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ON THE NONRADIAL PULSATIONS OF GENERAL RELATIVISTIC STELLAR MODELS[1]

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ABSTRACT

We present a nonsingular fourth-order system of equations which describe the nonradial pulsations of general relativistic stellar models. These equations represent an improvement over previous discussions in the literature which presented fifth-order systems or singular fourth-order systems to describe these pulsations. We also present the nonsingular power-series solutions to these equations near $r = 0$.

Subject headings: relativity — stars: pulsation

I. INTRODUCTION

The equations which describe the nonradial pulsations of general relativistic stellar models were first studied by Thorne and Campolattaro (1967*a*, *b*). They showed that Einstein's equations describing small, nonradial, quasi-periodic oscillations of general relativistic stellar models could be reduced to a fifth-order system of ordinary differential equations. While it was realized (see Ipser and Thorne 1973) that the true dynamics of a pulsating star must be governed by a fourth-order system of equations, Thorne and his coworkers did not explicitly reduce their equations to fourth order. We (Lindblom and Detweiler 1983) successfully reduced Thorne's equations to a fourth-order system and numerically integrated the equations to study the quadrupole oscillations of neutron stars. Since then, we have noticed that our fourth-order system of equations sometimes become singular inside the star when the pulsation frequency lies in a certain range. While the system of equations did not exhibit this singularity in over 90% of the cases studied earlier, and while the effect on the computed frequencies and gravitational radiation damping times was small even on those models where the system of equations became singular, we feel that it is worthwhile to show how the equations could be written in a nonsingular form. We also discuss the nonsingular power-series solutions to these equations near the center of the star ($r = 0$). (The power-series solutions are useful to anyone wishing to solve the equations numerically.)

A new computer code, which we have written, solves the nonsingular system of equations. On the basis of our new results we conclude that the frequencies and damping times for the quadrupole oscillations of the neutron star models published by us earlier were not severely affected by this singularity. The singularity appears to have the most effect on the computed frequencies when $\omega^2 \gtrsim \frac{1}{2}l(l+1)GMR^{-3}$. For the *f*-modes which we examined, this condition holds only for the most nonrelativistic of the models considered, with mass always less than 0.6 $M_\odot$ (typically less than 0.3 $M_\odot$). The largest errors that we found in the low-mass models satisfying $\omega^2 > \frac{1}{2}l(l+1)GMR^{-3}$ were 0.1% in the period of oscillation and 11% in the gravitational wave damping time. The errors in the larger mass models [with $\omega^2 < \frac{1}{2}l(l+1)GMR^{-3}$] was 3% in the gravitational wave damping time. The above condition, of course, always holds for some of the higher frequency *p*-modes which we did not examine.

The stellar models which we consider allow only for a barytropic equation of state, which limits us to zero-temperature, fluid models. And we have only examined the *p*- and *f*-modes of oscillation, which are those modes which couple most strongly to gravitational radiation. The *g*-modes, nonadiabatic effects, and torsional oscillations have not been considered. This is *not* because these latter effects are not important in their own right, but rather because their weak coupling to gravitational radiation generally requires a modified approach rather different from our own (cf. McDermott, Van Horn, and Scholl 1983).

II. THE NONSINGULAR SYSTEM OF EQUATIONS

We adopt the notation of Lindblom and Detweiler (1983) and describe the perturbed metric tensor of a general relativistic stellar model as

$$ds^2 = -e^{\nu}(1 + r^l H_0 Y^l_m e^{i\omega t})dt^2 - 2i\omega r^{l+1} H_1 Y^l_m e^{i\omega t} dt dr + e^{\lambda}(1 - r^l H_0 Y^l_m e^{i\omega t})dr^2 + r^2(1 - r^l K Y^l_m e^{i\omega t})(d\theta^2 + \sin^2\theta d\phi^2) . \quad (1)$$

The perturbation of the fluid in the star is described by the Lagrangian displacement vector ξ^a, having components

$$\xi^r = r^{l-1} e^{-\lambda/2} W Y^l_m e^{i\omega t} , \quad (2)$$

$$\xi^\theta = -r^{l-2} V \partial_\theta Y^l_m e^{i\omega t} , \quad (3)$$

$$\xi^\phi = -r^l (r \sin\theta)^{-2} V \partial_\phi Y^l_m e^{i\omega t} . \quad (4)$$

The metric perturbation functions H_0, H_1, and K, as well as the fluid perturbation functions V and W, are essentially the same (up to factors of i and r^l) as those used by Thorne and Campolattaro (1967*a*, *b*) and by Detweiler and Ipser (1973) in earlier studies of

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stellar pulsations. The functions e^{ν} and e^{λ} are the components of the metric of the unperturbed stellar model, while Y^l_m are the usual spherical harmonics.

The five perturbation function H_0, H_1, K, V, and W are not all independent, as Thorne and Campolattaro (1967*a*, *b*) showed. In fact, these functions must satisfy the following relationship as a consequence of Einstein's equation:

$$[3M + \tfrac{1}{2}(l+2)(l-1)r + 4\pi r^3 p]H_0 = 8\pi r^3 e^{-\nu/2}X - [\tfrac{1}{2}l(l+1)(M + 4\pi r^3 p) - \omega^2 r^3 e^{-(\lambda+\nu)}]H_1 + [\tfrac{1}{2}(l+2)(l-1)r - \omega^2 r^3 e^{-\nu} - r^{-1}e^{\lambda}(M + 4\pi r^3 p)(3M - r + 4\pi r^3 p)]K \,. \tag{5}$$

In this equation the perturbation function X is defined by

$$X = \omega^3(\rho + p)e^{-\nu/2}V - r^{-1}p' e^{(\nu-\lambda)/2}W + \tfrac{1}{2}(\rho + p)e^{\nu/2}H_0 \,, \tag{6}$$

where p and ρ are the pressure and density of the unperturbed stellar model, a prime denotes a derivative with respect to r, and $M = \frac{1}{2}r(1 - e^{-\lambda})$. The vacuum limit of this equation was given by Edelstein and Vishveshwara (1970). Because of this constraint, equation (5) (obtained by eliminating K' from eq. [A15] of Lindblom and Detweiler 1983), it is possible to completely eliminate one of the perturbation functions from the problem. Thorne and Campolattaro (1967*a*, *b*) eliminate H_1 from the problem using this constraint and write the resulting Einstein equations as a fifth-order system (for K, K', H_0, V, and W). Lindblom and Detweiler (1983) also use this constraint to eliminate H_1 from the equations and derive a fourth-order system (for K, H_0, X, and W). This is clearly the wrong thing to do! The coefficient of H_1 in equation (5) is not of definite sign. This coefficient can vanish for a given value of r whenever the frequency takes on the value

$$\omega^2 = \frac{1}{2}l(l+1)e^{\lambda+\nu}(r^{-3}M + 4\pi p) = \frac{1}{4}l(l+1)\frac{1}{r}\frac{d}{dr}(e^{\nu}) \,. \tag{7}$$

Consequently, if one chooses to express the perturbations in terms of K, H_0, X (or V), and W, then the system of equations will necessarily be singular whenever the frequency has a value such that equation (7) is satisfied somewhere inside the star. And this will always be true for some of the higher frequency *p*-modes.

Fortunately, equation (5) not only makes the difficulty obvious, but it makes the solution obvious as well. One should use equation (5) to eliminate one of the variables whose coefficient does not vanish. We chose to eliminate H_0, and to write the equations for the perturbations in terms of H_1, K, W, and X. Some obvious, but tedious, mathematical manipulation of equations (A14)–(A19) of Lindblom and Detweiler (1983) yield the following surprisingly simple set of equations for H_1, K, W, and X:

$$H_1' = -r^{-1}[l + 1 + 2Me^{\lambda}r^{-1} + 4\pi r^2 e^{\lambda}(p - \rho)]H_1 + r^{-1}e^{\lambda}[H_0 + K - 16\pi(\rho + p)V] \,, \tag{8}$$

$$K' = r^{-1}H_0 + \tfrac{1}{2}l(l+1)r^{-1}H_1 - [(l+1)r^{-1} - \tfrac{1}{2}\nu']K - 8\pi(\rho + p)e^{\lambda/2}r^{-1}W \,, \tag{9}$$

$$W' = -(l+1)r^{-1}W + re^{\lambda/2}[\gamma^{-1}p^{-1}e^{-\nu/2}X - l(l+1)r^{-2}V + \tfrac{1}{2}H_0 + K] \,, \tag{10}$$

$$X' = -lr^{-1}X + (\rho + p)e^{\nu/2}\{\tfrac{1}{2}(r^{-1} - \tfrac{1}{2}\nu')H_0 + \tfrac{1}{2}[r\omega^2 e^{-\nu} + \tfrac{1}{2}l(l+1)r^{-1}]H_1 + \tfrac{1}{2}(\tfrac{3}{2}\nu' - r^{-1})K - \tfrac{1}{2}l(l+1)\nu' r^{-2}V - r^{-1}[4\pi(\rho + p)e^{\lambda/2} + \omega^2 e^{\lambda/2-\nu} - \tfrac{1}{2}r^2(r^{-2}e^{-\lambda/2}\nu')']W\} \,. \tag{11}$$

In these equations γ is the adiabatic index, and V is to be thought of as the linear combination of H_1, K, W, and X obtained by eliminating H_0 between equations (5) and (6). This system of equations is manifestly nonsingular except near the center of the star, $r = 0$.

Another attractive feature of the variables H_1 and K (compared with H_0 and K) is their relationship with the purely gravitational perturbations in the exterior of the star. The transformation between H_1, K, and the Zerilli function is nonsingular (see Fackerell 1971), while the corresponding transformation involving H_0 and K may not be.

III. THE NONSINGULAR SOLUTIONS NEAR $r = 0$

While we have eliminated the singularity which occurs whenever equation (7) is satisfied, the system of equations is still singular near the center of the star at $r = 0$. To extract the physical nonsingular solutions from these equations, it is helpful to have the power-series solutions near $r = 0$. We seek solutions to the perturbation equations (8)–(11) which are of the form

$$H_1(r) = H_1(0) + \tfrac{1}{2}r^2 H_1''(0) + O(r^4) \,, \tag{12}$$

$$K(r) = K(0) + \tfrac{1}{2}r^2 K''(0) + O(r^4) \,, \tag{13}$$

$$W(r) = W(0) + \tfrac{1}{2}r^2 W''(0) + O(r^4) \,, \tag{14}$$

$$X(r) = X(0) + \tfrac{1}{2}r^2 X''(0) + O(r^4) \,. \tag{15}$$

The first-order constraints imposed on these functions by the perturbation equations are

$$X(0) = (\rho_0 + p_0)e^{\nu_0/2}\left\{\left[\frac{4\pi}{3}(\rho_0 + 3p_0) - \omega^2 e^{-\nu_0}/l\right]W(0) + \frac{1}{2}K(0)\right\} \,, \tag{16}$$

$$H_1(0) = \{2lK(0) + 16\pi(\rho_0 + p_0)W(0)\}/l(l+1) \,. \tag{17}$$

The constants ρ_0, p_0, and ν_0 which appear in these expressions are simply the first terms in the power-series expansions for the density, pressure, and gravitational potential:

$$\rho(r) = \rho_0 + \tfrac{1}{2}\rho_2 r^2 + O(r^4)\,, \tag{18}$$

$$p(r) = p_0 + \tfrac{1}{2}p_2 r^2 + \tfrac{1}{4}p_4 r^4 + O(r^6)\,, \tag{19}$$

$$\nu(r) = \nu_0 + \tfrac{1}{2}\nu_2 r^2 + \tfrac{1}{4}\nu_4 r^4 + O(r^6)\,. \tag{20}$$

The constants ρ_2, p_2, ν_2, p_4, and ν_4 in these expressions are related to the central values of the density and pressure (ρ_0 and p_0) by the structure equations for the unperturbed star. These relationships have been given in equations (B8)–(B12) of Lindblom and Detweiler (1983) and will not be repeated here. We note that equations (16) and (17) limit the number of linearly independent nonsingular solutions to two. We find that reasonably linearly independent solutions can be generated by taking $W(0) = 1$ and $K(0) = \pm(\rho_0 + p_0)$.

To find the second-order terms in the expansions [$H_1''(0)$, $K''(0)$, etc.] is somewhat more complicated. When the perturbation equations are evaluated to the second order, one obtains the following four constraints among the second-order coefficients:

$$\begin{aligned}
-\frac{1}{4}(\rho_0+p_0)K''(0) &+ \frac{1}{2}\left[p_2+(\rho_0+p_0)\frac{\omega^2(l+3)}{l(l+1)}e^{-\nu_0}\right]W''(0)+\frac{1}{2}e^{-\nu_0/2}X''(0)\\
&= \frac{1}{4}\nu_2 e^{-\nu_0/2}X(0)+\frac{1}{4}(\rho_2+p_2)K(0)+\frac{1}{4}(\rho_0+p_0)Q_0+\frac{1}{2}\omega^2(\rho_0+p_0)e^{-\nu_0}Q_1\\
&\quad -\left\{p_4-\frac{4\pi}{3}\rho_0 p_2+\frac{\omega^2}{2l}\left[\rho_2+p_2-(\rho_0+p_0)\nu_2\right]e^{-\nu_0}\right\}W(0)\,,
\end{aligned}\tag{21}$$

$$\frac{1}{2}(l+2)K''(0)-\frac{1}{4}l(l+1)H_1''(0)+4\pi(\rho_0+p_0)W''(0)=\frac{4\pi}{3}(\rho_0+3p_0)K(0)+\frac{1}{2}Q_0-4\pi\left[\rho_2+p_2+\frac{8\pi}{3}\rho_0(\rho_0+p_0)\right]W(0)\,, \tag{22}$$

$$\begin{aligned}
\frac{1}{2}(l+3)H_1''(0)&-K''(0)-8\pi(\rho_0+p_0)\frac{l+3}{l(l+1)}W''(0)\\
&= 4\pi\left[\frac{1}{3}(2l+3)\rho_0-p_0\right]H_1(0)+\frac{8\pi}{l}(\rho_2+p_2)W(0)-8\pi(\rho_0+p_0)Q_1+\frac{1}{2}Q_0\,,
\end{aligned}\tag{23}$$

$$\begin{aligned}
\frac{1}{2}(l+2)X''(0)&-\frac{1}{8}l(l+1)(\rho_0+p_0)e^{\nu_0/2}H_1''(0)-(\rho_0+p_0)e^{\nu_0/2}\left[\frac{1}{4}(l+2)\nu_2-2\pi(\rho_0+p_0)-\frac{1}{2}\omega^2e^{-\nu_0}\right]W''(0)\\
&= \frac{1}{2}\left[\rho_2+p_2+\frac{1}{2}(\rho_0+p_0)\nu_2\right]\frac{lX(0)}{\rho_0+p_0}+(\rho_0+p_0)e^{\nu_0/2}\left\{\frac{1}{2}\nu_2K(0)+\frac{1}{4}Q_0+\frac{1}{2}\omega^2e^{-\nu_0}H_1(0)-\frac{1}{4}l(l+1)\nu_2Q_1\right.\\
&\quad\left.+\left[\frac{1}{2}(l+1)\nu_4-2\pi(\rho_2+p_2)-\frac{16\pi^2}{3}\rho_0(\rho_0+p_0)+\frac{1}{2}\left(\nu_4-\frac{4\pi}{3}\rho_0\nu_2\right)+\frac{1}{2}\omega^2e^{-\nu_0}\left(\nu_2-\frac{8\pi}{3}\rho_0\right)\right]W(0)\right\}.
\end{aligned}\tag{24}$$

In these equations Q_0 and Q_1 stand for the following combinations of the first-order coefficients:

$$Q_0=\frac{4}{(l+2)(l-1)}\left\{8\pi e^{-\nu_0/2}X(0)-\left(\frac{8\pi}{3}\rho_0+\omega^2e^{-\nu_0}\right)K(0)-\left[\frac{2\pi}{3}l(l+1)(\rho_0+3p_0)-\omega^2e^{-\nu_0}\right]H_1(0)\right\}, \tag{25}$$

$$Q_1=\frac{2}{l(l+1)}\left[\frac{X(0)}{\gamma_0 p_0}e^{-\nu_0/2}+\frac{3}{2}K(0)+\frac{4\pi}{3}(l+1)\rho_0W(0)\right]. \tag{26}$$

While these expressions are messy, they are valuable and reasonably easy to use for numerical computations. Equations (21)–(24) have the following simple form:

$$\mathbf{T}Y''(0)=\mathbf{U}Y(0)\,, \tag{27}$$

where $Y(0)=[H_1(0), K(0), W(0), X(0)]$ is a vector representing the first-order perturbation coefficients; $Y''(0)$ is the second-order counterpart; and $\mathbf{T}$ and $\mathbf{U}$ are matrices which depend only on the constants ρ_0, p_0, l, ω, etc. These matrices can be evaluated numerically, and inverted to give values for the second-order coefficients according to the formula

$$Y''(0)=\mathbf{T}^{-1}\mathbf{U}Y(0)\,. \tag{28}$$

We find that these expansions are adequate to evaluate the solutions in a neighborhood of the center of the star large enough to allow us to continue the integration using traditional numerical methods.

IV. DISCUSSION

Having discovered the singularity in the system of stellar perturbation equations of Lindblom and Detweiler (1983), and having devised a means of eliminating the singularity by a proper choice of variables, we now wish to determine the extent to which the existence of the singularity affected the numerical oscillation frequencies published earlier. We note that a singularity occurred in our previous system only when the oscillation frequency had a value such that equation (7) is satisfied somewhere inside the star. The expression on the right-hand side of equation (7) is a decreasing function near the surface of the star, $r = R$, which approaches the value $\frac{1}{2}l(l + 1)MR^{-3}$ at the surface of the star. Consequently one would not expect a singularity to occur within the star if the frequency satisfies the inequality

$$\omega^2 < \tfrac{1}{2}l(l + 1)MR^{-3} . \tag{29}$$

For those models which we studied earlier this expression is satisfied if and only if the equations are singularity-free. In fact, over 90% of the stellar models had no problems with this singularity. Problems arose only for the relatively uninteresting, extremely Newtonian models (with masses generally less than 0.3 $M_\odot$).

To investigate further the reliability of those previously published results, we have written a new computer code which evaluates the eigenfrequencies of the quasi-normal modes of general relativistic stellar models by integrating the nonsingular equations (8)–(11). This code follows the algorithm detailed in Appendix A of Lindblom and Detweiler (1983) except for its use of the nonsingular equations. Using this new code, we have reevaluated the oscillation frequencies for a sampling of those stellar models previously published, including all those models which violate inequality (29). For those models which satisfy inequality (29) the largest errors observed in the sample of models examined were 0.3% in the period of oscillation and 3% in the gravitational radiation damping time. Most models had errors smaller than these extreme values by a factor of 10. For those models which violated inequality (29), 0.1% was the largest error in the oscillation period and 11% the largest error observed in a gravitational radiation damping time.

We use this opportunity to point out three typographical errors in the equations of Lindblom and Detweiler (1983). The right-hand side of equation (A17) should contain the additional term $-lr^{-1}X$. The term nr on the right-hand side of equation (A29) should be replaced by $nr(r - 2M)$. And equation (B15) is seriously missprinted but can be replaced by a linear combination of equations (21)–(24) in this paper. Finally, we had an error in the table for equation of state F. This led to substantial errors for these models with mass less than 1 $M_\odot$, but errors of only 1%, for all computed quantities, when the mass was above 1 $M_\odot$.

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