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ESTIMATES OF THE MAXIMUM ANGULAR VELOCITY OF ROTATING NEUTRON STARS

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ABSTRACT

The instabilities that limit the rotation rate of neutron stars are driven by gravitational radiation reaction and moderated by viscosity. This paper estimates the angular velocities where these instabilities set in. These estimates improve on previous work in two ways: (*a*) by approximating the influence of the dissipative mechanisms on the locations of the critical angular velocities, and (*b*) by using fully relativistic calculations of some of the relevant frequencies and time scales. This analysis indicates that the influence of gravitational radiation is somewhat greater than had been anticipated in earlier work, and as a result that an $m = 4$ or 5 (rather than $m = 3$ or 4) instability is probably responsible for limiting the angular velocity of rapidly rotating neutron stars. The minimum rotation periods are computed here for a sample of eight different equations of state for the nuclear matter, for a range of different viscosities, and for the allowed range of neutron star masses. These calculations show that the critical rotation periods are very insensitive to the value of the viscosity but depend strongly on the equation of state of the nuclear matter and on the mass of the neutron star.

Subject headings: dense matter — stars: neutron — stars: rotation

I. INTRODUCTION

Since the discovery of the pulsar PSR 1937+215 (Backer *et al.* 1982) with period 1.56 ms, a great deal of attention has been given to the problem of predicting the minimum period that is consistent with the widely accepted rotating neutron star model of pulsars. Friedman (1983) pointed out that the instability that limits the angular velocity of neutron stars occurs in a nonaxisymmetric mode that is driven by gravitational radiation reaction. While gravitational radiation causes these modes to grow in rapidly rotating stars, viscosity stabilizes them (Lindblom and Detweiler 1977). Thus the presence of viscosity causes an increase in the critical angular velocity where this gravitational radiation reaction instability sets in. For modes [having angular dependence exp $(im\varphi)$] with small values of m, the gravitational radiation reaction time scale is much shorter than the viscous time scale in neutron stars. The critical angular velocities for these modes are, consequently, effectively unchanged by the presence of viscosity. For large values of m the viscous time scale is much shorter than the gravitational radiation reaction time scale. The corresponding critical angular velocities are substantially increased in this case, which makes these modes effectively stable.

To predict the maximum angular velocity of neutron stars, therefore, one must perform two different analyses. First, the critical angular velocities where these instabilities set in must be determined, and their dependence on the viscous and gravitational radiation reaction time scales must be estimated for the relevant modes. Second, these dissipative time-scales must be estimated for realistic neutron stars. Friedman (1983) made estimates of both effects. He used the Newtonian Maclaurin spheroids as simple models of rotating stars to obtain estimates of the time scales for viscosity and gravitational radiation to influence the evolution of a mode. From these estimates he concluded that modes having $m \geq 5$ in neutron stars would be stabilized by the presence of viscosity. This led him to predict that the instability in the $m = 4$ mode would probably limit the angular velocity in neutron stars. In the Maclaurin spheroids this mode becomes unstable when t (the ratio of the rotational kinetic energy to the gravitational potential energy) has the value 0.08. Friedman estimated the critical angular velocities for this mode in realistic neutron stars by using the moment of inertia and gravitational binding energy of nonrotating relativistic neutron stars (see Arnett and Bowers 1977) to approximate t. The critical values of the energy ratio t have also been computed more recently for rigidly rotating Newtonian polytropes by Imamura, Friedman, and Durisen (1985) and by Managan (1985).

This paper adopts a rather different strategy for computing estimates of the dissipative time scales and estimates of the critical angular velocities of the relevant modes. The frequencies and gravitational radiation reaction time scales of the relevant modes ($2 \leq m \leq 5$) are computed here for fully relativistic but nonrotating neutron star models having "realistic" equations of state. The values of these quantities in rotating stars are then extrapolated from their nonrotating values using the Maclaurin spheroid formulae for their angular velocity dependence. These extrapolations suggest that gravitational radiation reaction is stronger in these stars than had been anticipated on the basis of Friedman's work. Consequently, the modes through $m = 4$ or $m = 5$ will be unstable to the gravitational radiation secular instability.

In § II of this paper the secular instabilities of the Maclaurin spheroids are reanalyzed The purpose here is to obtain the formulae for the angular velocity dependence of the frequencies and time scales that govern the secular instabilities. This analysis results in a very simple formula for the critical angular velocity of a given mode. In § III the accuracy of this formula for predicting the critical angular velocities in more general classes of stars than the Maclaurin spheroids is tested. These angular velocities are estimated for the $3 \leq m \leq 5$ modes of rigidly rotating Newtonian polytropes with indices $n = 1.0$ and $n = 1.5$. These estimates agree with the values deduced from computations of Imamura, Friedman, and Durisen (1985) and Managan (1985) to within 5%–10%. Finally, in § IV, the critical angular velocities are estimated for realistic neutron stars using this formula. The frequencies and gravitational radiation reaction time scales needed to use this formula are computed using a fully general relativistic analysis and a number of different "realistic" equations of state for the nuclear matter in the star. The critical rotation periods are computed for neutron stars having the maximum mass, and also for neutron stars having the minimum number of baryons (1.4 $N_\odot$) that can be formed astrophysically. The critical periods are computed for a range of different viscosities expected to span the possibilities for neutron star matter. These calculations show that the critical angular velocities depend very sensitively on the mass of the neutron star, and on the equation of state of the nuclear matter. They do not, however, depend very strongly on viscosity. The critical angular velocities, so predicted, are compared to the frequency of the pulsar PSR 1937+214. These calculations show that most of the critical rotation periods are shorter than the 1.56 ms period of this pulsar. Only the lowest mass models ($N = 1.4\ N_\odot$) in the stiffest equations of state in this sample are inconsistent with the existence of the 1.56 ms pulsar. These estimates suggest, however, that no pulsar having period shorter than ~0.6 ms is consistent with any of the realistic equations of state.

II. THE MACLAURIN SPHEROIDS

The secular instability (instability caused by the presence of dissipative forces) of the Maclaurin spheroids was first recognized on the basis of energy arguments by Thomson and Tait (1883). This early analysis did not, however, determine the time evolution of an unstable spheroid away from equilibrium. Roberts and Stewartson (1963) were the first to study the $l = 2$ modes of viscous Maclaurin spheroids. They explicitly calculated the frequencies of these modes and demonstrated the existence of an exponentially growing $m = -2$ mode in sufficiently rapidly rotating models. Chandrasekhar (1970*a*, *b*) demonstrated that an analogous instability to gravitational radiation reaction exists in the Maclaurin spheroids as well (in an $m = 2$ mode). Both these effects were first studied in the context of the modes of interest here (the general $l = m$ modes) by Comins (1979*a*, *b*).

The modes of the Maclaurin spheroid are taken to have the dependence exp $[i\sigma_m(\Omega)t + im\varphi - t/\tau_m(\Omega)]$ on the time t and the azimuthal angle φ. The frequencies $\sigma_m(\Omega)$ depend on the angular velocity Ω of the spheroid, as well as on the integer m. These frequencies, determined originally by Bryan (1889) and tabulated by Comins (1979*b*), have the remarkable property that they are essentially independent of angular velocity when viewed in the corotating frame of the spheroid. Thus the function

$$\alpha_m(\Omega) \equiv [\sigma_m(\Omega) + m\Omega]/\sigma_m(0) \tag{1}$$

is surprisingly independent of Ω, as can be seen in Figure 1. The frequencies of the nonrotating fluid spheres that appear in equation (1) were first computed by Thomson (1863) and are given by the expression

$$\sigma_m(0) = \left[\frac{GM}{R^3}\,\frac{2m(m-1)}{2m+1}\right]^{1/2}, \tag{2}$$

where G is Newton's constant, M the total mass, and R the radius of the corresponding nonrotating sphere (or equivalently the geometric mean of the principal axes of the rotating spheroid: $R^3 = a_1 a_2 a_3$).

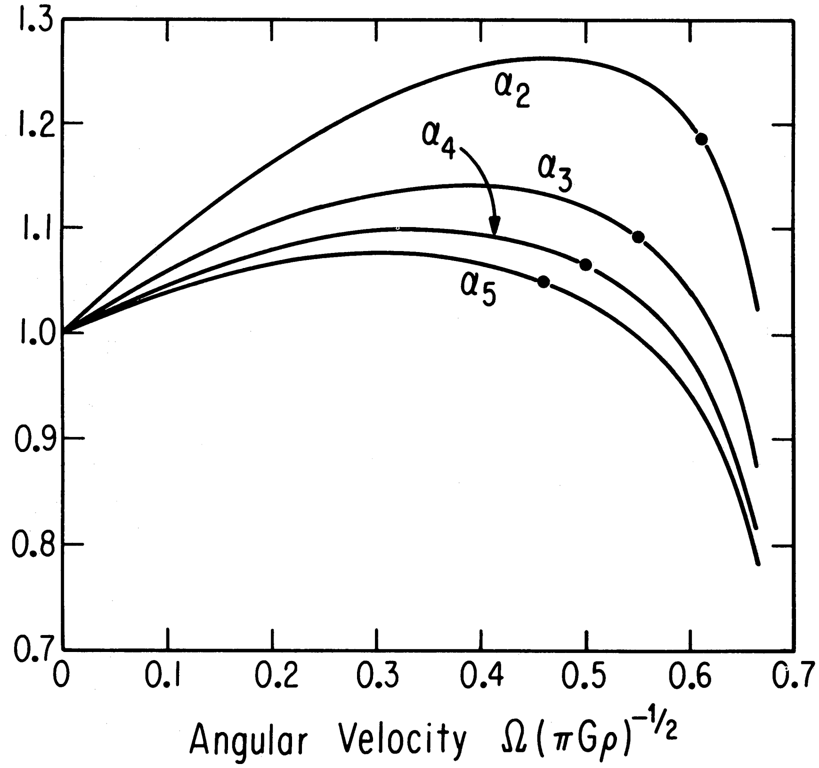

FIG. 1.—Maclaurin spheroid functions $\alpha_m(\Omega)$ for $2 \le m \le 5$. These functions, defined in equation (1), are the ratios of the frequencies of the $l = m$ modes (as measured in the rotating frame of the spheroid) to the frequency of the nonrotating sphere of the same density. The angular velocity is given in units of $(\pi G\rho)^{1/2}$, where ρ is the density of the spheroid.

The time scale $\tau_m(\Omega)$ that determines the rate at which a particular mode is damped (or amplified) by the dissipative processes was first computed by Comins (1979*a*) for these modes. His expression for this time scale can be written in the form

$$\frac{1}{\tau_m(\Omega)} = \beta_m(\Omega)\left\{\tau_{V,m}^{-1} + \tau_{GR,m}^{-1}\left[\frac{\sigma_m(\Omega)}{\sigma_m(0)\gamma_m(\Omega)}\right]^{2m+1}\right\}. \tag{3}$$

The viscous time scale $\tau_{V,m}$ for the corresponding nonrotating spheroid (first derived by Lamb 1881) is given by

$$\tau_{V,m}^{-1} = (2m+1)(m-1)\frac{\nu}{R^2}, \tag{4}$$

where ν is the kinematic viscosity. The corresponding time scale for gravitational radiation reaction (first computed by Detweiler 1975) is given by the expression

$$\tau_{GR,m}^{-1} = \frac{3}{2}\frac{(m+1)(m+2)}{(m-1)[(2m+1)!!]^2}\left(\frac{2m(m-1)}{2m+1}\right)^m\left(\frac{GM}{c^2R}\right)^{m+1}\frac{c}{R}, \tag{5}$$

where the constant c is the speed of light. The angular velocity dependence of the dissipative time scale $\tau_m(\Omega)$ has been absorbed into the two functions $\beta_m(\Omega)$ and $\gamma_m(\Omega)$. These functions have reasonably simple expressions in terms of the eccentricity $e(\Omega)$ of the spheroid of given angular velocity:

$$\beta_m(\Omega) = \frac{[1-e^2(\Omega)]^{1/3}[\sigma_m(\Omega)+m\Omega]}{\sigma_m(\Omega)+(m-1)\Omega} \tag{6}$$

and

$$\gamma_m(\Omega) \equiv \{\alpha_m(\Omega)[1-e^2(\Omega)]^{m/3}\}^{1/(2m+1)}. \tag{7}$$

These functions are depicted in Figures 2 and 3 for the first few values of m. These figures illustrate that none of the functions α_m, β_m, or γ_m deviate substantially from their nonrotating values $[\alpha_m(0) = \beta_m(0) = \gamma_m(0) = 1]$, even for very rapidly rotating spheroids. The circle on each curve represents the point at which the pure gravitational radiation reaction secular instability sets in to that mode.

The oscillations of the Maclaurin spheroids will be damped by the dissipative mechanisms as long as the time scale $\tau_m(\Omega)$ is positive. The critical angular velocity Ω_m (where instability to the mth mode sets in) occurs at the point where $\tau_m(\Omega)$ changes sign. Thus, Ω_m is simply the root of the equation

$$1/\tau_m(\Omega_m) = 0. \tag{8}$$

A more useful expression for these critical angular velocities can be obtained directly from equation (8) using equations (1) and (3). The result is:

$$\Omega_m = \frac{\sigma_m(0)}{m}\left[\alpha_m(\Omega_m) + \gamma_m(\Omega_m)\left(\frac{\tau_{GR,m}}{\tau_{V,m}}\right)^{1/(2m+1)}\right]. \tag{9}$$

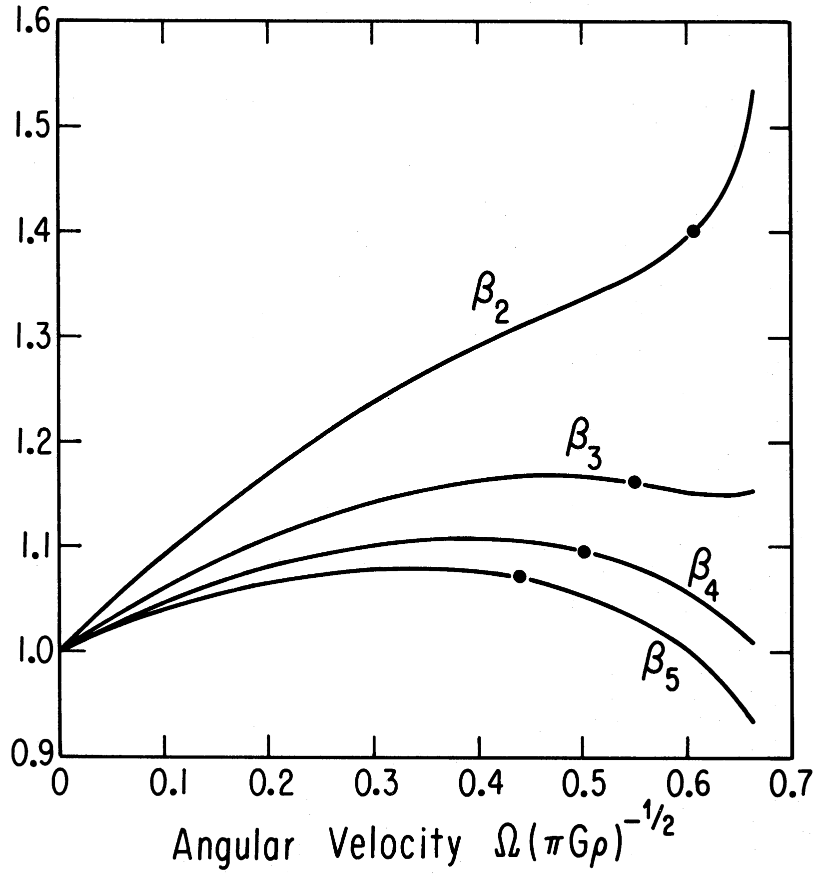

FIG. 2.—Maclaurin spheroid functions $\beta_m(\Omega)$ for $2 \le m \le 5$. These functions, defined in equation (6), give the angular velocity dependence of the viscous damping time for the $l = m$ modes. Angular velocity is given in same units as in Fig. 1.

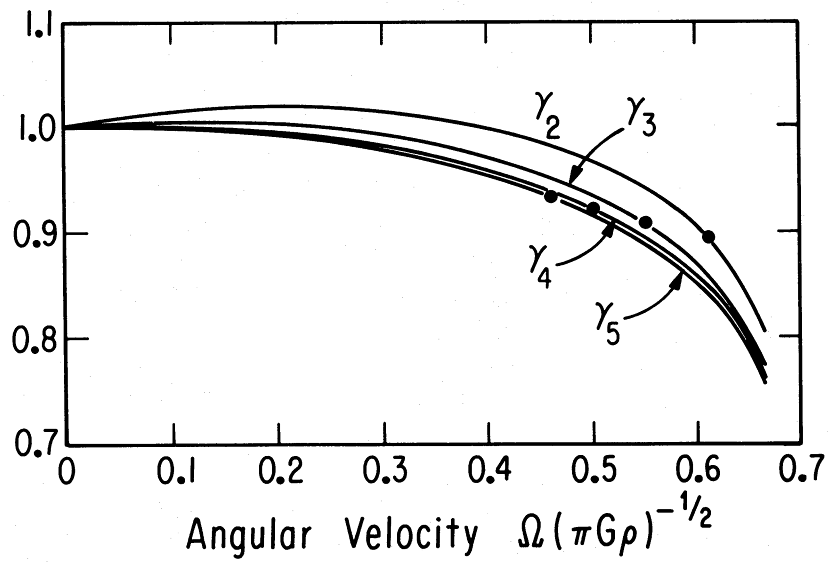

FIG. 3.—Maclaurin spheroid functions $\gamma_m(\Omega)$ for $2 \le m \le 5$. These functions, defined in eqn. (7), give the angular velocity dependence of the gravitational radiation reaction damping time for the $l = m$ modes. Angular velocity is given in same units as in Fig. 1.

This equation is very easy to solve numerically because the functions α_m and γ_m are nearly independent of Ω. Thus, an excellent approximation to the critical angular velocity is simply

$$\Omega_m \approx \frac{\sigma_m(0)}{m}\left[1 + \left(\frac{\tau_{GR,m}}{\tau_{V,m}}\right)^{1/(2m+1)}\right]. \tag{10}$$

Since this expression depends only on the properties of the nonrotating stellar model, it is extremely easy to compute. This approximation is accurate to within 5%–10% for the $3 \le m \le 5$ modes of the Maclaurin spheroids, and the approximation gets better and better for larger values of m.

In neutron stars the gravitational radiation reaction time scale is much shorter than the viscous time scale, $\tau_{GR,m} \ll \tau_{V,m}$, for small values of m ($m = 2$ or 3). For these small values of m, the second term in equation (9) will be negligible compared to the first so that $\Omega_m \approx \Omega_{GR,m}$, where

$$\Omega_{GR,m} = \frac{\sigma_m(0)}{m}\,\alpha_m(\Omega_{GR,m})\,. \tag{11}$$

These "viscosity-free" critical angular velocities have decreasing magnitudes as m gets larger. Thus, each successive mode is unstable over a wider range of angular velocities. For larger values of m ($m \gtrsim 6$), the viscous time scale is shorter than the gravitational radiation time scale: $\tau_{GR,m} > \tau_{V,m}$. In this case the second term in equation (9) is larger than the first. The critical angular velocities are consequently significantly increased. Therefore, viscosity stabilizes these modes. Since the critical angular velocities Ω_m are decreasing with m for small m and increasing with m for larger m, it follows that a minimum critical angular velocity will exist. For neutron stars this minimum occurs for $m = 4$ or 5, as we argue in § IV. This minimum critical angular velocity will be the maximum angular velocity of the stable Maclaurin spheroids.

The time scale with which a given mode grows is given by equation (3). The gravitational radiation reaction term in this equation is very small over much of the relevant angular velocity range because it is proportional to the expression $(\Omega - \Omega_{GR,m})^{2m+1}$, where $\Omega_{GR,m}$ is the viscosity-free critical angular velocity. Since the viscous term is nearly independent of angular velocity, it is the viscous time scale that will essentially determine the secular evolution of these modes. Of course, at the critical angular velocity this time scale goes to infinity. However, because of the presence of viscosity it does so more slowly than in the pure gravitational radiation case. Near the critical angular velocity, the first term in the Taylor series for $1/\tau_m(\Omega)$ can be evaluated to obtain the expression

$$\frac{1}{\tau_m(\Omega)} \approx -\frac{2m+1}{\tau_{GR,m}}\left[\frac{m}{\sigma_m(0)} - \frac{d\alpha_m}{d\Omega} - \left(\frac{\tau_{GR,m}}{\tau_{V,m}}\right)^{1/(2m+1)}\frac{d\gamma_m}{d\Omega}\right] \times \beta_m(\Omega_m)\left[\frac{\Omega - \Omega_m}{\gamma_m(\Omega_m)}\right]\left(\frac{\tau_{GR,m}}{\tau_{V,m}}\right)^{2m/(2m+1)}. \tag{12}$$

Since α_m and γ_m are decreasing functions and since $\gamma_m < 1$ when evaluated at the critical angular velocities, $\beta_m > 1$ and an upper bound for $\tau_m(\Omega)$ is given by

$$\left|\frac{1}{\tau_m(\Omega)}\right| \gtrsim \frac{m(2m+1)|\Omega - \Omega_m|}{\tau_{GR,m}\,\sigma_m(0)}\left(\frac{\tau_{GR,m}}{\tau_{V,m}}\right)^{2m/(2m+1)}. \tag{13}$$

These expressions are linear in $\Omega - \Omega_m$, while the corresponding expressions in the viscosity-free case go like $(\Omega - \Omega_m)^{2m+1}$. Thus the characteristic time scale for the growth of these instabilities is shorter in the presence of viscosity than one would have anticipated on the basis of the pure gravitational radiation instability.

III. ROTATING NEWTONIAN POLYTROPES

At present the analysis of the secular instabilities in rapidly rotating but nonuniform density stellar models is not nearly as well advanced as that described in § II. This more general analysis must be completely numerical, since analytic solutions for the equilibrium structures of these stars do not even exist, let alone analytic solutions of the perturbation equations. In fact, at the present time, the only models for which the critical angular velocities have been calculated (for $m > 2$ modes) are the rapidly rotating Newtonian polytropes. Imamura, Friedman, and Durisen (1985) and Managan (1985) used slightly

TABLE 1

CRITICAL ANGULAR VELOCITIES IN RIGIDLY ROTATING NEWTONIAN POLYTROPES[a]

n	m	t_c	$\Omega_c(\pi G\bar{\rho}_0)^{-1/2}$	$\sigma_m(\pi G\bar{\rho}_0)^{-1/2}$	$\Omega_{GR}(\pi G\bar{\rho}_0)^{-1/2}$
1.0	3	0.079	0.60	1.950	0.64
1.0	4	0.058	0.56	2.335	0.58
1.0	5	0.044	0.50	2.648	0.53
1.5	3	0.056	0.61	2.219	0.66
1.5	4	0.043	0.57	2.587	0.62
1.5	5	0.034	0.53	2.880	0.56

[a] The critical angular velocities Ω_c were deduced from the critical values of the energy ratio t_c tabulated by Imamura, Friedman, and Durisen 1985 and by Managan 1985. The critical angular velocities Ω_{GR} were deduced from the frequencies of nonrotating polytropes σ_m by solving eqn. (11). All frequencies are expressed in units of $(\pi G\bar{\rho}_0)^{1/2}$, where $\bar{\rho}_0$ is the average density of the corresponding nonrotating polytrope.

different variational principles to estimate the critical angular velocities in the $2 \leq m \leq 5$ modes for polytropes having indices $n = 1.0$ and $n = 1.5$. The results of their calculations are summarized in Table 1. The primary result of their computation is the value of t_c, the ratio of the rotational kinetic energy T to the gravitational potential energy W of the star when a particular mode becomes unstable. I have converted their critical values of this energy ratio into values for the critical angular velocity (listed as Ω_c in Table 1) using unpublished data on the structure of rigidly rotating polytropes supplied to me by J. Friedman from J. Imamura.

It would be desirable to have a more straightforward method for computing the critical angular velocities in stellar models having more general equations of state. Clearly the critical angular velocities, even in the general case, will be the roots of an equation which is the analog of equation (9) for the Maclaurin spheroids:

$$\Omega_m = \frac{\sigma_m(0)}{m}\left[\tilde{\alpha}_m(\Omega_m) + \tilde{\gamma}_m(\Omega_m)\left(\frac{\tau_{GR,m}}{\tau_{V,m}}\right)^{1/(2m+1)}\right]. \qquad (14)$$

In this equation $\sigma_m(0)$, $\tau_{GR,m}$ and $\tau_{V,m}$ represent the frequency and dissipative time scales of the mth mode of the nonrotating but otherwise general stellar models. The functions $\tilde{\alpha}_m$ and $\tilde{\gamma}_m$ will depend on the details of the structure of the corresponding general rotating stellar models. While the properties of nonrotating stars [e.g., $\sigma_m(0)$, $\tau_{GR,m}$, $\tau_{V,m}$] are reasonably easy to compute, the functions $\tilde{\alpha}_m$ and $\tilde{\gamma}_m$ that contain information about the pulsations of the rotating models are not easy to determine.

In the case of the Maclaurin spheroids, we saw in § II that the functions α_m and γ_m were nearly independent of the angular velocity. In that case the values of the critical angular velocities could be determined with reasonable accuracy for the $m \geq 3$ modes by replacing both these functions by their nonrotating values of 1. It seems likely, therefore, that the functions $\tilde{\alpha}_m$ and $\tilde{\gamma}_m$ will not depend strongly on the angular velocity in general stellar models either. It should be possible to obtain reasonably accurate estimates of the critical angular velocities in general rotating stellar models, therefore, by solving equation (14) using the correct values for $\sigma_m(0)$, $\tau_{GR,m}$, and $\tau_{V,m}$, but using the Maclaurin spheroid functions α_m and γ_m.

This approximation method was tested on the modes of the rotating Newtonian polytropes. The relevant frequencies of the nonrotating polytropes were computed using a computer code described in Lindblom and Detweiler (1983) and Balbinski *et al.* (1985). Those frequencies are listed in Table 1 and are accurate to about 0.1%. Using these frequencies and the Maclaurin spheroid functions α_m, the "viscosity-free" critical angular velocities Ω_{GR} were determined by solving equation (11) numerically. The results of this computation are also listed in Table 1. The actual values of the critical angular velocities Ω_c agree with the estimates to within 5%–10%. Given the ease with which Ω_{GR} can be computed (compared to the difficulty of determining Ω_c), this degree of accuracy is perhaps better than one would have anticipated.

IV. REALISTIC NEUTRON STARS

The study of the structure and stability of rapidly rotating stars in general relativity is in an even more primitive state of development than is the analogous theory for Newtonian stars. Numerical models of rapidly rotating relativistic polytropes have been constructed by Butterworth and Ipser (1976) and Butterworth (1976), and rapidly rotating models of realistic neutron stars have been constructed more recently by Friedman, Ipser, and Parker (1985). Friedman (1978) demonstrated the existence of the gravitational radiation reaction secular instability in rotating general relativistic stars; and Lindblom and Hiscock (1983) showed that viscosity tends to suppress the instability in sufficiently slowly rotating stars as it does in Newtonian stars. To date, however, no one has computed the frequencies or the dissipative time scales for rotating relativistic stars.

The purpose of this final section is to estimate the critical angular velocities of realistic general relativistic neutron stars. The plan is to solve equation (14) using fully general relativistic values of $\sigma_m(0)$ and $\tau_{GR,m}$ while using the Maclaurin spheroid expressions for $\tau_{V,m}$, $\tilde{\alpha}_m$, and $\tilde{\gamma}_m$. Since this method of estimating the critical angular velocities in rotating Newtonian polytropes proved to be accurate to within 5%–10%, the anticipated accuracy of the present estimates should be comparable to that.

The frequencies and gravitational radiation damping times have been computed for the $2 \leq l = m \leq 5$ modes of fully relativistic neutron stars using a computer code that is described in Lindblom and Detweiler (1983) and Detweiler and Lindblom (1985). These frequencies and time scales are reported in Table 2 for neutron stars based on eight different models of the supernuclear density equation of state of nuclear matter. For each equation of state two different neutron stars are considered. The first corresponds to the model containing 1.4 $N_\odot$ baryons ($N_\odot = 1.19 \times 10^{57}$). This is the minimum mass neutron star that can be formed astrophysically. The other neutron star model listed in Table 2 is the maximum mass model for each equation of state. Since the equation of state is not well known in the density range needed for neutron stars, a number of different models of it are used to gain some feeling for the uncertainty in the frequencies caused by this ignorance. The equations of state are denoted by capital letters (A, B, C, etc.), and Table 3 defines each one in terms of a reference to the literature (see also Lindblom and Detweiler 1983). The frequencies reported in Table 2 are given as ratios with the Kelvin frequency $\sigma_m(0)$ (see eq. [2]) and the Detweiler time scale $\tau_{GR,m}$ (see eq. [5]) for a star having the same mass and radius. Presenting these data in this way gives an indication of the accuracy one can expect to achieve when performing "back of the envelope" calculations based on these simple frequency and time scale formulae.

TABLE 2

Frequencies of General Relativistic Neutron Star Models

Equation of State	$M/M_\odot$	R (km)	ω_2[a]	ω_3	ω_4	ω_5	χ_2[a]	χ_3	χ_4	χ_5
M	1.277	16.057	1.367	1.271	1.224	1.195	0.82	1.27	1.83	2.56
	1.759	11.903	1.339	1.167	1.095	1.055	1.86	5.32	12.9	30.0
L	1.311	14.944	1.292	1.212	1.175	1.152	0.99	1.58	2.34	3.68
	2.661	13.619	1.112	0.994	0.947	0.923	4.69	17.1	54.6	207.
N	1.385	13.784	1.252	1.171	1.136	1.115	1.18	2.04	3.26	5.14
	2.563	12.270	1.077	0.961	0.916	0.893	6.51	26.6	95.6	279.
O	1.282	12.798	1.260	1.177	1.140	1.120	1.17	2.01	3.20	5.47
	2.380	11.581	1.082	0.967	0.923	0.900	5.94	23.6	81.5	248.
C	1.317	12.027	1.317	1.205	1.154	1.124	1.16	2.23	3.90	6.64
	1.852	9.952	1.180	1.046	0.992	0.963	3.88	13.7	41.7	124.
F	1.262	10.325	1.306	1.183	1.129	1.097	1.35	2.87	5.50	9.81
	1.463	7.966	1.220	1.071	1.010	0.976	3.79	13.6	41.8	123.
A	1.246	9.783	1.251	1.147	1.103	1.078	1.53	3.24	6.20	11.5
	1.653	8.427	1.132	1.008	0.959	0.933	4.84	18.1	58.9	183.
B	1.223	8.209	1.259	1.134	1.080	1.049	1.95	5.03	11.6	25.9
	1.412	7.000	1.135	1.009	0.959	0.930	5.72	22.8	79.4	263.

[a] The frequencies Σ_m and gravitational radiation damping times T_m of these neutron stars are presented here as ratios with the Newtonian expressions for these quantities given in the text; thus $\omega_m = \Sigma_m/\sigma_m(0)$ and $\chi_m = T_m/\tau_{GR,m}$, where $\sigma_m(0)$ and $\tau_{GR,m}$ are the Kelvin frequency and the Detweiler time scale, which are defined in equations (2) and (5). Two stellar models are given for each equation of state: the first containing 1.4 $N_\odot$ baryons, and the other having the maximum possible mass for that equation of state.

TABLE 3

Equations of State

Equation of State	Reference
A	Pandharipande 1971 (neutron)
B	Pandharipande 1971 (hyperonic; model C)
C	Bethe and Johnson 1974 (model I)
F	Arponen 1972
L	Pandharipande, Pines, and Smith 1976 (mean field)
M	Pandharipande, Pines, and Smith 1976 (tensor)
N	Serot 1979*a*, *b*
O	Bowers, Gleeson, and Pedigo 1975

The effect of viscosity on the modes of general relativistic stars has not been computed to date. Such computations are currently in progress; however, the present analysis will be done using Lamb's formula (eq. [4]) for the viscous time scale. The viscosity ν of neutron star matter has been estimated by Friedman (1983) to lie in the range $1 < \nu < 100$ cm^2 s^{-1}. Using this range for the kinematic viscosity and the neutron star radii listed in Table 2, the range of viscous time scales have been computed for each $2 \leq m \leq 5$ mode. These time scale ranges are depicted graphically in Figure 4 for the 1.4 $N_\odot$ neutron star models. The ranges of the gravitational radiation time scales for these models from Table 2 are also depicted in this figure for comparison. This figure shows that the gravitational

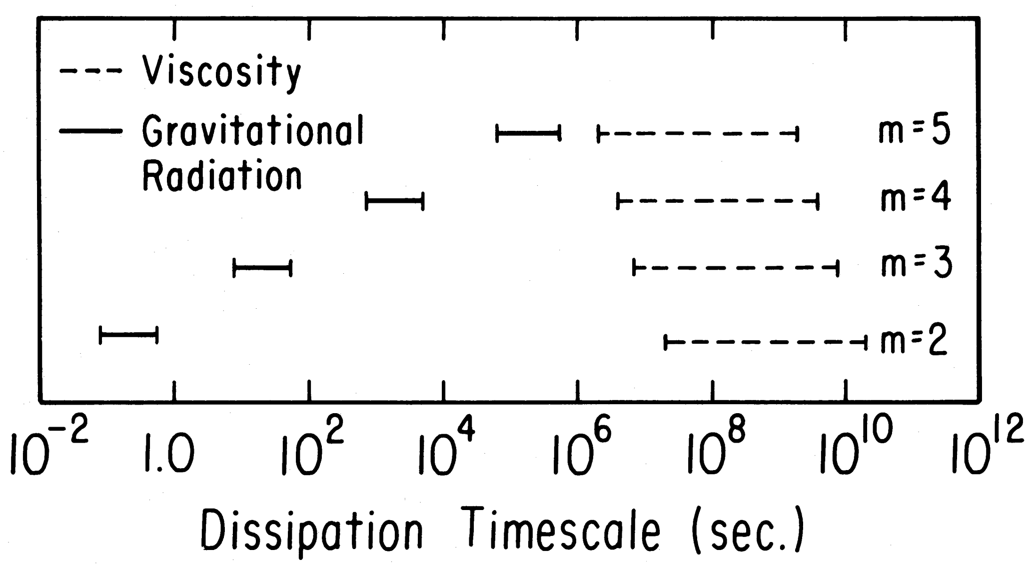

Fig. 4.—Dissipation time scales (viscous and gravitational radiation) for the 1.4 $N_\odot$ realistic neutron star models. The viscous time scales are based on the assumption that the kinematic viscosity for neutron star matter lies in the range $1 < \nu < 100$ cm^2 s^{-1}. The gravitational radiation time scales are based on a fully relativistic calculation of the nonradial modes of these stars.

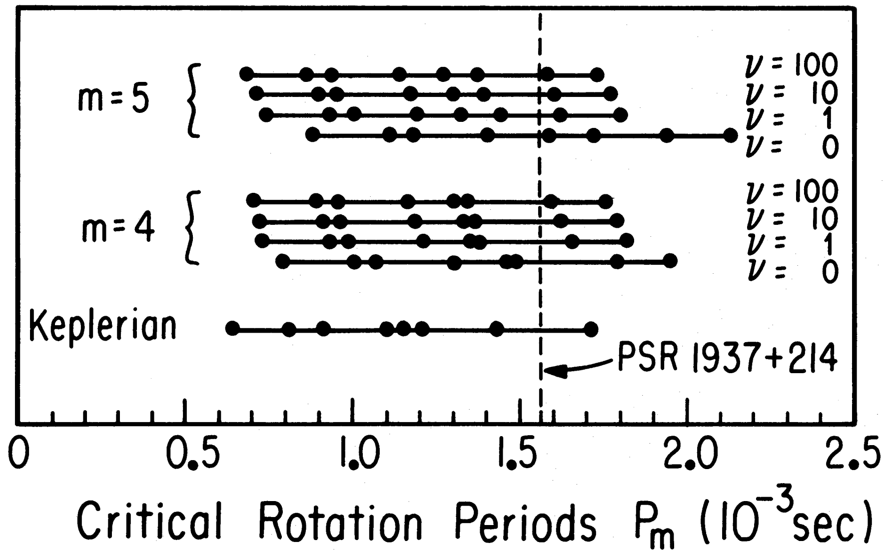

FIG. 5.—Critical rotation periods for the $m = 4$ and $m = 5$ modes of 1.4 $N_\odot$ realistic neutron star models. These critical rotation periods are the solutions of equation (14) using the dissipation time scales shown in Fig. 4. The critical periods have been determined for several values of the kinematic viscosity ν (cm^2 s^{-1}). For comparison, the period of the pulsar PSR 1937+214 and the periods of the "Keplerian" angular velocity for these stellar models have been included.

radiation time scales increase by about a factor of 100 with each increase in the value of m. This figure also shows that for small values of m ($m \leq 5$) the gravitational radiation time scale is shorter than the viscous time scale: $\tau_{GR,m} < \tau_{V,m}$. For larger values of m ($m > 6$) it is also clear that the viscous time scales will be shorter than the gravitational radiation time scales. The time scales for the maximum mass models are somewhat shorter (up to a factor of 5) than those shown in Figure 4, both for viscosity and for gravitational radiation.

Using these frequencies and time scales, the critical angular velocities for these relativistic neutron stars have been estimated by solving equation (14). This analysis indicates that the minimum critical angular velocity will occur in an $m = 4$ or an $m = 5$ mode for viscosities in the range $1 < \nu < 100$ cm^2 s^{-1}. The critical rotation periods $P_m = 2\pi/\Omega_m$ for these modes are depicted in Figure 5 for the 1.4 $N_\odot$ neutron stars having several different values of the viscosity. The circles in this figure represent the estimated critical periods for each of the equations of state used. Also included in this figure are the "Keplerian" periods for these same 1.4 $N_\odot$ stellar models as computed by Friedman, Ipser, and Parker (1985). Stars rotating faster than this "Keplerian" angular velocity are unstable to shedding mass from their equators. Consequently no equilibrium stellar models (even unstable ones) rotating faster than this limit are possible.

Figure 6 illustrates the variation in the critical angular velo-

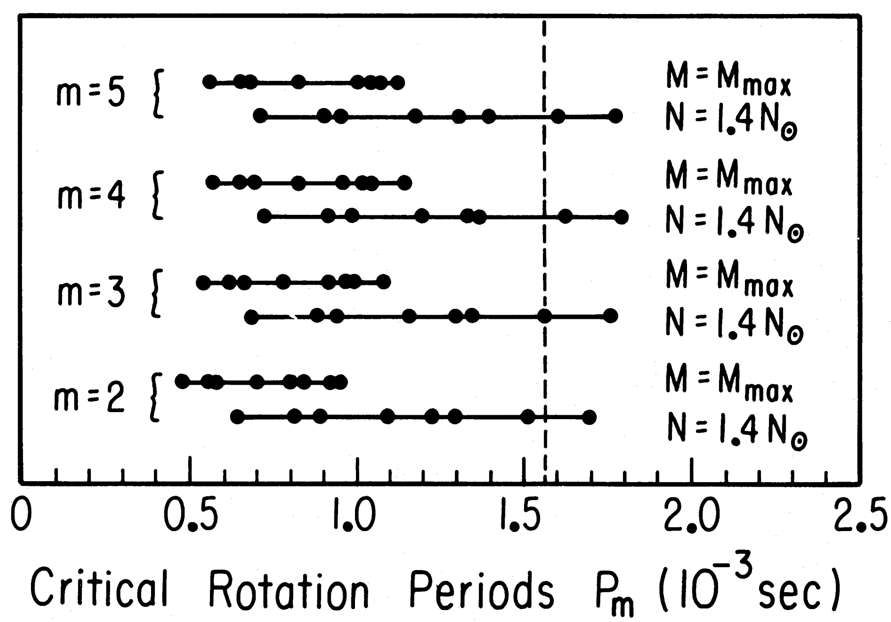

FIG. 6.—Critical rotation periods for the modes $2 \leq m \leq 5$ of realistic neutron stars. These critical periods were computed with the kinematic viscosity having the value $\nu = 10$ cm^2 s^{-1}. The critical periods are shown for two sets of neutron star models, one set consisting of stars having the maximum mass allowed for each equation of state, and the other set consisting of stars containing 1.4 $N_\odot$ baryons.

cities that can come about due to differences in the masses of neutron stars. This figure shows the critical rotation periods computed with a viscosity of $\nu = 10\ \mathrm{cm^2\ s^{-1}}$. For each of the modes, $2 \leq m \leq 5$, the critical periods are given for the maximum mass neutron stars in each equation of state, and for the models containing 1.4 $N_\odot$ baryons. This figure illustrates quite clearly that the critical rotation periods are maximum for the $m = 4$ or the $m = 5$ modes in all of these models.

Figures 5 and 6 illustrate the considerable uncertainty in the critical angular velocities that arises from the uncertainty in the supernuclear density equation of state. By comparison, the critical periods are remarkably insensitive to the value of the viscosity. (This excuses in part the use of the Lamb formula for the viscous time scale.) These periods change by only a few percent as the viscosity varies over the expected range $1 < \nu < 100\ \mathrm{cm^2\ s^{-1}}$. Figure 6 illustrates that the critical periods also depend sensitively on the masses of the neutron stars. Current observational evidence is consistent with the expectation that all neutron stars have masses near 1.4 $M_\odot$ (see, e.g., Shapiro and Teukolsky 1983). On this basis one might expect that the 1.4 $N_\odot$ models more accurately reflect the properties of real neutron stars than do the maximum mass models. However, since the masses of most neutron stars have not been determined to date, it is probably unsound to dismiss at this time the possibility of having larger mass stars.

Also included in Figures 5 and 6 is the pulsation period of the pulsar PSR 1937+214 for comparison. This calculation indicates that the existence of this pulsar is inconsistent with the stiffer equations of state (M and L) and the assumptions (*a*) that the mass of the pulsar is near 1.4 $N_\odot$ and (*b*) that the viscosity is in the range $1 < \nu < 100\ \mathrm{cm^2\ s^{-1}}$. These calculations also indicate that no pulsar having a period shorter than ~0.6 ms would be consistent with any of the equations of state in this sample.

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