THE ASTROPHYSICAL JOURNAL, 267:402–408, 1983 April 1

NECESSARY CONDITIONS FOR THE STABILITY OF ROTATING NEWTONIAN STELLAR MODELS[1]

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Received 1982 June 4; accepted 1982 September 23

ABSTRACT

It is shown that the positivity of a certain energy functional for the perturbations of a rotating stellar model is a necessary condition for stability. The stellar models considered are composed of viscous and thermally conducting fluid, whose equilibrium states are necessarily rigidly rotating and isothermal. The perturbations are not restricted to be adiabatic or axisymmetric. This work generalizes the results of Barston who proved an analogous necessity theorem for nonrotating dissipative fluid stellar models.

Subject headings: hydrodynamics — instabilities — stars: interiors — stars: rotation

I. INTRODUCTION

The stability of rotating Newtonian stellar models has been studied using certain appropriate energy functionals. These energy functionals are constructed from the quantities which describe the deviations of a perturbed fluid star from its equilibrium configuration; these energies decrease monotonically with time. As monotonic functions of time these energies are effectively Lyapunov functionals (see Chetaev 1961 for example) which can be used to test the stability of the equilibrium stellar models. If the energy is positive for all allowable initial perturbations, the star is stable. The energy may only decrease and is bounded below by zero in this case. If, on the other hand, the energy is negative for some initial perturbation, the energy has no lower bound and may decrease forever. The star is unstable if this occurs.

Energy functionals for rotating fluid stars have been constructed and used to discuss stability by a number of authors: Lyttleton (1953), Clement (1964), Lynden-Bell and Ostriker (1967), Howard and Siegman (1969), Hunter (1977), Friedman and Schutz (1978*a*, *b*), Lindblom (1979), and others. Little attention has been given, however, to the question of whether these energy functionals actually form a rigorous test of the stability of rotating stars. If the energy functional is positive for all allowable initial perturbations, is the star really stable? How does the positivity of the energy guarantee the boundedness of the fluid perturbations? If, on the other hand, the energy is negative for some initial perturbation, does this really mean that the perturbation will grow without bound?

For the case of nonrotating fluid stellar models, rigorous theorems do exist which show that energy functionals can be used to determine whether a stellar model is unstable. A method of analysis developed by Laval, Mercier, and Pellat (1965) has been extended by Barston (1969, 1970) and Eisenfeld (1970) to prove the necessity of the energy functional stability criteria for nonrotating stellar models. They have shown that any perturbation having negative energy will grow without bound. The purpose of the present work is to provide a rigorous extension of these results for rotating stellar models. In particular, we show that the energy functional stability criterion given by Lindblom (1979) for viscous and thermally conducting rotating fluid stellar models is a necessary condition for the stability of these models. Since the equilibrium configurations of these stellar models are necessarily rigidly rotating and isothermal, they are more appropriate models of stars at the endpoint of stellar evolution (white dwarfs or neutron stars) than normal main-sequence stars.

Sufficient conditions for stability seem to be more difficult to obtain. Even for nonrotating stellar models it has not been determined whether or not the positivity of the energy functional is sufficient to guarantee the boundedness of all of the perturbation functions (see Eisenfeld 1970). One can show that the velocity perturbation is bounded if the energy is positive, however. We extend this result to the case of rotating stellar models.

II. THE ENERGY FUNCTIONAL

In this section the equations of motion for a Newtonian fluid stellar model are reviewed, and the energy functional previously discussed by Lindblom (1979) for these models is recalled.

The stellar models considered here are those described by the solutions of the Newtonian equations of a viscous heat-conducting fluid. The four fundamental quantities which describe the state of the fluid are the fluid velocity v^i,

[1] Work supported by the National Science Foundation grant PHY 81-18387.

the mass density ρ, the entropy per unit mass s, and the gravitational potential ϕ. These quantities satisfy the following equations of motion:

$$\rho(\partial_t v^i + v^j \nabla_j v^i) = -\nabla^i p + \rho \nabla^i \phi + \nabla_j(\eta \sigma^{ij}) + \nabla^i(\zeta \theta) , \tag{1}$$

$$\rho T(\partial_t s + v^i \nabla_i s) = \nabla^i(\kappa \nabla_i T) + \zeta \theta^2 + \tfrac{1}{2}\eta \sigma_{ij} \sigma^{ij} , \tag{2}$$

$$\partial_t \rho + \nabla_i(\rho v^i) = 0 , \tag{3}$$

$$\nabla^i \nabla_i \phi = -4\pi G \rho . \tag{4}$$

To specify the additional thermodynamic properties of the fluid, the above equations must be supplemented by an equation of state. The internal energy density of the fluid ϵ is assumed to be a given function of ρ and s:

$$\epsilon = \epsilon(\rho, s) . \tag{5}$$

The temperature T, pressure p, and chemical potential μ are then defined as follows:

$$T = \frac{1}{\rho}\left(\frac{\partial \epsilon}{\partial s}\right)_\rho , \tag{6}$$

$$p = \rho\left(\frac{\partial \epsilon}{\partial \rho}\right)_s - \epsilon , \tag{7}$$

$$\mu = \frac{\epsilon + p}{\rho} - Ts . \tag{8}$$

Other kinematic properties of the fluid are described by the shear tensor σ^{ij} and the expansion θ:

$$\sigma^{ij} = \nabla^i v^j + \nabla^j v^i - \tfrac{2}{3} g^{ij} \theta , \tag{9}$$

$$\theta = \nabla_i v^i . \tag{10}$$

Finally, the dissipative processes in the fluid are governed by the coefficients of viscosity, η and ζ, and the heat conduction coefficient κ. These are assumed to be positive functions of ρ and s.

The unperturbed equilibrium solutions to equations (1)–(4) are assumed to be stationary:

$$\partial_t v^i = \partial_t \rho = \partial_t s = 0 . \tag{11}$$

It follows that the solutions must also be axisymmetric, rigidly rotating, and isothermal (see Lindblom 1978, § 3); therefore,

$$\sigma^{ij} = \theta = v^i \nabla_i \rho = v^i \nabla_i s = \nabla_i T = 0 . \tag{12}$$

The equations which describe the deviations of a perturbed stellar model from its equilibrium configuration will be discussed here in the Eulerian framework. The difference between a fluid quantity Q in the perturbed fluid and its value in the equilibrium model will be denoted δQ. Thus, the δQ are the usual Eulerian perturbation quantities. The first-order equations of motion for these perturbation quantities are obtained from equations (1)–(4). These equations are given by

$$\rho(\partial_t \delta v^i + \delta v^j \nabla_j v^i + v^j \nabla_j \delta v^i) = -\nabla^i \delta p + \rho^{-1} \delta \rho \nabla^i p + \rho \nabla^i \delta \phi + \nabla_j(\eta \delta \sigma^{ij}) + \nabla^i(\zeta \delta \theta) , \tag{13}$$

$$\rho T(\partial_t \delta s + v^i \nabla_i \delta s + \delta v^i \nabla_i s) = \nabla^i(\kappa \nabla_i \delta T) , \tag{14}$$

$$\partial_t \delta \rho + v^i \nabla_i \delta \rho + \nabla_i(\rho \delta v^i) = 0 , \tag{15}$$

and

$$\nabla_i \nabla^i \delta \phi = -4\pi G \delta \rho . \tag{16}$$

From these equations it is clear that the velocity perturbation δv^i, the density perturbation $\delta \rho$, and the specific entropy perturbation δs may be freely specified at an initial instant of time. Equations (13)–(16) determine how such initial data are evolved with time.

Associated with the perturbed motion of the star are certain conserved quantities: the changes in the total mass δM, linear and angular momentum $\delta P(\boldsymbol{\lambda})$, and the total entropy of the star δS. These quantities are defined as follows:

$$\delta M = \int \delta \rho d^3x , \tag{17}$$

$$\delta S = \int (s \delta \rho + \rho \delta s) d^3x , \tag{18}$$

and

$$\delta P(\boldsymbol{\lambda}) = \int (\delta\rho v^i + \rho\delta v^i)\lambda_i \, d^3x \,. \tag{19}$$

These integrals are to be performed over all space. The vector field λ^i is any of the generators of the symmetries of space (a translation or rotation) which satisfies Killing's equation:

$$\nabla_i \lambda_j + \nabla_j \lambda_i = 0 \,. \tag{20}$$

Each of these quantities is conserved as a consequence of the equations of motion (13)–(16) as long as the net heat flux across the surface of the star is negligible:

$$\frac{d}{dt}(\delta M) = \frac{d}{dt}(\delta S) = \frac{d}{dt}[\delta P(\boldsymbol{\lambda})] = 0 \,. \tag{21}$$

To insure that the perturbed stellar model represents the same star as the corresponding unperturbed model, we demand that the perturbations in these conserved quantities vanish:

$$\delta M = \delta S = \delta P(\boldsymbol{\lambda}) = 0 \,. \tag{22}$$

These equations (22) place eight constraints on the otherwise arbitrary initial data for the functions $\delta\rho$, δs, and δv^i.

The following energy functional for these stellar models was previously discussed by Lindblom (1979):

$$E = \frac{1}{2}\int \left[\rho\delta v^i \delta v_i + \frac{1}{\rho}\left(\frac{\partial\rho}{\partial p}\right)_s (\delta p)^2 + \frac{1}{\rho}\left(\frac{\partial\rho}{\partial s}\right)_p \left(\frac{\partial p}{\partial s}\right)_T (\delta s)^2 - \frac{1}{4\pi G}\nabla_i \delta\phi \nabla^i \delta\phi\right] d^3x \tag{23}$$

(see also Howard and Siegman 1969 and Friedman and Schutz 1978*a*, *b*). The time derivative of this energy may be computed by using the perturbation equations of motion (13)–(16) with the result:

$$\frac{dE}{dt} = -\int \left[\frac{1}{2}\eta\delta\sigma_{ij}\delta\sigma^{ij} + \zeta(\delta\theta)^2 + \frac{\kappa}{T}\nabla_i(\delta T)\nabla^i(\delta T)\right] d^3x \,. \tag{24}$$

Thus, the energy functional is monotonically decreasing as long as the dissipation coefficients (η, ζ, and κ) and the temperature T are positive.

The remainder of this paper will be devoted to showing that the positivity of the energy functional E (for all allowed initial perturbations) is a necessary condition for the stability of these stellar models.

III. THE NECESSITY OF THE STABILITY CONDITION

To demonstrate the necessity of the energy functional stability condition, it must be shown that any stellar model which violates the condition is necessarily unstable. Thus, it must be shown that if a stellar model admits an initial perturbation ($\delta\rho$, δs, δv^i) for which the energy E is negative, then that perturbation will grow without bound as a consequence of the evolution equations. The argument proceeds in the following way. The first step is to determine the set of perturbations for which the energy is stationary. The second step is to determine the value of the energy functional for these stationary perturbations. We find that the energy functional vanishes for these perturbations. It follows that the energy cannot be stationary for any perturbation having negative energy; furthermore, it follows that this perturbation cannot evolve to a state where the energy is stationary. Thus, the energy will decrease without bound. The final step is to show that unboundedness of the energy is sufficient to guarantee the unboundedness of some perturbation function ($\delta\rho$, δs, δv^i).

The first step in this argument is to determine the set of perturbations for which the energy functional is stationary. This is a straightforward exercise, an outline of which is presented in the Appendix. We find that these energy functional stationary perturbations must satisfy the following three conditions: (*a*) the perturbed temperature δT must be a constant; (*b*) the perturbed velocity δv^i must be a constant linear combination of the six Killing vector fields (see eq. [A3]); (*c*) the perturbations must satisfy the following Bernoulli-like equation

$$c = v_i \delta v^i + \delta\phi - \delta\mu \,, \tag{25}$$

where c is a constant.

The second step is to use these properties of the perturbations to determine the value of the energy functional. We begin by rewriting the energy functional after integrating the last term in equation (23) by parts:

$$E = \frac{1}{2}\int \left[\rho\delta v_i \delta v^i + \frac{1}{\rho}\left(\frac{\partial\rho}{\partial p}\right)_s (\delta p)^2 + \frac{1}{\rho}\left(\frac{\partial\rho}{\partial s}\right)_p \left(\frac{\partial p}{\partial s}\right)_T (\delta s)^2 - \delta\phi\delta\rho\right] d^3x \,. \tag{26}$$

To evaluate this expression for the energy functional stationary perturbations obtained above, we replace $\delta\phi$ in equation (26) by the expression for these perturbations in equation (25). The most complicated set of terms in the resulting expression are the three thermodynamics terms on the left-hand side of equation (27):

$$\frac{1}{\rho}\left(\frac{\partial\rho}{\partial p}\right)_s(\delta p)^2+\frac{1}{\rho}\left(\frac{\partial\rho}{\partial s}\right)_p\left(\frac{\partial p}{\partial s}\right)_T(\delta s)^2-\delta\rho\delta\mu=(s\delta\rho+\rho\delta s)\delta T\;. \tag{27}$$

To convert these terms to the expression on the right-hand side of equation (27), one performs a series of the usual thermodynamic change of variable tricks and makes use of the Maxwell relation

$$\left(\frac{\partial\rho}{\partial s}\right)_p=-\rho^2\left(\frac{\partial T}{\partial p}\right)_s\;. \tag{28}$$

With this simplification the energy takes the form:

$$E=\frac{1}{2}\int\left[(\rho\delta v^i+\delta\rho v^i)\delta v_i+(s\delta\rho+\rho\delta s)\delta T-c\delta\rho\right]d^3x\;. \tag{29}$$

This expression may be rewritten by making use of the definitions of the conserved quantities in equations (17)–(19), the fact that c and δT are constants, and the fact that δv^i is a Killing vector field:

$$E=\tfrac{1}{2}[\delta P(\boldsymbol{\delta v})+\delta T\delta S-c\delta M]\;. \tag{30}$$

Each term in this expression for the energy vanishes because of the constraints in equation (22). Therefore, the energy functional is identically zero for these perturbations. To summarize, we have shown that the kernel of the functional dE/dt is a subset of the kernel of the functional E.

Assume that there is an initial perturbation $(\delta\rho_0, \delta s_0, \delta v_0^i)$ for which the initial value of the energy functional is less than zero. The evolution of the energy by equation (24) requires the energy to decrease monotonically. The continuity of the functional E in a suitable function space (such as the Hilbert space of square integrable functions with square integrable first derivatives) guarantees that the evolution of the initial perturbation $(\delta\rho_0, \delta s_0, \delta v_0^i)$ will remain outside an open neighborhood of the kernel of E. Thus, the evolution will also remain outside an open neighborhood of the kernel of dE/dt. The continuity of the functional dE/dt then guarantees that the value of dE/dt will be bounded above by some negative number. Consequently, the value of the energy functional E will decrease without bound in the evolution of the initial data $(\delta\rho_0, \delta s_0, \delta v_0^i)$.

The final step in the proof of the necessity of the energy functional stability criterion is to show that unboundedness of the energy E implies unboundedness of the perturbation functions also. To achieve this, we will derive a lower estimate for the energy. The only term in the energy (see eq. [26]) which is not expressed simply in terms of the perturbation functions $(\delta\rho, \delta s, \delta v^i)$ is the single term $\delta\phi\delta\rho$. We can derive an estimate for this term by constructing the Fourier series for these functions in a box length $2L$ which completely encloses the star (see Hunter 1977). Thus, we let

$$\delta\phi=\sum_{\boldsymbol{k}}A(\boldsymbol{k})\exp\left[i\pi(\boldsymbol{k}\cdot\boldsymbol{x})/L\right]\,, \tag{31}$$

and

$$\delta\rho=(\pi/4GL^2)\sum_{\boldsymbol{k}}(\boldsymbol{k}\cdot\boldsymbol{k})A(\boldsymbol{k})\exp\left[i\pi(\boldsymbol{k}\cdot\boldsymbol{x})/L\right]\,, \tag{32}$$

where the components of the vector $\boldsymbol{k}$ are integers. It is straightforward to compute the following integrals using these expansions:

$$\int\delta\rho\delta\phi d^3x=(2\pi L/G)\sum_{\boldsymbol{k}}(\boldsymbol{k}\cdot\boldsymbol{k})|A(\boldsymbol{k})|^2\,, \tag{33}$$

$$\int(\delta\rho)^2d^3x=(\pi^2/2G^2L)\sum_{\boldsymbol{k}}(\boldsymbol{k}\cdot\boldsymbol{k})^2|A(\boldsymbol{k})|^2\;. \tag{34}$$

From these expressions follows the desired inequality:

$$\int\delta\rho\delta\phi d^3x\le(4GL^2/\pi)\int(\delta\rho)^2d^3x\;. \tag{35}$$

Consequently, the energy functional may be bounded from below by the following expression:

$$E\ge\frac{1}{2}\int\left[\rho\delta v_i\delta v^i+\frac{1}{\rho}\left(\frac{\partial\rho}{\partial p}\right)_s(\delta p)^2+\frac{1}{\rho}\left(\frac{\partial\rho}{\partial s}\right)_p\left(\frac{\partial p}{\partial s}\right)_T(\delta s)^2-\frac{4GL^2}{\pi}(\delta\rho)^2\right]d^3x\;. \tag{36}$$

The first term in this integral is positive definite as long as the fluid density is positive. Thus, the energy is also bounded below by the expression:

$$E \geq \frac{1}{2}\int \left\{\frac{1}{\rho}\left(\frac{\partial p}{\partial \rho}\right)_s \left[\delta\rho - \left(\frac{\partial \rho}{\partial s}\right)_p \delta s\right]^2 + \frac{1}{\rho}\left(\frac{\partial \rho}{\partial s}\right)_p \left(\frac{\partial p}{\partial s}\right)_T (\delta s)^2 - \frac{4GL^2}{\pi}(\delta\rho)^2\right\} d^3x \,. \tag{37}$$

Whenever the energy decreases without bound as it evolves, it follows that the integral on the right-hand side of equation (37) also decreases without bound. Therefore, either δs or $\delta\rho$ must become unbounded in this case also.

To summarize, we have shown that if the energy functional is negative for some perturbation, then that perturbation will grow without bound. We accomplished this by showing that the time derivative of the energy is bounded away from zero for negative energy perturbations. The resulting decrease without bound of the energy was shown to imply that the perturbation functions $\delta\rho$ or δs were also unbounded. Thus, the positivity of the energy functional E is a necessary condition for stability.

IV. SUFFICIENCY OF THE STABILITY CONDITION

Complete results do not exist on whether or not the positivity of the energy functional is sufficient to guarantee the boundedness of all of the perturbation functions. We can show, however, that the perturbed velocity δv^i is bounded in this case.

Assume that the energy functional E is positive for all possible initial perturbations ($\delta\rho$, δs, δv^i). In § III we showed that the time derivative of E vanishes only when E itself vanishes. Thus, any initial perturbation will evolve toward a state having zero energy. Since the energy cannot decrease beyond zero, these zero-energy perturbations must be equivalent to the stationary-energy perturbations determined in the Appendix.

The energy functional E in equation (23) contains two distinct groups of terms $E = E_1 + E_2$, where

$$E_1 = \frac{1}{2}\int \rho\delta v^i \delta v_i \, d^3x \,, \tag{38}$$

and

$$E_2 = \frac{1}{2}\int \left[\frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_s (\delta p)^2 + \frac{1}{\rho}\left(\frac{\partial \rho}{\partial s}\right)_p \left(\frac{\partial p}{\partial s}\right)_T (\delta s)^2 - \delta\rho\delta\phi\right] d^3x \,. \tag{39}$$

The functional E_1 depends only on the velocity perturbation δv^i, while E_2 depends only on $\delta\rho$ and δs. Since each of the perturbation functions ($\delta\rho$, δs, δv^i) may be freely specified initially, it follows that E_1 and E_2 must each be positive definite if the energy E is positive definite. Similarly, when the energy vanishes, each of the functionals E_1 and E_2 must vanish. This implies that the velocity perturbation δv^i vanishes when the energy vanishes. Since the energy must evolve toward the zero energy state, the velocity perturbation must evolve toward zero also. Thus, we have shown that the positivity of the energy functional is sufficient to guarantee the boundedness of the velocity perturbation.

It is not clear at this time whether or not the positivity of E is sufficient to guarantee the boundedness of the other perturbation functions $\delta\rho$ and δs. We can show, however, that these functions may not diverge very quickly with time. The equations of motion for $\delta\rho$ and δs are given by

$$(\partial_t + v^i\nabla_i)\delta\rho = -\nabla_i(\rho\delta v^i) \,, \tag{40}$$

and

$$\rho T(\partial_t + v^i\nabla_i)\delta s = -\rho T\delta v^i \nabla_i s + \nabla_i(\kappa\nabla^i \delta T) \,. \tag{41}$$

The right-hand sides of these equations go to zero as the perturbation evolves toward the zero-energy state. Consequently, the time derivatives of $\delta\rho$ and δs (in the corotating frame of the unperturbed star) goes to zero as the perturbation evolves. Thus, these functions may diverge only very slowly. Even a linear divergence with time is prohibited by these equations.

APPENDIX

PERTURBATIONS WITH STATIONARY ENERGY

The purpose of this appendix is to solve the perturbation equations (13)–(16) for the class of solutions which have a time-independent energy functional. Thus, we must find the perturbations which make the integral on the right-hand side of equation (24) vanish. Since the integrand is the sum of positive terms, it follows that

$$\delta\sigma_{ij} = \delta\theta = \nabla_i \delta T = 0 \tag{A1}$$

if and only if the integral vanishes. The vanishing of the shear and expansion of the perturbation imply that the perturbed velocity δv^i satisfies Killing's equation:

$$\nabla_i \delta v_j + \nabla_j \delta v_i = 0 . \tag{A2}$$

There are exactly six linearly independent solutions to this equation, three translations and three rotations:

$$\begin{aligned} &\boldsymbol{e}_x = (1, 0, 0) , \quad \boldsymbol{e}_y = (0, 1, 0) , \quad \boldsymbol{e}_z = (0, 0, 1) , \\ &\boldsymbol{r}_x = (0, -z, y) , \quad \boldsymbol{r}_y = (z, 0, -x) , \quad \boldsymbol{r}_z = (-y, x, 0) . \end{aligned} \tag{A3}$$

The perturbation of the velocity field δv^i must be a linear combination of these six vectors with coefficients that may depend only on time.

Equation (A1) also reveals that the perturbation of the temperature δT may depend only on time. The perturbation δT can be related to the perturbation functions $\delta\rho$ and δs by the expansion

$$\delta T = \left(\frac{\partial T}{\partial \rho}\right)_s \delta\rho + \left(\frac{\partial T}{\partial s}\right)_\rho \delta s . \tag{A4}$$

This equation and the evolution equations (14) and (15) for $\delta\rho$ and δs guarantee that δT must be independent of time also.

To learn more about the perturbations which keep the energy stationary, we must consider the perturbed Navier-Stokes equation (13). The most complicated nonvanishing terms in equation (13) arise from the perturbed buoyant force. These can be simplified by using the thermodynamic indentity (see eqs. [6]–[8])

$$\nabla_i p = \rho s \nabla_i T + \rho \nabla_i \mu \tag{A5}$$

and its corresponding perturbation equation. The result is the expression

$$\rho \nabla_i \delta\mu = \nabla_i \delta p - \rho^{-1} \delta\rho \nabla_i p . \tag{A6}$$

Using this expression and equation (A2), the perturbed Navier–Stokes equation assumes the simple form

$$\partial_t \delta v_i = \nabla_i (v^j \delta v_j + \delta\phi - \delta\mu) . \tag{A7}$$

The right-hand side of equation (A7) is a gradient, implying that $\partial_t \delta v^i$ is curl free. Thus, the coefficients of the rotations in δv^i must be independent of time.

By using the constraints on the total momentum of the perturbed star, we can also show that $\partial_t \delta v^i$ vanishes identically. To see this, we consider the expression for the conservation of momentum generated by one of the translation Killing vector fields λ^i:

$$\frac{d}{dt}[\delta P(\boldsymbol{\lambda})] = \int (\partial_t \delta\rho v^i \lambda_i + \rho \partial_t \delta v^i \lambda_i) d^3x = 0 . \tag{A8}$$

The equation of motion for $\delta\rho$ (see eq. [15]) can be used to simplify the first term in equation (A8), while the fact that both λ^i and $\partial_t \delta v^i$ are translations can be used to factor the second term:

$$\frac{d}{dt}[\delta P(\boldsymbol{\lambda})] = \int (\delta\rho v^i + \rho \delta v^i) \nabla_i (\lambda^j v_j) d^3x + (\partial_t \delta v^i \lambda_i) \int \rho d^3x . \tag{A9}$$

The gradient $\nabla_i(\lambda^j v_j)$ is a translation Killing vector since λ^i is a translation and v^i is a rotation. Therefore, the first integral on the right of equation (A9) is just the perturbed linear momentum $\delta P[\nabla(\boldsymbol{v} \cdot \boldsymbol{\lambda})]$, which vanishes because of the constraint equation (22). The requirement that the second term in equation (A9) vanish, together with the freedom to choose λ^i as any translation gives the desired result:

$$\partial_t \delta v^i = 0 . \tag{A10}$$

We are now prepared to obtain the first integral of the perturbed Navier-Stokes equation for these perturbations. We use the condition in equation (A10) together with equation (A7) to obtain the result:

$$c = v_i \delta v^i + \delta\phi - \delta\mu , \tag{A11}$$

where c may depend only on time. In fact it is straightforward to show that c does not depend on time either. One uses equations (14)–(16) together with the requirement that the gravitational potential ϕ and its perturbation $\delta\phi$ must vanish asymptotically.

To summarize, we have shown that any perturbation whose energy functional E is constant in time must satisfy the following conditions: (a) the perturbed temperature δT is a constant; (b) the perturbed velocity δv^i is a constant linear combination of the six Killing vector fields (eq. [A3]); (c) the perturbations must satisfy the Bernoulli-like equation (A11).

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