rotation, gravitational radiation, and viscosity on the relevant modes of the neutron star. Lindblom (1986) showed that the critical angular velocities, Ω_l, could be determined approximately by solving the following equation:

$$\Omega_l = \frac{\omega_l}{l}\left[\alpha_l(\Omega_l) + \gamma_l(\Omega_l)\left(\frac{\tau_{G,l}}{\tau_{v,l}}\right)^{1/(2l+1)}\right]. \tag{26}$$

In this equation ω_l is the frequency of the lth mode of the corresponding nonrotating star; α_l and γ_l are functions described in Lindblom (1986) which are nearly equal to one over their entire domain; the time scales $\tau_{G,l}$ and $\tau_{\eta,l}$ in this equation refer to the gravitational radiation and viscous time scales of the corresponding nonrotating star. This equation, therefore, contains all of the information about the effects of rotation on the relevant modes in the functions α_l and γ_l.

We have solved this equation for the critical angular velocities of the neutron stars constructed from the eight equations of state considered here. We have used the new viscosity time scales computed in this paper, along with the gravitational radiation time scales computed by Lindblom (1986). The results of this analysis are presented in Table 2. There we give the critical rotation periods (measured in milliseconds) for the maximum mass neutron star model, and the model containing 1.4 $N_\odot$ baryons for each equation of state. We compute the critical rotation period for temperatures between 10^7 K and 10^{10} K. For these computations we used the nonsuperfluid viscosities. Superfluidity only decreases these critical periods by $\sim 1\%$ at the temperatures listed. Since this change in the critical rotation periods is so small compared with the changes due to the uncertainty in the equation of state, it would not be possible to distinguish superfluid neutron stars from those containing normal fluid simply by observing maximum rotation rates.

TABLE 2

CRITICAL ROTATION PERIODS FOR NEUTRON STARS[a]

Equation of State	$M/M_\odot$	R (km)	T (K) 10^{10}	10^9	10^8	10^7
M	1.277	16.057	1.81	1.75	1.71	1.69*
	1.759	11.903	1.03	0.96	0.94	0.92*
L	1.311	14.944	1.63	1.57	1.53	1.50*
	2.661	13.619	1.13	1.05	1.00	0.94
N	1.385	13.784	1.43	1.37	1.33	1.30
	2.563	12.270	0.99	0.92	0.87	0.82
O	1.282	12.798	1.32	1.27	1.23	1.20*
	2.380	11.581	0.94	0.87	0.83	0.78
C	1.317	12.027	1.18	1.13	1.10	1.08*
	1.852	9.952	0.81	0.76	0.70	0.69
F	1.262	10.325	0.97	0.92	0.89	0.88
	1.463	7.966	0.64	0.60	0.57	0.55*
A	1.246	9.783	0.91	0.86	0.84	0.82
	1.653	8.427	0.67	0.63	0.60	0.57
B	1.223	8.209	0.71	0.67	0.65	0.63*
	1.412	7.000	0.54	0.51	0.48	0.46*

[a] Periods given in milliseconds. A number that is followed by an asterisk indicates that the period is essentially equal (within 0.1%) to the minimum possible rotation period that is estimated for this stellar model. In this case the instability that limits the rotation rate is not expected to be the secular instability considered here.

An interesting feature of these results is the temperature dependence of the critical rotation periods. For sufficiently low temperatures, we find that viscosity completely stabilizes these modes. In Table 2 we have marked with an asterisk the critical periods which are equal to the minimum rotation periods for the associated Maclaurin spheroid on which the estimates of the critical rotation periods are based. The secular instability would not be expected to play a role in limiting the angular velocity of these stars. At a temperature of $T = 10^6$ K we find that all of the stellar models are immune to these gravitational radiation induced secular instabilities.

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APPENDIX

APPROXIMATE TIME SCALES

The purpose of this Appendix is to derive approximate formulae for the time scales that govern the dissipation of energy from the nonradial oscillations of neutron stars.

We have found that the nonradial oscillations of a neutron star are rather well described (within factors of ~ 2 in the high-density region of the star) by the Lagrangian displacement

$$\xi_a = \frac{\epsilon R^2}{l} e^{i\omega t}\,\nabla_a\left[\left(\frac{r}{R}\right)^l Y^l_m\right], \tag{A1}$$

where ϵ is a (small) dimensionless parameter and R is the total radius of the star. This corresponds to $We^{\lambda/2} = -lV = \epsilon R^{2-l}$ in the notation of § II, and is precisely the Lagrangian displacement for the nonradial Kelvin oscillations of an incompressible Newtonian star. The velocity perturbation associated with such a Lagrangian displacement is given by

$$\delta v^a = i\omega\xi^a\,. \tag{A2}$$

From these expressions we can evaluate the energy and dissipation integrals needed to obtain approximate dissipation time scales. The kinetic energy associated with these oscillations is, in the Newtonian limit, simply

$$E_K = \tfrac{1}{2}\int \rho \delta v^a \delta v_a^* d^3x \,, \tag{A3}$$

where δv_a^* is the complex conjugate of δv_a. The energy in an oscillating star consists of kinetic and potential energy pieces. Since the system is basically a harmonic oscillator, in the linear perturbation limit considered here, the kinetic and potential contributions to the energy are expected to have equal magnitudes; thus, the total energy is given by $E = 2E_K$. The integral in equation (A3) can be explicitly evaluated for the Lagrangian displacement from equation (A1) if we assume that the density, ρ, is uniform throughout the star ($\rho = \bar{\rho} \equiv 3M/4\pi R^3$). For neutron stars, the density is in fact reasonably uniform. Under these circumstances the total energy contained in the oscillations is given by

$$E = l^{-1}\rho\omega^2\epsilon^2 R^5 \,. \tag{A4}$$

To complete our evaluation of the dissipative time scale, τ, we must evaluate the dissipation integrals:

$$\frac{dE}{dt} = -\frac{2E}{\tau} = -\int\left(2\eta\delta\sigma^{ab}\delta\sigma_{ab}^* + \zeta\,|\,\delta\theta\,|^2 + \frac{\kappa}{T}\,\nabla_a\delta T\nabla^a\delta T^*\right)d^3x \,, \tag{A5}$$

where η, ζ, and κ are the viscosity coefficients and thermal conductivity, T is the temperature of the star and, $\delta\sigma^{ab}$ and $\delta\theta$ are the shear and expansion of the perturbed fluid motion. Unfortunately, within the context of the uniform density Newtonian fluids for which equation (A1) is the appropriate Lagrangian displacement, two of the terms in equation (A5) vanish identically. The expansion, $\delta\theta$, is identically zero for this Lagrangian displacement and $\nabla_a \delta T$, vanishes for adiabatic perturbations of ideal incompressible fluids (see, e.g., Lindblom 1979). We can, however, evaluate the shear viscosity contribution to the dissipation of the energy in this approximation. The shear of the perturbed fluid motion is related to the Lagrangian displacement given in equation (A1) by

$$\delta\sigma_{ab} = i\omega\nabla_a\xi_b \,. \tag{A6}$$

If the shear viscosity η is uniform throughout the star, then it is straightforward to evaluate the shear portion of the integral. We will assume, therefore that $\eta = \bar{\eta} \equiv \eta(\bar{\rho}, T_c)$ is uniform for this approximation, where T_c is the central temperature of the star. It follows that

$$\int 2\eta\delta\sigma^{ab}\delta\sigma_{ab}^*\, d^3x = 2l^{-1}(l-1)(2l+1)\omega^2\epsilon^2 R^3\eta \,. \tag{A7}$$

This expression can now be used together with equation (A4) and (A5) to obtain an approximate formula for the shear viscosity time scale τ_η:

$$\frac{1}{\tau_\eta} = \frac{(l-1)(2l+1)\eta}{\rho R^2} \,. \tag{A8}$$

This formula is identical to the one derived by Lamb (1881) to describe the viscous damping of the nonradial oscillations of incompressible Newtonian stars.

To obtain (nonvanishing) expressions for the contributions of bulk viscosity and thermal conductivity to the damping of neutron star oscillations, we must go beyond the incompressible fluid approximation used to obtain equation (A8). It is convenient to use the Lagrangian change in the pressure, Δp, as the fundamental variable for this part of the discussion. This variable vanishes identically in the incompressible fluid approximation. To go beyond that approximation we will posit the following form for Δp:

$$\frac{\Delta p}{p} = -\frac{\epsilon l}{\gamma}\left(\frac{r}{R}\right)^l e^{i\omega t}Y_m^l \,, \tag{A9}$$

where γ is the adiabatic index of the fluid and ϵ is the same small dimensionless parameter as that given in equation (A1). This formula correctly approaches zero as the fluid becomes incompressible (i.e., as γ goes to infinity) and agrees with the perturbations of the pressure in our realistic neutron star models to within about a factor of 2 (except in the low-density outer layers of the star where $\Delta p/p$ vanishes for realistic neutron stars).

To proceed with our estimation of the dissipation integrals in equation (A5), we need to relate the variable Δp to the quantities that appear in those expressions: $\delta\theta$ and δT. The conservation of mass (or in relativistic neutron stars, the conservation of baryon number) implies the following relation between the expansion of the fluid and Δp for adiabatic perturbations:

$$\delta\theta = -i\omega\Delta p/\gamma p \,. \tag{A10}$$

(The relativistic analog of this equation also contains the redshift factor $e^{-\nu/2}$ on the right-hand side.) The temperature perturbation is also simply related to Δp for adiabatic perturbations about an isothermal background:

$$\delta T = \Delta T = \left(\frac{\partial T}{\partial p}\right)_s \Delta p \,. \tag{A11}$$

We can now complete our evaluation of the dissipation integrals if we assume that the dissipation coefficients ζ and κ, the temperature T, and the thermodynamic derivatives γ and $\beta \equiv \partial \log T/\partial \log p$ are uniform throughout the star. Under these assumptions the following integrals are straightforward consequences of equations (A9)–(A11):

$$\int \zeta |\delta\theta|^2 d^3x = \epsilon^2 \zeta \frac{l^2 \omega^2 R^3}{(2l+3)\gamma^4}, \tag{A12}$$

$$\int \frac{\kappa}{T} \nabla_a \delta T \nabla^a \delta T^* d^3x = \epsilon^2 \kappa T \left(\frac{\beta}{\gamma}\right)^2 l^3 R. \tag{A13}$$

These integrals can now be used with equations (A4) and (A5) to obtain the desired dissipative time scales:

$$\frac{1}{\tau_\zeta} = \frac{l^3 \zeta}{2(2l+3)\rho\gamma^4 R^2}, \tag{A14}$$

$$\frac{1}{\tau_\kappa} = \frac{l^4 \kappa T \beta^2}{2\rho\omega^2\gamma^2 R^4}. \tag{A15}$$

To use these formulae we need estimates of the thermodynamic derivatives β and γ, as well as an estimate of the frequency ω. To estimate the thermodynamic derivatives we will assume that the neutron star matter is an ideal Fermi gas of neutrons. The neutrons are nonrelativistic over most of the relevant density range for neutron stars, and the thermal energies are low compared to the Fermi energy of the neutrons. It is reasonable, therefore, to adopt the nonrelativistic zero temperature Fermi gas model for the neutron matter. In this case the thermodynamic derivatives β and γ can easily be evaluated, with the results:

$$\beta = \tfrac{2}{5}, \tag{A16}$$

$$\gamma = \tfrac{5}{3}. \tag{A17}$$

The frequency ω is given with reasonable accuracy (see Lindblom 1986) by Thomson's (1863) formula:

$$\omega^2 = \frac{2l(l-1)}{2l+1} \frac{GM}{R^3}. \tag{A18}$$

So, for neutron stars the dissipative time scales are given approximately by

$$\frac{1}{\tau_\zeta} = \frac{1}{2}\left(\frac{3}{5}\right)^4 \frac{l^3}{2l+3} \frac{\zeta}{\rho R^2}, \tag{A19}$$

$$\frac{1}{\tau_\kappa} = \frac{3}{4\pi}\left(\frac{3}{25}\right)^2 \frac{l^3(2l+1)}{l-1} \frac{\kappa T}{G\rho^2 R^4}. \tag{A20}$$

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