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The accuracy of the quadrupole approximation for the gravitational radiation from pulsating stars

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Summary. Two methods for computing the damping of the non-radial oscillations of stars due to gravitational radiation emission are compared: (i) solving the complete linearized equations of general relativity and (ii) using an approximate method based on the quadrupole formula. It is shown that the results of the fully relativistic calculation approach the results based on the quadrupole formula for sufficiently non-relativistic stars (i.e. GM/c^2R sufficiently small). The accuracy of the approximation method is investigated by considering polytropic stars with a range of polytropic indices. The approximation is worse for larger polytropic indices, i.e. for greater central condensation. We conclude that when Newtonian models are used to approximate relativistic ones, models with the same value of GM/c^2R (rather than, say, central density) should be compared.

1 Introduction

In the past decade there has been considerable interest in the question of whether the so-called quadrupole formula for gravitational radiation is really a valid approximation to general relativity in the Newtonian limit. Now that (i) a consensus among theoreticians is emerging in support of the standard quadrupole approximation (described in more detail below), and (ii) its predictions have proved to be consistent with observations of the binary pulsar PSR 1913+16 (Weisberg & Taylor 1984), we ask in this paper the natural next question: *how good* is the approximation? That is, how relativistic does a system have to be before the quadrupole approximation is in error by, say, 50 per cent in its prediction of radiation? This question is of crucial importance to some astrophysical calculations, yet it has so far received almost no attention in the literature. We study here the case of the radiation emitted by non-radially pulsating stars. We find that, although the quadrupole formula is strikingly good in some respects, its accuracy depends on the equation of state and on ambiguities in the manner in which a Newtonian approximation is defined for a relativistic system. As a rough guide, it seems to be accurate to within 50 per cent for stars with surface redshifts of less than 5 per cent.

There are really two quadrupole formulas. One, for the radiation given off by the source, is called the Landau–Lifshitz formula (Landau & Lifshitz 1975). It is an extension to self-gravitating Newtonian systems of the slow-motion approximation of linearized theory (see Misner, Thorne & Wheeler 1973), which is that the luminosity in gravitational waves is

$$L_{\mathrm{GW}}=\frac{1}{5}\,\frac{G}{c^5}\left\langle\sum_{jk}|\dddot{\bar{I}}_{jk}|^2\right\rangle, \tag{1}$$

where we define

$$\bar{I}_{jk}=I_{jk}-\tfrac{1}{3}\delta_{jk}I_{ll}$$

$$I_{jk}(t)=\int\varrho(t,y)y_iy_j\,d^3y,$$

and where angle brackets represent an average over one period of the (assumed nearly periodic) Newtonian system. The other quadrupole formula describes radiation-reaction effects in the emitting body itself. First systematically derived within the post-Newtonian approximation by Chandrasekhar & Esposito (1970), it may be most compactly described by the reaction potential (Thorne 1969b; Burke 1971)

$$\Phi_{\mathrm{react}}=\frac{G}{5c^5}x_ix_j\,d^5\bar{I}_{ij}/dt^5, \tag{2}$$

where we sum on repeated indices. When this term is added to the usual Newtonian potential, it produces a force which dissipates over one period the same amount of energy as turns up in radiation, equation (1). We will use equation (2) in this paper.

The early derivations of the quadrupole formulas quoted above were much criticized for their lack of mathematical rigour (Ehlers *et al.* 1976). Further work (e.g. Anderson & DeCanio 1975; Kerlick 1980a, b) led to a deeper understanding of the approximation scheme they were derived from (Walker & Will 1980). Although some calculations of specific systems, for example Cooperstock (1982) and Rosenblum (1983), have been troubled by differences in interpretation, a number of recent derivations of the formulae, taking very different viewpoints, have been more satisfactory (Futamase 1983; Anderson *et al.* 1982; Damour 1983), so that now there can be little doubt that both equations (1) and (2) represent asymptotic approximations to general relativity in the Newtonian limit.

As with any asymptotic approximation, the usefulness depends on the size of the error terms. These may be estimated as follows. In equation (1) we may estimate that a time-derivative will produce a frequency of order $(G\varrho)^{1/2}\sim(GMR^{-3})^{1/2}$, where ϱ is a typical density, M the system's mass, and R its typical size. Then it is not hard to show that

$$L_{\mathrm{GW}}=\alpha\frac{c^5}{G}\left(\frac{GM}{c^2R}\right)^5, \tag{3}$$

where α is a dimensionless number which ought to be of order one if the system is executing large-amplitude motion. We can see from equation (3) that if we approximate a relativistic system by a Newtonian one of, say, the same mass M but whose radius is different by δR (we generally cannot match them both), then the luminosity might have a relative error of $5\,\delta R/R$. We will see that errors of this type do seem to set the accuracy of the approximation based on equation (3).

Rough calculations of this type must be supplemented by detailed calculations of some examples before we can have confidence in using equations (1) and (2). The system we treat here, the non-radial normal modes of a spherical star, provides a clear straightforward test of the accuracy of the quadrupole approximation. This problem is solvable in general relativity because of the approximation that the amplitude of pulsation is small. This produces linear equations (Thorne & Campolattero 1967) which fully incorporate all the effects of gravitational radiation. The normal modes have complex eigenfrequencies whose (generally small) imaginary parts damp the pulsation as the waves carry energy away. For stars that are not very relativistic one would expect that the real parts of the eigenfrequencies would be close to the (real) eigenfrequencies that one would calculate for the normal modes of a Newtonian star of similar mass and radius. Then by treating equation (2) as a perturbation of the Newtonian operator whose eigenvalues are the Newtonian eigenfrequencies, one can calculate by standard perturbation theory the changes in the eigenfrequencies due to radiation reaction. This is our measure of the accuracy of the quadrupole approximation: how close does the imaginary part of the perturbed Newtonian eigenfrequency come to the exact relativistic one?

An earlier comparison by Balbinski & Schutz (1982) using the relativistic normal modes calculated by Thorne (1969a) and Detweiler (1973) revealed a striking disparity: the radiation-reaction damping took place three times faster than the relativistic calculations suggested for a star with a surface redshift of only 3 per cent. But the relativistic calculations used for this comparison were troubled by numerical difficulties in the low-redshift case (where the imaginary part of the eigenfrequency can be less than 10^{-3} times the real part), and more recent calculations by Lindblom & Detweiler (1983) reduced but did not remove the discrepancy. In this paper we report new calculations specifically designed to facilitate a comparison of the two approaches. We eliminate unwanted equation-of-state effects by studying *polytropes*,

$$p = K\varrho^{1+1/n}, \tag{4}$$

where in the relativistic case we take ϱ to be the total mass-energy density. In Section 2 we outline our methods of finding the eigenfrequencies of pulsation for both Newtonian and relativistic models and in Section 3 we present our results and draw our conclusions.

2 Description of the method

To assess the accuracy of the quadrupole formula approximation, two distinct calculations are necessary. In one calculation (performed by E. Balbinski and B. Schutz) the quadrupole formula is used along with the equations for the structure and pulsations of Newtonian stellar models to estimate the rate that gravitational radiation damps stellar oscillations. In the other calculation (performed by S. Detweiler and L. Lindblom) the linearized Einstein equations are solved to find the general relativistic prediction for the damping of stellar oscillations. The techniques used to perform each of these calculations have been described in detail elsewhere; see Balbinski (1982), Balbinski & Schutz (1982), Lindblom & Detweiler (1983) and Detweiler & Lindblom (1985). We merely summarize those calculations here briefly.

2.1 THE QUADRUPOLE FORMULA APPROXIMATION

The quadrupole formula states that a nearly Newtonian system radiates energy in gravitational radiation according to equation (1). The damping time, τ, is the inverse of the imaginary part of the eigenfrequency and is related to the rate with which the system loses energy,

$$\tau = -\frac{1}{2}\left[\frac{1}{E}\frac{dE}{dt}\right]^{-1} = \frac{5}{2}\frac{c^5}{G}E\Big/\left\langle \sum_{jk} |\dddot{I}_{jk}|^2 \right\rangle. \tag{5}$$

The right-hand side of the expression can be evaluated in a straightforward manner when the system is a pulsating stellar model. The equations describing the periodic oscillations of a Newtonian stellar model are solved (numerically) to find the eigenfrequencies and eigenfunctions describing the motion of the fluid in the fundamental quadrupole (f) mode. These frequencies and eigenfunctions are then inserted into the integrals for the energy E and quadrupole moment tensor I_{jk} to evaluate the right-hand side of equation (5). For a star undergoing sinusoidal oscillations the characteristic damping time τ defined in this way is independent of time. The details of this computation are described in Balbinski (1982) and Balbinski & Schutz (1982).

2.2 GENERAL RELATIVISTIC CALCULATION

To find the general relativistic value of the damping time τ, defined above, one looks for solutions to the equations of general relativity which describe small-amplitude oscillations of a general relativistic stellar model. The normal mode solutions to these equations [with time-dependence $\exp(i\omega t)$] are not strictly periodic. The eigenfrequencies, ω, for this system will be complex numbers when outgoing gravitational wave-boundary conditions are imposed at infinity. The real part of ω describes the pulsation frequency of the star, while the imaginary part of ω describes the characteristic time, $\tau=1/\mathrm{Im}(\omega)$ at which the pulsations are damped due to the emission of gravitational radiation. To find τ, in the general relativistic context, reduces then to the solution of the eigenvalue problem for the fourth-order system of equations which describe the pulsations of general relativistic stars. The details of solving this eigenvalue problem are described in Lindblom & Detweiler (1983) and Detweiler & Lindblom (1985).

3 Results

It is a fairly straightforward numerical exercise to solve for the pulsation frequency ω and the gravitational radiation damping time τ for the quadrupole oscillation mode of a given (Newtonian or relativistic) stellar model following the outline described above. We have chosen to perform these calculations for stellar models constructed from the simple polytropic equations of state (4).

In Newtonian theory, the equations describing the structure and oscillations of stars with a polytropic equation of state have a scale invariance which allows one to construct the following dimensionless frequency and damping time, which depend only on the polytropic index n,

$$\omega=c_\omega(n)\left(\frac{GM}{R^3}\right)^{1/2}, \tag{6}$$

$$\tau=c_\tau(n)\,\frac{R}{c}\left(\frac{GM}{c^2R}\right)^{-3}. \tag{7}$$

In these expressions M and R are the total mass and radius of the star, while G and c are Newton's constant and the speed of light. The coefficients c_ω and c_τ depend only on the polytropic index n. Therefore, once the coefficients c_ω and c_τ are known for different polytropic indices, equations (6) and (7) allow one to compute the frequencies and damping times for all possible Newtonian polytropic stars. We have determined these coefficients for a range of polytropic indices $0.5\leqslant n\leqslant 2.5$ and the results are summarized in Table 1.

The general relativistic equations for the structure and pulsations of stellar models are not invariant under the scalings found in the Newtonian theory. Consequently c_ω and c_τ as defined by equations (6) and (7) are not constants for a given n in general relativity but functions of K and ϱ_c.

Table 1. Frequencies and damping times for Newtonian polytropes based on the quadrupole formula.

n	c_ω	c_τ
0.50	1.042	13.02
0.75	1.127	10.60
1.00	1.227	8.46
1.25	1.332	6.87
1.50	1.457	5.54
1.75	1.587	4.51
2.00	1.743	3.72
2.25	1.899	3.20
2.50	2.069	2.91

Table 2. Properties of general relativistic $n=1$ polytropes.

Central Density (10^{15} gm/cm^3)	$M/M_\odot$	GM/c^2R	$\omega\left[\frac{R^3}{GM}\right]^{1/2}$	$\tau\,\frac{c}{R}\left[\frac{GM}{c^2R}\right]^3$
3.00	1.266	0.211	1.160	34.43
2.00	1.126	0.172	1.187	24.31
1.00	0.802	0.109	1.214	15.76
0.70	0.635	0.0833	1.220	13.49
0.50	0.495	0.0631	1.224	12.04
0.30	0.326	0.0403	1.226	10.62
0.20	0.228	0.0278	1.227	9.92
0.10	0.1198	0.01435	1.227	9.20
0.07	0.0851	0.01015	1.227	8.97
0.05	0.0614	0.00730	1.227	8.80

We have chosen to examine in detail a particular $n=1$ equation of state in the relativistic context. We have taken the constant K to have the value 100 when Gp/c^4 and $G\varrho/c^2$ are measured in units of $1/\text{km}^2$:

$$Gp/c^4=100\,(G\varrho/c^2)^2.$$

(This choice of K gives stellar models which are similar to realistic neutron stars.) Table 2 summarizes our calculations on this equation of state. From this table it is clear that the relativistic models have oscillation frequencies which approach the Newtonian values for stars having small

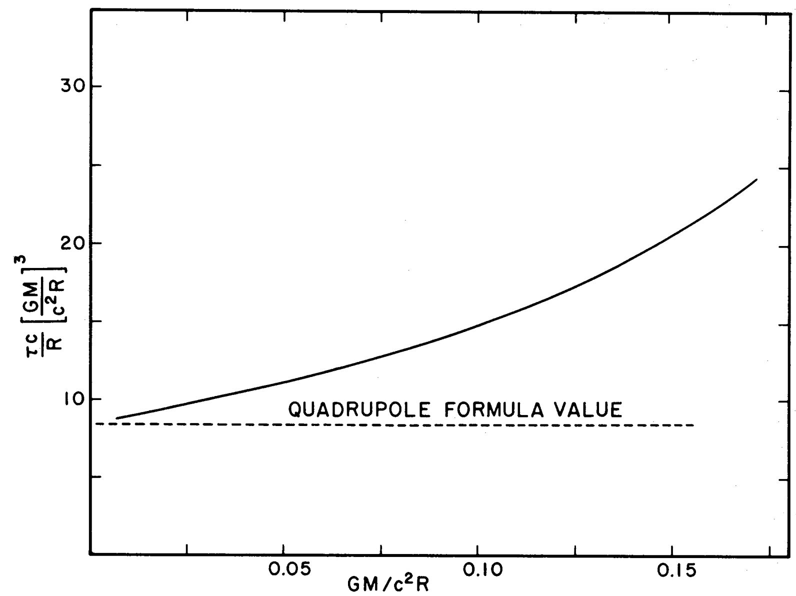

Figure 1. Gravitational radiation damping times for relativistic $n=1$ polytropes.

Table 3. Properties of general relativistic polytropes having $M=1.0\,M_\odot$ and $GM/c^2R=0.03$.

Polytropic Index n	$\omega\left[\frac{R^3}{GM}\right]^{1/2}$	$\tau\,\frac{c}{R}\left[\frac{GM}{c^2R}\right]^3$
0.50	1.044	15.04
0.75	1.131	12.03
1.00	1.227	10.04
1.25	1.334	8.20
1.50	1.453	6.73
1.75	1.587	5.59
2.00	1.734	4.75
2.25	1.894	4.22
2.50	2.062	4.07

GM/c^2R: i.e. $\omega^2[GM/R^3]^{-1/2}\rightarrow c_\omega(1)$. The gravitational radiation damping times τ approach the Newtonian value for stars having small GM/c^2R as well. This fact is illustrated in Fig. 1. This calculation confirms the validity of the quadrupole formula for estimating the gravitational radiation damping time in sufficiently Newtonian (i.e. sufficiently small GM/c^2R) stellar models.

We have also investigated the dependence of the accuracy of the quadrupole formula estimate on the internal structure of the star. We have computed a series of relativistic stellar models, each of which has $M=1.0\,M_\odot$ and $GM/c^2R=0.03$, but which have differing polytropic index n. The central densities and polytropic constants K in the models were adjusted until the masses and radii had the prescribed values to better than one part in 10^6. These models have the same overall compactness but have rather different variations in their density. The results of these calculations are presented in Table 3 and Fig. 2. For these models we see that the Newtonian constant $c_\omega(n)$

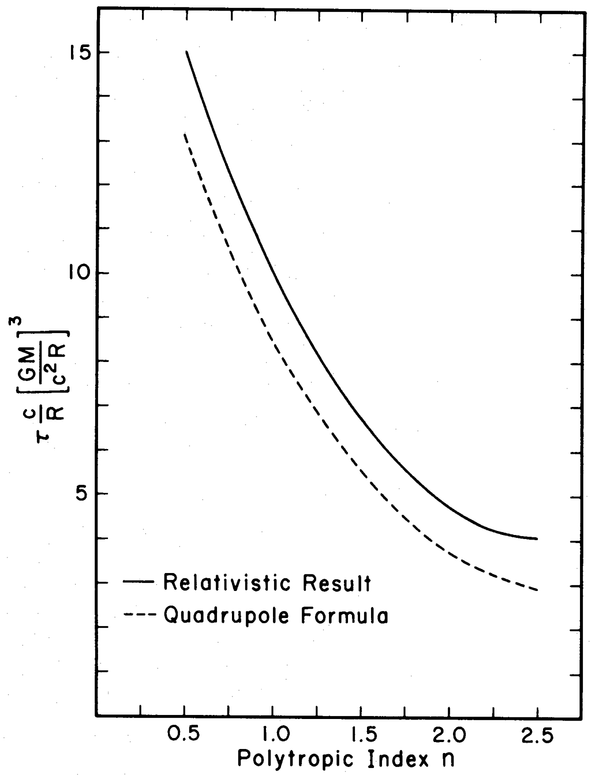

Figure 2. Gravitational radiation damping times for polytropes with $M=1.0\,M_\odot$ and $GM/c^2R=0.03$.

Table 4. Properties of Harrison–Wheeler equation of state stellar models.

a) Newtonian Model with Quadrupole Formula Damping Time

Central Density (10^{14}gm/cm^3)	$M/M_\odot$	R(km)	$\frac{GM}{c^2R}$	Period (10^{-3}sec)	Damping Time (sec)
3.0	0.473	21.6	0.0323	1.184	4.3

b) General Relativistic Models

Central Density (10^{14}gm/cm^3)	$M/M_\odot$	R(km)	$\frac{GM}{c^2R}$	Period (10^{-3}sec)	Damping Time (sec)
2.5	0.380	22.25	0.0252	1.304	10.19
3.0	0.404	20.81	0.0287	1.197	7.43
3.5	0.424	19.73	0.0317	1.114	5.77
4.0	0.440	18.87	0.0344	1.048	4.66
4.5	0.455	18.16	0.0370	0.993	3.88
5.0	0.468	17.56	0.0394	0.946	3.30
5.5	0.480	17.04	0.0416	0.904	2.85

agrees fairly uniformly with the appropriate relativistic pulsation frequency $\omega[GM/R^3]^{-1/2}$ to within a few tenths of a per cent. The gravitational radiation damping times $\tau c(GM/c^2R)^3/R$ for the relativistic models do not agree nearly so well with the Newtonian $c_\tau(n)$. For the models with less variation in the density (i.e. those with smaller polytropic index n) the damping times agree to about 10 per cent while for the more centrally condensed models the disagreement grows to about 30 per cent for $n=2.5$. All of these disagreements are larger than one might have expected for stars having $GM/c^2R=0.03$. Thus we see that a stellar model must be very nearly Newtonian before the quadrupole formula may be used with accuracy, in the sense that even if the frequencies are estimated well, damping times will not be unless this is so. It appears that the quadrupole formula is less accurate for centrally condensed stars than for more uniform ones.

Finally, we reconsider the stellar model found by Balbinski & Schutz (1982) to have a gravitational radiation damping time estimated by the quadrupole formula which was strongly at variance with the existing, fully relativistic calculations. This stellar model was the Harrison–Wheeler equation of state (see Hartle & Thorne 1968) and has $GM/c^2R\approx0.03$. The parameters of both the Newtonian model computed in Balbinski & Schutz (1982) as well as new relativistic models based on the algorithm of Detweiler & Lindblom (1985) are presented in Table 4. A fundamental question becomes apparent upon examination of Table 4: which Newtonian stellar model should be associated with which general relativistic stellar model? – Should one compare models having the same mass, or the same radius, or the same ratio M/R or the same central density, or...? It is clear that there exists an entire range of relativistic models corresponding to the single Newtonian model computed by Balbinski & Schutz. The gravitational wave damping times range from about 9 s for models having the same radius to about 3 s for models having the same mass. In the light of this considerable ambiguity, it is not surprising to find rather large discrepancies between a particular Newtonian model and a particular relativistic model. We note that something of a 'best fit' can be obtained by comparing models with the same ratio GM/c^2R. These models have masses and radii which agree to about 10 per cent and damping times which agree to about 20 per cent. This level of agreement is consistent with the values obtained for similarly centrally condensed polytropes ($n=2$) with similar values of GM/c^2R. Thus we conclude that the discrepancies found by Balbinski & Schutz (1982) between the quadrupole formula and fully relativistic values of the damping time were not solely due to numerical inaccuracies in the original relativistic calculations (Detweiler & Lindblom 1985) but also to the ambiguity in associating a particular Newtonian star with its appropriate relativistic analogue. The latter problem is a manifestation of the fundamentally asymptotic nature of the formula, but we have found that estimates using the approximation are improved when the parameter GM/Rc^2 is used to relate Newtonian models to relativistic ones.

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