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ON THE STABILITY OF ROTATING STELLAR MODELS IN GENERAL RELATIVITY THEORY[1]

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ABSTRACT

We investigate the effects of viscosity and thermal conductivity on the stability of rotating stellar models in general relativity theory. The equations of motion for the perturbed fluid stellar model (including nonadiabatic and dissipative effects) are used to construct an energy functional for the perturbed motion of the star. This energy is used to investigate the stability of rotating stellar models. The most interesting results of our investigation are (1) that the generic gravitational radiation-induced secular instability (discovered by J. L. Friedman in rotating perfect fluid stars) does not exist in slowly rotating stars having nonzero dissipation coefficients; and (2) three conditions necessary for the stability of these models are (*a*) the Schwarzschild criterion, (*b*) subluminal sound velocity, and (*c*) the dissipation coefficients not being too large.

Subject headings: relativity — stars: interiors — stars: rotation

I. INTRODUCTION

In this paper we continue the study of the stability of rotating stellar models in general relativity theory as begun, for example, by Chandrasekhar and Friedman (1972), Friedman and Schutz (1975), and Friedman (1978). Our main interests here are the effects of viscosity, thermal conductivity, and nonadiabatic processes on stability. Our stability analysis is conducted completely within the context of general relativity theory and the semiempirical theory of dissipative fluids given by Eckart (1940). We are able to study in a self consistent manner, therefore, the interaction between the effects of the fluid dissipative processes and the effects of gravitational radiation on the stability of rotating stars. Roberts and Stewartson (1963) were the first to demonstrate that a fluid dissipative process (viscosity) could cause a secular instability in rotating stars, while Chandrasekhar (1970) first showed that gravitational radiation caused a similar instability. Subsequent analysis of the gravitational radiation instability by Friedman and Schutz (1978*b*) suggested that all perfect fluid rotating stars were unstable to the gravitational radiation secular instability. This generic gravitational radiation instability occurs, however, only for very short length scale perturbations in slowly rotating stars. The analysis of this instability by Friedman and Schutz (1978*b*), therefore, is not self-consistent since the interaction of short wavelength gravitational radiation is outside the domain of validity of their Newtonian analysis. It was Friedman (1978) who first showed that all perfect fluid rotating stars were unstable to a gravitational radiation secular instability within the context of general relativity theory. We extend the analysis of the generic gravitational radiation secular instability by including the effects of the fluid dissipation processes. We find two interesting results: (1) the gravitational radiation secular instability does not occur in slowly rotating stars composed of dissipative fluid. Thus, the gravitational radiation secular instability is not generic. (2) The fluid dissipation coefficients (e.g., the thermal conductivity) must be bounded above by certain thermodynamic expressions (see § VI) to prevent a generic dissipative secular instability (within the context of Eckart's semiempirical theory of the dissipative processes).

Our analysis is based on an energy functional stability criterion. A functional which describes the energy of the perturbed motion of the star can be used to examine the stability of the star if the energy is a monotonically decreasing function of time. Given such an energy functional, we examine it to determine whether there exist any perturbations having negative energy. If no such negative energy perturbations exist, the star is stable. The energy must decrease and is bounded below by zero in this case. Thus, no perturbation function may grow without bound. If the star admits negative energy perturbations, then the energy has no lower bound. The amplitude of such a negative energy perturbation may grow without bound as the energy decreases to negative infinity.

Sections II–IV of this paper are devoted to the construction of an energy functional which is appropriate for the study of the stability of dissipative relativistic fluid stellar models. This energy is the relativistic analog of the

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corotating canonical energy used by Friedman and Schutz (1978*a*, *b*) and Lindblom (1979) in the study of dissipative Newtonian fluid stars. We show that this energy, unlike Friedman's (1978) perfect fluid energy functional, is invariant under the gauge group of trivial Lagrangian perturbations. We also show that the effects of the fluid dissipation processes always make the energy decrease with time. These properties would make our energy an appropriate tool for the study of the stability of these stellar models, except that the energy is also affected by the flux of gravitational energy. This flux has no definite sign (even for purely outgoing radiation) so that our energy is not necessarily a decreasing function of time for all possible perturbations.

Our energy can be used to test the stability of stellar models with respect to those classes of perturbations for which the energy is monotonically decreasing. Thus, any perturbation whose gravitional energy flux is strictly positive or any perturbation whose fluid dissipative processes dominate the gravitational dissipation of energy may be tested for stability using our energy. In § V we show that one interesting class of perturbations which meets this criterion is the "local" or "short length scale" class. For these perturbations the fluid dissipation mechanisms dominate the gravitational dissipation of the perturbed energy.

In § VI we use our energy to analyze the stability of the short length scale perturbations. We find that negative energy perturbations always exist, if arbitrarily short wavelengths are considered. However, our equations only describe the motion of real stellar material to the extent that that material behaves like a fluid. Real matter does not satisfy the fluid equations of motion for arbitrarily short wavelength motions. Wavelengths must be longer than the average interparticle separation and the mean free path of particles in the fluid, for example. Restricting our attention then to those wavelengths for which our equations are physically relevant, we arrive at three conditions which are necessary for the stability of a star: (1) The star must satisfy the relativistic Schwarzschild criterion. (2) The adiabatic sound speed must be less than the velocity of light. (3) The thermal conductivity coefficient must be bounded above by an expression described in § VI. We show that this criterion is trivially satisfied for normal materials. Our analysis suggests that analogous constraints may exist for the viscosity coefficients, but these do not emerge from the simple short wavelength analysis presented here. The simple theory of the dissipative processes used here is known to contain unphysical superluminal propagation of short wavelength thermal fluctuations (see, e.g., Zumino 1957). The theory is, however, thought to be well behaved in the region of applicability of the hydrodynamic equations themselves (see Weymann 1967). Since the three conditions for stability were derived within the domain of applicability of the hydrodynamic equations, they should be correct within this context.

Our analysis also shows that the gravitational radiation secular instability shown to exist by Friedman (1978) in perfect fluid stellar models is not generic in stars made of dissipative fluid. If the star is rotating sufficiently slowly (see § VI), then the perturbations found to be unstable by Friedman are a subset of the short length scale perturbations. However, in a dissipative fluid star satisfying certain thermodynamic conditions (see § VI), all short length scale perturbations are stable. Thus, the generic gravitational radiation secular instability does not exist in slowly rotating stars if the fluid dissipation coefficients are nonzero.

II. EQUILIBRIUM STELLAR MODELS

In the following sections we analyze the stability of rotating general relativistic stellar models. It is appropriate to describe here some of the relevant properties of the equilibrium models whose stability we will analyze. These equilibrium models are stationary, axisymmetric, rigidly rotating, and have vanishing thermal currents. Thus, there exist timelike and axial Killing vector fields τ^a and ϕ^a:

$$\mathscr{L}_\tau g_{ab} = \mathscr{L}_\phi g_{ab} = 0 \,, \tag{1}$$

where g_{ab} is the spacetime metric and $\mathscr{L}_v$ is the Lie derivative along the vector field v^a. The star is rigidly rotating in the sense that the unit four-velocity of the fluid, u^a, is proportional to $k^a = \tau^a + \Omega\phi^a$, with Ω a constant:

$$u^a = \lambda k^a \,. \tag{2}$$

The vanishing of the relativistic thermal current implies that the temperature T must satisfy the following "isothermality" condition:

$$u^a \nabla_a u^b = -\nabla^b \log T \,. \tag{3}$$

The equilibrium stellar models are solutions of Einstein's equations,

$$G^{ab} = 8\pi T^{ab} \equiv 8\pi\rho u^a u^b + 8\pi p q^{ab} \,, \tag{4}$$

where ρ and p are the energy density and pressure of the fluid, G^{ab} is the Einstein curvature tensor, and q^{ab} is the projection tensor defined by

$$q^{ab} = g^{ab} + u^a u^b \,. \tag{5}$$

Euler's equation,

$$\nabla^b p = -(\rho + p) u^a \nabla_a u^b \,, \tag{6}$$

is a consequence of equation (4) and Bianchi identities. The thermodynamic state of the fluid is most conveniently described in terms of n and s, the number density of particles in the fluid and the entropy per particle, respectively. The equation of state of the fluid is specified by the function $\rho = \rho(n, s)$. The first law of thermodynamics then requires the relationships:

$$\left(\frac{\partial \rho}{\partial n}\right)_s = \frac{\rho + p}{n}, \tag{7}$$

$$\left(\frac{\partial \rho}{\partial s}\right)_n = Tn, \tag{8}$$

$$\mu = \frac{\rho + p}{n} - Ts, \tag{9}$$

where μ is the chemical potential of the fluid. It follows that these functions also satisfy the Maxwell relation:

$$\left(\frac{\partial p}{\partial s}\right)_n = n^2\left(\frac{\partial T}{\partial n}\right)_s. \tag{10}$$

From the isothermality condition (3) and Euler's equation (6), it follows that

$$\frac{dp}{dT} = \frac{\rho + p}{T}, \tag{11}$$

where dp/dT is the derivative computed in the barotropic equilibrium configuration. Furthermore, it follows that all of the thermodynamic quantities are invariant under the symmetries

$$\mathscr{L}_\tau n = \mathscr{L}_\phi n = \mathscr{L}_\tau s = \mathscr{L}_\phi s = 0. \tag{12}$$

(See, e.g., Lindblom 1976).

III. EQUATIONS OF MOTION FOR THE PERTURBED STELLAR MODELS

The functions describing the differences between perturbed and unperturbed quantities have been represented in two ways (see Friedman and Schutz 1975 and Friedman 1978 for a thorough discussion of relativistic perturbation theory). We denote by δQ the Eulerian change in a quantity Q, that is, the change in the value of the quantity Q at a given point of spacetime. We denote by ΔQ the Lagrangian change in Q, that is, the change in Q observed by a given particle of fluid. The changes in all physical quantities can be represented in terms of three fundamental fields: ξ^a, Δs, and h_{ab}. The dislocation of a particle of fluid in the perturbed stellar model is represented by ξ^a, Δs is the Lagrangian change of the specific entropy, and $h_{ab} = \delta g_{ab}$ is the Eulerian change in the spacetime metric tensor. The changes in the other physical quantities are related to these by using the expressions

$$\delta u^a = \tfrac{1}{2}u^a u^b u^c h_{bc} + \lambda q^a{}_b \mathscr{L}_k \xi^b, \tag{13}$$

$$\delta n = -\tfrac{1}{2} n q^{ab} h_{ab} - q^a{}_b \nabla_a (n\xi^b). \tag{14}$$

The Lagrangian change in any quantity Q can be expressed in terms of the Eulerian change by the relationship

$$\Delta Q = \delta Q + \mathscr{L}_\xi Q. \tag{15}$$

Also, changes in composite quantities such as Δp can be computed to first order by treating δ as a differential operator which satisfies the chain rule; thus, for example,

$$\Delta p = \frac{\partial p}{\partial n}\Delta n + \frac{\partial p}{\partial s}\Delta s = -\tfrac{1}{2} p\gamma q^{ab}\Delta g_{ab} + \frac{\partial p}{\partial s}\Delta s, \tag{16}$$

where $\gamma \equiv \partial \log (p)/\partial \log (n)$ is the standard adiabatic index.

Another useful quantity is the change in the Einstein tensor

$$\delta G^{ab} = -\tfrac{1}{2}\epsilon^{aceg}\epsilon^{bdf}{}_g \nabla_{(c}\nabla_{d)} h_{ef} + G^{abcd}h_{cd}, \tag{17}$$

where ϵ^{abcd} is the covariantly constant antisymmetric tensor (parentheses surrounding indices denote symmetrization) and

$$G^{abcd} = \tfrac{1}{2}R^{a(cd)b} + \tfrac{1}{2}[g^{ab}R^{cd} - g^{b(c}R^{d)a} - g^{a(c}R^{d)b}] + \tfrac{1}{4}R[g^{ac}g^{bd} + g^{ad}g^{bc}]; \tag{18}$$

the tensors R_{abcd}, R_{ab}, and R are the Riemann, Ricci, and scalar curvatures, respectively, of the background metric g_{ab}.

The change in the stress energy tensor will be decomposed into a part representing adiabatic ($\Delta s = 0$) changes to a perfect fluid stress tensor $\delta_{\rm PF} T^{ab}$ and portions resulting from the presence of viscosity and thermal conductivity:

$$\delta T^{ab} = \delta_{\rm PF} T^{ab} + \left(nTu^a u^b + \frac{\partial p}{\partial s} q^{ab}\right)\Delta s - 2\eta\delta\sigma^{ab} - \zeta q^{ab}\delta\theta + u^a \delta q^b + u^b \delta q^a . \tag{19}$$

The scalars η and ζ are the coefficients of viscosity, while $\delta\sigma^{ab}$, $\delta\theta$, and δq^a represent the changes in the shear, expansion, and heat flow vector. These quantities are related to the fundamental perturbation quantities by the expressions:

$$\delta\theta = \nabla_a(\tfrac{1}{2}u^a q^{bc} h_{bc} + \lambda q^a{}_b \mathscr{L}_{\boldsymbol{k}} \xi^b) , \tag{20}$$

$$\delta\sigma_{ab} = \tfrac{1}{2}\lambda q_{ac} q_{bd}(\nabla^c \mathscr{L}_{\boldsymbol{k}} \xi^d + \nabla^d \mathscr{L}_{\boldsymbol{k}} \xi^c) + \tfrac{1}{2}\lambda q_{ac} q_{bd} \mathscr{L}_{\boldsymbol{k}} h^{cd} - \tfrac{1}{3} q_{ab} \delta\theta , \tag{21}$$

$$\delta q^a = -\kappa T q^{ab}\left[\nabla_b\left(\frac{\Delta T}{T}\right) + \lambda u^c(\mathscr{L}_{\boldsymbol{k}} h_{bc} + \nabla_b \mathscr{L}_{\boldsymbol{k}} \xi_c + \nabla_c \mathscr{L}_{\boldsymbol{k}} \xi_b) - \tfrac{1}{2}\nabla_b(u^c u^d \Delta g_{cd})\right] , \tag{22}$$

where κ is the coefficient of thermal conductivity. The perfect fluid change in the stress tensor is given by

$$\delta_{\rm PF} T^{ab} = (\rho + p)(u^a \delta u^b + u^b \delta u^a) - p h^{ab} + \left[(\rho + p)\frac{\Delta n}{n} - \xi^c \nabla_c \rho\right] u^a u^b + \left[p\gamma \frac{\Delta n}{n} - \xi^c \nabla_c p\right] q^{ab} . \tag{23}$$

The perturbed Einstein equations are given by

$$\delta G^{ab} = 8\pi \delta T^{ab} . \tag{24}$$

These equations determine the evolution of the metric perturbations, h_{ab}. The evolution of the fluid dislocation ξ^a is determined by the first-order change in the relativistic Navier-Stokes equations:

$$\Delta(\nabla_a T^{ab}) = 0 . \tag{25}$$

Finally, the evolution of the entropy perturbation is governed by

$$n u^a \nabla_a(\Delta s) = -\nabla_a(\delta q^a / T) . \tag{26}$$

It is convenient to split the Navier-Stokes equation into an adiabatic perfect fluid part and a force term resulting from the nonadiabatic and dissipative terms. Thus, equation (25) may be written

$$\Delta_{\rm PF}(\nabla_a T^{ab}) = F^b . \tag{27}$$

The force term is given by

$$F^a = \nabla_b(\delta_{\rm PF} T^{ab} - \delta T^{ab}) = \nabla_b\left[2\eta\delta\sigma^{ab} + \zeta q^{ab}\delta\theta - u^a\delta q^b - u^b \delta q^a - \left(nTu^a u^b + \frac{\partial p}{\partial s} q^{ab}\right)\Delta s\right] , \tag{28}$$

while the adiabatic perfect fluid portion of the equation has the form

$$\Delta_{\rm PF}(\nabla_b T^{ab}) = \nabla_b(\delta_{\rm PF} T^{ab}) + \tfrac{1}{2} g^{ab} T^{cd}(g_{bc}\nabla_d h^e{}_e + 2\nabla_c h_{bd} - \nabla_b h_{cd}) . \tag{29}$$

The above discussion has introduced the system of equations needed to describe arbitrary nonadiabatic perturbations of a rotating general relativistic stellar model. It will also be useful in the discussion that follows to introduce certain objects which were found to be of fundamental importance in the study of adiabatic perfect fluid perturbations by Friedman and Schutz. For example, one useful function, $L(\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{h}}; \boldsymbol{\xi}, \boldsymbol{h})$, is defined by

$$L(\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{h}}; \boldsymbol{\xi}, \boldsymbol{h}) = U^{abcd}\nabla_a \hat{\xi}_b \nabla_c \xi_d - T^{ab} R_{acbd} \hat{\xi}^c \xi^d + V^{abcd}(\hat{h}_{ab} \nabla_c \xi_d + h_{ab} \nabla_c \hat{\xi}_d) - \frac{1}{32\pi} \epsilon^{aceg} \epsilon^{bdf}{}_g \nabla_c \hat{h}_{ab} \nabla_d h_{ef}$$
$$+ \frac{1}{2}\left(W^{abcd} - \frac{1}{8\pi} G^{abcd}\right)\hat{h}_{ab} h_{cd} - \tfrac{1}{2}\nabla_c T^{ab}(\hat{h}_{ab}\xi^c + h_{ab}\hat{\xi}^c) , \tag{30}$$

where

$$U^{abcd} = (\rho + p) u^a u^c q^{bd} + p(g^{ab} g^{cd} - g^{ad} g^{bc}) - \gamma p q^{ab} q^{cd} , \tag{31}$$

$$V^{abcd} = \tfrac{1}{2}(\rho + p)(u^a u^c q^{bd} + u^b u^c q^{ad} - u^a u^b q^{cd}) - \tfrac{1}{2}\gamma p q^{ab} q^{cd} , \tag{32}$$

and

$$W^{abcd} = \tfrac{1}{2} U^{abcd} - \tfrac{1}{2} T^{ac} g^{bd} . \tag{33}$$

The function L is symmetric in its arguments:

$$L(\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{h}}; \boldsymbol{\xi}, \boldsymbol{h}) = L(\boldsymbol{\xi}, \boldsymbol{h}; \hat{\boldsymbol{\xi}}, \hat{\boldsymbol{h}}) , \tag{34}$$

and can be used as a Lagrangian for adiabatic perfect fluid perturbations. Another important vector field $R^a(\hat{\xi}, \hat{h}; \xi, h)$ is defined by

$$R^a(\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{h}}; \boldsymbol{\xi}, \boldsymbol{h}) = U^{abcd}\hat{\xi}_b \nabla_c \xi_d + V^{cdab} h_{cd}\hat{\xi}_b - \frac{1}{32\pi}\epsilon^{aceg}\epsilon^{bdf}{}_g \hat{h}_{cd}\nabla_b h_{ef} . \tag{35}$$

These functions satisfy the following fundamental identity (see Friedman and Schutz 1975):

$$\hat{\xi}_a \Delta_{\rm PF}(\nabla_b T^{ab}) + \frac{1}{16\pi}\hat{h}_{ab}(\delta G^{ab} - 8\pi\delta_{\rm PF} T^{ab}) = -L(\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{h}}; \boldsymbol{\xi}, \boldsymbol{h}) + \nabla_a R^a(\hat{\boldsymbol{\xi}}, \hat{\boldsymbol{h}}; \boldsymbol{\xi}, \boldsymbol{h}) . \tag{36}$$

IV. AN ENERGY FUNCTIONAL FOR ROTATING STELLAR MODELS

In the theory of adiabatic perfect fluid perturbations of rotating stars one finds it useful to introduce the following energy associated with the perturbations:

$$\tilde{E}(\Sigma) = \frac{1}{2}\int_\Sigma \left\{R^a(\mathscr{L}_k \boldsymbol{\xi}, \mathscr{L}_k \boldsymbol{h}; \boldsymbol{\xi}, \boldsymbol{h}) - R^a(\boldsymbol{\xi}, \boldsymbol{h}; \mathscr{L}_k \boldsymbol{\xi}, \mathscr{L}_k \boldsymbol{h}) + \frac{1}{16\pi}\nabla_b(k^{[a}\epsilon^{b]ceg}\epsilon^{ldf}{}_g h_{cd}\nabla_l h_{ef})\right\} d\Sigma_a , \tag{37}$$

where square brackets surrounding a pair of indices indicate antisymmetrization and the integral is performed over a spacelike surface Σ. The pure divergence term has been included for reasons that will become apparent later. By virtue of equations (34) and (36) this energy is conserved (that is, independent of which spacelike surface, Σ, the integral is performed upon) as long as the perfect fluid field equations are satisfied:

$$\tilde{E}(\Sigma_2) - \tilde{E}(\Sigma_1) = 0 . \tag{38}$$

This energy can be generalized to be of use in the discussion of the more general fluid perturbations of interest here. From our experience with the Newtonian analogues of this energy (see Lindblom 1979), we know that extra contributions from the nonadiabatic perturbations are to be expected. We find that by adding the following extra pieces, we obtain an energy functional which has several attractive features:

$$E(\Sigma) = \tilde{E}(\Sigma) + \int_\Sigma u^a \frac{n}{\lambda}\Delta s\left(\frac{\partial T}{\partial n}\Delta n + \frac{1}{2}\frac{\partial T}{\partial s}\Delta s\right) d\Sigma_a + \int_\Sigma \frac{1}{\lambda}\left[u_b \Delta u^b(\delta q^a + u^a n T\Delta s) + \frac{\Delta T}{T}\delta q^a\right] d\Sigma_a$$
$$+ \int_\Sigma \left[\mathscr{L}_k \xi_b \delta_D T^{ab} + \frac{1}{4\lambda}u^a h_{bc}\delta_D T^{bc} - \frac{1}{2\lambda}u^a\xi_b\nabla_c(\delta_D T^{bc})\right] d\Sigma_a , \tag{39}$$

where $\delta_D T^{ab} = \delta T^{ab} - \delta_{\rm PF} T^{ab}$ represents the portions of the perturbed stress tensor arising from the dissipative and nonadiabatic terms.

The time derivative of this energy is computed with the aid of the fundamental identity equation (36). Using the full perturbed equations of motion equations (24), (27), and (28), it follows that

$$\tilde{E}(\Sigma_2) - \tilde{E}(\Sigma_1) = -\int \nabla_a \left[\mathscr{L}_k \xi_b \delta_D T^{ab} + \frac{1}{4\lambda}u^a h_{bc}\delta_D T^{bc} - \frac{1}{2\lambda}u^a \xi_b \nabla_c(\delta_D T^{bc})\right] d^4x$$
$$+ \frac{1}{2}\int (\mathscr{L}_k h_{ab} + \nabla_a \mathscr{L}_k \xi_b + \nabla_b \mathscr{L}_k \xi_a)\delta_D T^{ab} d^4x . \tag{40}$$

The integrals on the right-hand side of equation (40) are performed in the four-dimensional region bounded by Σ_1 and Σ_2. This expression accounts for the contributions to E from the last integral in equation (39). Next one uses the explicit form of the dissipative stress tensor $\delta_D T^{ab}$ from equation (19) together with the detailed expressions for $\delta\theta$, $\delta\sigma^{ab}$, and δq^a from equations (20)–(22). A lengthy calculation which uses these facts (in the second integral which appears in eq. [40]) together with the equation of motion for Δs [eq. (26)] and various bits of thermodynamic trivia from § II yields the result:

$$E(\Sigma_2) - E(\Sigma_1) = -\int \frac{1}{\lambda}\left[2\eta\delta\sigma_{ab}\delta\sigma^{ab} + \zeta(\delta\theta)^2 + \frac{1}{\kappa T}q^{ab}\delta q_a \delta q_b\right] d^4x . \tag{41}$$

The integrand in equation (41) is positive definite whenever the viscosity and heat conduction coefficients are positive. Therefore, the energy functional E is a monotonically decreasing function of time, whose time derivative is the natural relativistic generalization of the Newtonian expression for the time derivative of the corresponding energy (see Lindblom 1979).

The second attractive feature of the energy functional E is the fact that it is invariant under the trivial group of Lagrangian perturbations. These trivial Lagrangian perturbations are those which leave the physical state of the star unchanged; that is, all of the Eulerian perturbations must vanish everywhere for these trivial perturbations. Equations (13)–(15) place the following restrictions on the trivial perturbation $(\Delta\sigma, \eta^a) = (\Delta s, \xi^a)$:

$$\Delta\sigma = \eta^a \nabla_a s \,, \tag{42}$$

$$0 = q^a{}_b \mathscr{L}_{\boldsymbol{k}} \eta^b \,, \tag{43}$$

$$0 = q^a{}_b \nabla_a (n\eta^b) \,. \tag{44}$$

In analogy with Friedman's (1978) discussion of the adiabatic situation, it follows that

$$\eta^a = \Psi k^a + \frac{1}{n} \epsilon^{abcd} u_b \nabla_c \Upsilon_d \,, \tag{45}$$

where Ψ is an arbitrary scalar function and the vector field Υ^d may be specified arbitrarily on a spacelike slice, but must be propagated off the slice by the condition

$$\mathscr{L}_{\boldsymbol{k}} \Upsilon_a = 0 \,. \tag{46}$$

The trivial entropy perturbation is given by equation (42). We see that the trivial group in this more general non-adiabatic situation is considerably larger than the adiabatic group studied by Friedman.

To see that the energy functional E is invariant under trivial changes in the perturbation quantities it is sufficient that E be written in a form which involves only Eulerian perturbation quantities. In Appendix A, we describe how equation (39) for the energy may be transformed into the following form:

$$\begin{aligned} E(\Sigma) = \int_\Sigma \Bigg\{ & \frac{1}{\lambda} \delta T^a{}_b \delta u^b - \frac{1}{2\lambda} (\rho + p) u^a g^{bc} \delta u_b \delta u_c + \frac{1}{\lambda} \delta q^a \frac{\delta T}{T} + \frac{1}{2\lambda} u^a \frac{1}{\rho + p} \left[\left(\frac{\partial \rho}{\partial p}\right)_s (\delta p)^2 + \left(\frac{\partial \rho}{\partial s}\right)_p \frac{dp}{ds} (\delta s)^2 \right] \\ & + \frac{1}{32\pi\lambda} u^a [G^{bcde} - 4\pi(\rho + p) u^b (u^c g^{de} - 4 u^d g^{ce}) + 8\pi p g^{bd} g^{ce}] h_{bc} h_{de} \\ & - \frac{1}{32\pi} (\delta^a{}_l \mathscr{L}_{\boldsymbol{k}} h_{cd} - \tfrac{1}{2} k^a \nabla_l h_{cd}) \epsilon^{lceg} \epsilon^{bdf}{}_g \nabla_b h_{ef} \Bigg\} d\Sigma_a \,. \end{aligned} \tag{47}$$

We have neglected to write a number of pure divergence terms in this expression. These terms vanish identically whenever the boundary of Σ is outside of the support of the fluid (see Appendix A for details).

Since this expression for the energy depends only on Eulerian perturbation quantities, it follows that E is invariant under the trivial gauge transformations. We note that unlike the Newtonian version of this energy, E depends explicitly on the viscosities and thermal conductivity through the terms involving $\delta T^a{}_b$ and δq^a in equation (47).

Before leaving our discussion of the general properties of the energy functional E, it is appropriate to look at the form which the energy takes in the exterior region of the stellar model. Here the contributions to the energy come only from the perturbations in the gravitational field, h_{ab}, and can be thought of as the energy associated with gravitational radiation. From equations (35) and (37) this contribution to the energy is given by

$$E_{\rm GW}(\Sigma) = \frac{1}{64\pi} \int_\Sigma \{ -\epsilon^{aceg} \epsilon^{bdf}{}_g (\mathscr{L}_{\boldsymbol{k}} h_{cd} \nabla_b h_{ef} - h_{cd} \nabla_b \mathscr{L}_{\boldsymbol{k}} h_{ef}) + 2\nabla_b (k^{[a} \epsilon^{b]ceg} \epsilon^{ldf}{}_g h_{cd} \nabla_l h_{ef}) \} d\Sigma_a \,. \tag{48}$$

This form of the gravitational wave energy appears to depend on first and second derivatives of the metric perturbation h_{ab}. In fact, however, the second derivative terms may all be eliminated using the perturbation equations so that the energy may be evaluated purely in terms of the initial data h_{ab} and $\nabla_c h_{ab}$. The independence of $E_{\rm GW}$ from second derivatives may be made explicit by using the definition, equation (48), and the perturbed vacuum Einstein equations, equation (17). The resulting expression is

$$E_{\rm GW}(\Sigma) = \int_\Sigma t^a(\boldsymbol{k}) d\Sigma_a \,, \tag{49}$$

where

$$32\pi t^a(\boldsymbol{k}) = -\epsilon^{aceg} \epsilon^{bdf}{}_g \mathscr{L}_{\boldsymbol{k}} h_{cd} \nabla_b h_{ef} + \tfrac{1}{2} k^a [\epsilon^{lceg} \epsilon^{bdf}{}_g \nabla_l h_{cd} \nabla_b h_{ef} + R^{e(cd)b} h_{ed} h_{cb}] \,. \tag{50}$$

The independence of t^a from second derivatives motivated the inclusion of the pure divergence in the original definition of the energy, equation (37).

The vector field t^a defined above makes good sense as the momentum of the gravitational field for the following reasons. Consider the Lagrangian $L(\boldsymbol{h})$ for the perturbation in the gravitational field, evaluated in the exterior of the star:

$$L(\boldsymbol{h}) = \tfrac{1}{2}L(0, \boldsymbol{h}; 0, \boldsymbol{h}) = -\frac{1}{64\pi}\epsilon^{aceg}\epsilon^{bdf}{}_{g}\nabla_c h_{ab}\nabla_d h_{ef} - \frac{1}{64\pi}R^{a(bc)d}h_{ac}h_{bd}\ . \tag{51}$$

We see that the vector t^a is related to this Lagrangian by the following expression:

$$t^a = \frac{\partial L}{\partial \nabla_a h_{bc}}\mathscr{L}_{\boldsymbol{k}} h_{bc} - k^a L\ . \tag{52}$$

Thus, t^a is the canonical momentum associated with the Killing vector k^a (see Trautmann 1964). The conservation of this vector field $\nabla_a t^a = 0$ is the expression of the Noether conservation law for the symmetry which the Killing vector k^a generates.

Another conserved energy-momentum vector associated with the Killing vector k^a can be obtained from the symmetric stress energy tensor associated with the Lagrangian $L(\boldsymbol{h})$. The Lagrangian $L(\boldsymbol{h})$ depends not only upon the perturbation of the metric h_{ab}, but also upon the background metric itself. Thus, one can vary $L(\boldsymbol{h})$ with respect to the background metric to obtain the symmetric stress tensor:

$$t^{ab} = -2\delta L/\delta g_{ab} - g^{ab}L\ . \tag{53}$$

This tensor is conserved, $\nabla_a t^{ab} = 0$, whenever h_{ab} satisfies the linearized field equations $\delta G^{ab} = 0$. As a consequence the energy momentum vector,

$$T^a = t^{ab}k_b\ , \tag{54}$$

is also conserved: $\nabla_a T^a = 0$. Since T^a could be used to define a conserved energy, it is fortunate that it is related to t^a in the following way (see Trautman 1964):

$$T^a = t^a + \nabla_b s^{[ab]}\ , \tag{55}$$

where the antisymmetric tensor $s^{[ab]}$ is defined by

$$s^{[ab]} = +\frac{1}{16\pi}\epsilon^{abeg}\epsilon^{fdc}{}_{g}\nabla_d h_{ec}h_{fl}k^l\ . \tag{56}$$

Therefore, an energy defined in terms of T^a would differ from that defined in terms of t^a only by a surface integral. From equation (56) it is clear that T^a depends on first and second derivatives of the metric; consequently, it cannot be evaluated strictly in terms of the initial data for the field h_{ab}. For these reasons, we believe that the best representation of the energy-momentum contained in the perturbed gravitational field (as seen by an observer moving along the trajectories of k^a) is given by t^a. This choice agrees with that made by Palmer (1978) by somewhat different arguments.

The Killing vector field k^a is necessarily timelike within the stellar model; however, far away from the rotation axis it necessarily becomes spacelike if the star is rotating. Therefore, the current t^a only makes sense as an energy-momentum current in the region of spacetime near the rotation axis where k^a is timelike. It is appropriate therefore to restrict the domain of integration in the definition of the energy functional (e.g., in eqs. [39] and [47]) to include only a subset of the surface Σ where k^a is timelike. We choose to limit the domain of integration to include only the intersection of the surface Σ with the interior of the unperturbed stellar model.

Restricting the domain of integration of the energy functional to include only the interior of the star has certain advantages and certain disadvantages, compared to extending the integration to spatial infinity. The energy E represents the energy contained in the perturbed stellar model plus the energy in gravitational radiation. By restricting the domain of integration, the energy functional describes the energy in the perturbed star itself while ignoring the energy in gravitational radiation in the rest of the universe. When the energy is restricted in this way, it becomes a more appropriate tool for the study of the stability of the star itself. The disadvantage of truncating the domain of integration in the energy is that the timelike boundary integral in the calculation of the time derivative of the energy does not vanish, unless this boundary occurs at spatial infinity. Consequently, the expression for the time derivative of the energy given in equation (41) must be modified as follows:

$$E(\Sigma_2) - E(\Sigma_1) = -\int_\Lambda t^a d\Lambda_a - \int_V \frac{1}{\lambda}\left[2\eta\delta\sigma_{ab}\delta\sigma^{ab} + \zeta(\delta\theta)^2 + \frac{1}{\kappa T}\delta q^a\delta q_a\right]dV\ . \tag{57}$$

The integrals on the right-hand side of equation (57) are performed over the timelike surface of the star Λ and the interior of the stellar model V, as depicted in Figure 1. Since the flux of gravitational energy through the surface of the star does not have definite sign, the first integral on the right-hand side of equation (57) has no definite sign. Consequently, the energy functional need no longer be monotonically decreasing.

The ambiguity in the sign of the integral of t^a arises from two distinct causes. First, the perturbed star may emit

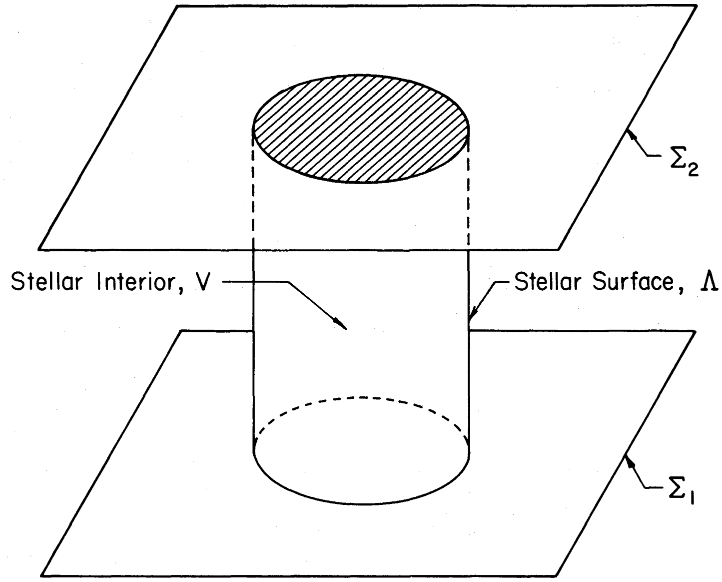

FIG. 1.—This is a spacetime diagram of an unperturbed stellar model. The vertical direction is timelike in this figure. The surfaces Σ_1 and Σ_2 are arbitrary, nonintersecting spacelike surfaces. The portion of the stellar interior to the future of Σ_1 and the past of Σ_2 is denoted V, and Λ represents the portion of the surface of the star which lies between Σ_1 and Σ_2.

gravitational radiation into the surrounding universe, or it could absorb radiation if the radiation was carefully aimed at the star. Second, the stress energy of the gravitational field is so poorly defined locally that two timelike observers at a given point in spacetime may not even agree on the sign of the flux of energy across a small element of surface.

The first type of ambiguity is physical and can be eliminated simply by limiting consideration to purely outgoing solutions to the perturbation equations. These can be identified by a careful examination of the asymptotic behavior of the solutions near null infinity ($\mathscr{I}$). The second type of ambiguity cannot be eliminated as far as we know. The nature of the ambiguity can be more fully explained, however. The gravitational momentum flux $t^a(\boldsymbol{k})$ defined in equation (50) is the canonical momentum associated with the Killing vector field k^a. Analogous conserved momenta can be defined for the other Killing vector fields, e.g., τ^a and ϕ^a. The energy flux integrals over the stellar surface, as seen by the two sets of observers moving along k^a and τ^a differ by an angular momentum flux integral:

$$\int_\Lambda t^a(\boldsymbol{k})d\Lambda_a = \int_\Lambda t^a(\boldsymbol{\tau})d\Lambda_a + \Omega \int_\Lambda t^a(\boldsymbol{\phi})d\Lambda_a \,. \tag{58}$$

The angular momentum flux can have either sign, and its magnitude can exceed that of the energy flux. Consequently, a positive energy flux as determined by $t^a(\boldsymbol{\tau})$ could have a negative energy flux as determined by $t^a(\boldsymbol{k})$. (Indeed, this is the case for the class of perturbations found to exhibit the generic gravitational radiation instability by Friedman 1978.)

This lengthy discussion of the gravitational contribution to the energy flux has revealed the following situation: the energy $E(\Sigma)$ will not be monotonically decreasing for all classes of perturbations of all stellar models. The energy will be monotonically decreasing, however, for those perturbations where the gravitational energy flux term is positive and also for those perturbations whose dissipation is dominated by the viscosities and thermal conductivity, i.e., when

$$\int_V \frac{1}{\lambda}\left[2\eta\delta\sigma_{ab}\,\delta\sigma^{ab} + \zeta(\delta\theta)^2 + \frac{1}{\kappa T}\,q^{ab}\delta q_a\,\delta q_b\right]d^4x \geq \left|\int_\Lambda t^a(\boldsymbol{k})d\Lambda_a\right| . \tag{59}$$

In the other limit, when gravitational radiation dominates the dissipation, the energy functional used by Friedman (1978) (the analog of eq. [37] where the globally timelike Killing vector τ^a is substituted for k^a) will be monotonically decreasing. For intermediate cases, no monotonically decreasing energy of this type may exist at all.

V. SHORT LENGTH SCALE PERTURBATIONS

A class of perturbations for which the energy constructed in the last section is monotonically decreasing is the "local" perturbations. These perturbations have the defining property that the metric (gravitational) perturbations are much smaller than the fluid perturbations. Seguin (1975) and Kandrup (1982) argue that all perturbations having sufficiently short characteristic length scale have this property; Friedman (1978) rigorously shows that a certain set of short length scale perturbations of perfect fluid stars have this property. For this reason, we will hereafter call these perturbations "short length scale" perturbations.

In the limit that the metric perturbations are negligible, the energy functional, equation (47), reduces to the following

$$E(\Sigma) = \int_\Sigma \left\{ \tfrac{1}{2}(\rho + p)k^a q_{bc}\, \delta u^b \delta u^c + \frac{1}{\lambda}\, \delta u^a \delta p + \tfrac{1}{2}k^a \frac{1}{\rho + p}\left[\left(\frac{\partial \rho}{\partial p}\right)_s (\delta p)^2 + \left(\frac{\partial \rho}{\partial s}\right)_p \frac{dp}{ds}(\delta s)^2\right] - \frac{1}{\lambda}(2\eta \delta\sigma^a{}_b + \zeta q^a{}_b \delta\theta - u^a \delta q_b)\delta u^b + \frac{1}{\lambda}\, \delta q^a \frac{\delta T}{T}\right\} d\Sigma_a \,. \tag{60}$$

We note that this expression still depends explicitly on the viscosities and thermal conductivity. The time derivative of this expression will be determined by equation (57). The gravitational momentum flux t^a is (by definition) negligible compared to the fluid dissipation terms. Consequently, the following inequality will be satisfied:

$$\left| \int_\Lambda t^a d\Lambda_a \right| \ll \int_V \frac{1}{\lambda}\left[2\eta \delta\sigma_{ab}\delta\sigma^{ab} + \zeta(\delta\theta)^2 + \frac{1}{\kappa T}\,\delta q^a \delta q_a\right] dV \,, \tag{61}$$

and the energy will be monotonically decreasing unless there are short length scale perturbations for which the fluid dissipation terms vanish identically. To investigate the possibility of such nondissipative short length scale perturbations, we must find the solutions to the system of equations:

$$0 = \delta\theta = \delta\sigma_{ab} = \delta q^a = h_{ab} \,. \tag{62}$$

Using equations (20)–(22), we find the following:

$$0 = q_a{}^c q_b{}^d[\nabla_c(\mathscr{L}_{\boldsymbol{k}}\, \xi_d + f k_d) + \nabla_d(\mathscr{L}_{\boldsymbol{k}}\, \xi_c + f k_c)] \,, \tag{63}$$

$$0 = q_a{}^c u^d[\nabla_c(\mathscr{L}_{\boldsymbol{k}}\, \xi_d + f k_d) + \nabla_d(\mathscr{L}_{\boldsymbol{k}}\, \xi_c + f k_c)] \,, \tag{64}$$

where

$$f = -\Delta T/T + u^a u^b \nabla_a \xi_b \,. \tag{65}$$

The first of these, equation (63), follows from the vanishing of the shear and expansion, while equation (64) follows from the vanishing of the heat current. Furthermore, the Lagrangian changes Δs and Δn are constant along the fluid flow lines for these perturbations,

$$u^a \nabla_a \Delta s = u^a \nabla_a \Delta n = 0 \,. \tag{66}$$

Thus, $\Delta T/T$ is also constant along the flow lines. This fact and the identity

$$u^a u^b \nabla_a(\mathscr{L}_{\boldsymbol{k}}\, \xi_b + k_b u^c u^d \nabla_c \xi_d) = 0 \,, \tag{67}$$

along with equations (63)–(64) imply that $\mathscr{L}_{\boldsymbol{k}}\, \xi^a + f k^a$ is a Killing vector field. Since the spacetime of the unperturbed star admits exactly two Killing vector fields, we find that

$$\mathscr{L}_{\boldsymbol{k}}\, \xi^a = -f k^a + \omega_1 k^a + \omega_2 \phi^a \tag{68}$$

for constants ω_1 and ω_2. The Eulerian change in the fluid velocity for these nondissipative perturbations is given by

$$\delta u^a = \lambda \omega_2\, q^a{}_b \phi^b \,. \tag{69}$$

This perturbation corresponds to a rigid rotation of the star, i.e., a change in the angular velocity Ω. Such a perturbation is certainly not of the short length scale class. Consequently, *there are no nondissipative short length scale perturbations*. Therefore, inequality (61) is satisfied for all short length scale perturbations, and the time evolution of the energy is given by

$$E(\Sigma_2) - E(\Sigma_1) = -\int_V \frac{1}{\lambda}\left[2\eta \delta\sigma_{ab}\delta\sigma^{ab} + \zeta(\delta\theta)^2 + \frac{1}{\kappa T}\,\delta q^a \delta q_a\right] dV \,. \tag{70}$$

The energy is monotonically decreasing for short length scale perturbations.

To demonstrate the existence of such local or short length scale perturbations, we consider the perturbations used by Friedman (1978) in his analysis of the stability of perfect fluid stellar models. Consider the set of perturbations which have sinusoidal dependence in the azimuthal coordinate ϕ. Thus, we let

$$\xi^a = \alpha^a \cos m\phi \,, \tag{71}$$

$$\mathscr{L}_{\boldsymbol{k}}\, \xi^a = \beta^a \cos m\phi + \gamma^a \sin m\phi \,, \tag{72}$$

where the vectors α^a, β^a, and γ^a are arbitrary vector fields which are constant along the vector field ϕ^a:

$$\mathscr{L}_\phi \alpha^a = \mathscr{L}_\phi \beta^a = \mathscr{L}_\phi \gamma^a = 0 .$$

The vector field α^a is taken to be independent of the integer m, while β^a and γ^a may depend on m only linearly. (Friedman 1978 chose β^a independent of m and $\gamma^a = -m\Omega\alpha^a + \hat{\gamma}^a$, with $\hat{\gamma}^a$ independent of m.) The metric perturbation h_{ab} is also taken to have sinusoidal dependence:

$$h_{ab} = h^0{}_{ab} \sin m\phi + h^1{}_{ab} \cos m\phi , \tag{73}$$

$$\mathscr{L}_k h_{ab} = h^2{}_{ab} \sin m\phi + h^3{}_{ab} \cos m\phi . \tag{74}$$

The tensors $h^0{}_{ab}$, $h^1{}_{ab}$, ..., have zero Lie derivatives along ϕ^a. The m dependence of these tensors is determined by the Einstein field equations (24). A proper rigorous analysis of this dependence must be done in the context of the appropriate weighted Sobolev spaces. Friedman (1978) has performed this analysis for the perfect fluid case. We choose here to present only an heuristic analysis.

The perturbed Einstein tensor, equation (17), is a second-order differential operator acting on h_{ab}. The terms in this expression having the largest m dependence will be an m^2 coefficient of h_{ab} and a coefficient of $\mathscr{L}_k h_{ab}$ linear in m. The perturbed perfect fluid stress tensor contains terms of the form $m\xi^a$ and $\mathscr{L}_k \xi^a$. Thus, the perturbed perfect fluid stress tensor for these perturbations contains terms which are at most linear in m. The perturbed Einstein equations (24) require then that the perturbed metric may not increase with m faster than $h_{ab} \sim m^{-1}$ and $\mathscr{L}_k h_{ab} \sim m^0$. This naive analysis ignores the existence of the two gravitational wave degrees of freedom. On a particular spacelike slice one is always free to specify the amplitudes of these two gravitational wave degrees of freedom, and one can always choose them to be large compared to the fluid perturbations. In the large m (or short wavelength) limit of primary concern here, these gravitational wave degrees of freedom completely decouple from the matter perturbations. In the short wavelength limit, a small region of the star appears to the wave to be nearly homogeneous and isotropic. The standard cosmological analysis (see Weinberg 1972) reveals the decoupling of the matter perturbations from the gravitational wave degrees of freedom in this case. Thus, the presence of these short wavelength gravitational waves does not affect the structure or stability of the star at all. In the discussion that follows, we will restrict the amplitudes of these free gravitational degrees of freedom to be consistent with the amplitudes of the constrained portions of the gravitational field. Thus, in the case of perfect fluid stars these amplitudes will be taken to obey $h_{ab} \sim m^{-1}$ and $\mathscr{L}_k h_{ab} \sim m^0$. For these perturbations we can see that the metric perturbations become negligible compared to the fluid perturbations for sufficiently large values of m (i.e., very short length scales).

The perturbations involving a dissipative fluid are more subtle. The initial value problem does not appear to be well posed for these fluids. It does not appear to be possible to evaluate δq^a, $\delta\sigma^{ab}$, and $\delta\theta$ strictly in terms of the initial data ξ^a and $\mathscr{L}_k \xi^a$. For the wavelike solutions considered here, time derivatives are generally smaller than spatial derivatives by a factor of the propagation velocity. These time derivatives should exhibit the same m dependence as spatial derivatives, however. Consequently, the perturbed stress tensor for the perturbations in equations (71)–(72) contains terms such as $m^2\xi^a$ and $m\mathscr{L}_k \xi^a$ which are proportional to m^2. From the perturbed Einstein equations one is led therefore to the conclusion that the perturbed metric will vary like $h_{ab} \sim m^0$ and $\mathscr{L}_k h_{ab} \sim m$. Thus, it is no longer possible to conclude *a priori* that h_{ab} will be small compared to ξ^a for large values of m. In fact, such a relationship continues to hold for reasonable values of the dissipation coefficients. The coefficient of the highest power of m in the perturbed Einstein equations has the following qualitative form:

$$m^2 h_{ab} \sim m^2 \eta \xi^a .$$

If the dissipation coefficient is sufficiently small, then the perturbed metric will still remain much smaller than the perturbed fluid variables. To see how severe this constraint on the dissipation coefficient is, we introduce a length scale l characteristic of the unperturbed star: $|\xi^a| < l$. In units where the speed of light and the gravitational constant have not been set to unity, the required constraint on the viscosity coefficient is given by

$$Gc^{-3}\eta l = 2.5 \times 10^{-39} \eta(\text{poise}) l(\text{cm}) \ll 1 .$$

For normal stars, $l \lesssim 10^{10}$ cm which yields the constant $\eta \ll 4 \times 10^{28}$ poise. This constraint is easily satisfied by any known material. Analogous constraints exist for ζ the bulk viscosity and for κT the thermal conductivity times the unperturbed temperature. The constraint is even weaker on smaller objects like neutron stars. Thus, we see that the metric perturbations will be negligible compared to the fluid perturbations for sufficiently short length scale perturbations, even if dissipation is present in the fluid.

Let us now examine how the fluid dissipation mechanisms dominate the time evolution of the energy for these perturbations. For fixed (nonzero) values of the dissipation coefficients, the perturbed stress energy tensor will depend on the integer m quadratically for the perturbations in equations (71) and (72). The dissipative contribution to the time derivative of the energy (the right-hand side of eq. [61]) will depend on m quartically. The gravitational momentum flux t^a will depend on m at most linearly for these perturbations. Thus, for fixed values of the dissipation coefficients, the time derivative of the energy will be dominated by the fluid dissipation terms for all $|m| > C$, for some finite constant C. The constant C will depend on the structure of the unperturbed star and the values of the dissipation coefficients. Therefore, for large enough values of m, the energy E will be a monotonically decreasing function of time for these perturbations.

VI. A STABILITY CRITERION FOR ROTATING STARS

A monotonic decreasing energy functional can be used to test the stability of rotating stellar models. If such an energy is positive for all possible sets of initial data for the perturbation functions (ξ^a, Δs, h_{ab}), then the stellar model must be stable. In this case the energy is bounded below by zero, so that the perturbation of the star must remain bounded and would presumably decrease to zero as it evolved. Alternatively, if the energy functional were negative for some appropriate initial data (ξ^a, Δs, h_{ab}), then it would be possible for such data to grow without bound, while still allowing the energy to decrease continuously to negative infinity.

No one has yet proven that stability criteria based on an energy argument, such as that outlined above, give necessary or sufficient conditions for the stability of general relativistic stellar models. Rigorous theorems do exist, however, for some related physical problems. The method of Laval, Mercier, and Pellat (1965) (see also Barston 1970) can be used to show that the positivity of a certain energy functional is a necessary condition for the stability of axisymmetric perturbations of rotating Newtonian stars. Lindblom (1983) has shown that the positivity of the Newtonian limit of E (eq. [39]) is a necessary condition for the stability (with respect to any perturbation) of rotating Newtonian stars composed of dissipative fluids. These works also show that the positivity of the energy is sufficient to bound the velocity perturbations of the fluid, but it is not known whether this guarantees boundedness of all the physical perturbations. Similarly, Kandrup (1982) has argued that the positivity of an energy functional is a necessary condition for the stability of short length scale axisymmetric perturbations of general relativistic rotating stars. Finally, Friedman (1982) has argued that the positivity of an energy functional is a necessary condition for the stability of rotating general relativistic stellar models to all axisymmetric perturbations. Thus, while no one has yet proven that the stability criteria based on energy functionals for more general systems (such as the one considered here, or the one considered by Friedman 1978) are infallible tests for stability, it is generally expected that they are reliable tests.

The energy functional constructed by us in § IV is not monotonically decreasing for all physical perturbations of all stellar models. Consequently, it cannot be used, in general, to test the stability of rotating stellar models. There are, however, interesting classes of perturbations in all stellar models for which our energy is monotonically decreasing (such as the short length scale perturbations discussed in § V). The stability of perturbations which belong to these classes may be tested using our energy. Also, there may exist some stellar models for which all perturbations have strictly decreasing energy (for example slowly rotating stars).

Let us use the stability criterion and the energy functional constructed in § IV to examine the stability of the short length scale perturbations. The energy for these perturbations is given by equation (60). A more useful version of this energy may be obtained by introducing the normal vector to the spacelike surface, n^a, and the velocity of the observers along the trajectories n^a relative to the fluid:

$$v^a = (\delta^a{}_b + u^a u_b)n^b/u_c n^c \; . \tag{75}$$

It follows that the energy may be written as

$$E(\Sigma) = \frac{1}{2}\int_\Sigma \left\{(\rho + p)\gamma_{bc}\,\delta u^b \delta u^c + \frac{1}{\rho + p}\left(\frac{\partial \rho}{\partial s}\right)_p \frac{dp}{ds}(\delta s)^2 + (\rho + p)^{-1}\left(\frac{\partial \rho}{\partial p}\right)_s \left[\delta p + (\rho + p)\left(\frac{\partial p}{\partial \rho}\right)_s v_b \delta u^b\right]^2 \right.$$
$$\left. + (\rho + p)\left[1 - \left(\frac{\partial p}{\partial \rho}\right)_s v_b v^b\right](v_c \delta u^c)^2 (v_d v^d)^{-1} + 2\delta q_b\left[\delta u^b + v^b\left(\frac{\delta T}{T}\right)\right] - 4\eta\,\delta\sigma_{bc} v^b \delta u^c - 2\zeta\,\delta\theta\, v_b \delta u^b\right\} k^a d\Sigma_a \, , \tag{76}$$

where γ_{ab} is the two-dimensional metric orthogonal to u^a and n^a:

$$\gamma_{ab} = g_{ab} + u_a u_b - v_a v_b (v_c v^c)^{-1} \; . \tag{77}$$

When one chooses the usual $t =$ constant spacelike surfaces, then γ_{ab} reduces to the metric on the two-dimensional surfaces which are orthogonal to the orbits of the Killing vector fields. We see that the perfect fluid portions of the energy, in this limit, are positive definite as long as the Schwarzschild criterion,

$$\left(\frac{\partial \rho}{\partial s}\right)_p \frac{dp}{ds} > 0 \, , \tag{78}$$

and the additional inequality,

$$\left(\frac{\partial p}{\partial \rho}\right)_s v_a v^a \le 1 \, , \tag{79}$$

are satisfied. As pointed out by Kandrup (1982), the latter inequality will be automatically satisfied whenever the adiabatic sound speed $(\partial p/\partial \rho)_s$ is less than unity (the speed of light in our units) since $0 \leq v^a v_a \leq 1$. In fact, the requirement of subluminal sound propagation is strict. One is always free to choose which surface Σ is to be used, and $v^a v_a$ can be made arbitrarily close to one in an area large enough to enclose these local perturbations. The rotating energy functional presented by Kandrup (1982) appears to be a noncovariant version of our equation (76) when the dissipation coefficients (η, ζ, and κ) are negligible.

The terms in equation (76) which are proportional to the dissipation coefficients do not have definite sign. To see this we note that the unperturbed stellar model is unchanged when the Killing vector fields are transformed by the "parity" transformation as follows: $\tau^a \to -\tau^a$ and $\phi^a \to -\phi^a$. Consider a spacelike surface Σ with normal vector n^a and perturbation data (ξ^c, Δs, h_{ab}) on this surface. We apply the "parity" transformation to the perturbation quantities to obtain a new set of data. Under the "parity" transformation, we note that the following transformations hold:

$$(\xi^a, \Delta s, h_{ab}, \delta n, k^a d\Sigma_a, \delta q^a) \to (\xi^a, \Delta s, h_{ab}, \delta n, k^a d\Sigma_a, \delta q^a)\,, \tag{80}$$

$$(k^a, v^a, \delta u^a, n^a, \delta\theta, \delta\sigma_{ab}) \to -(k^a, v^a, \delta u^a, n^a, \delta\theta, \delta\sigma_{ab})\,. \tag{81}$$

It follows therefore that the dissipative terms in the energy functional change sign under this parity transformation while the nondissipative terms do not. A necessary condition for stability therefore is that the nondissipative terms themselves must be positive definite. We have seen that this condition reduces to the relativistic Schwarzschild criterion, equation (78), and the causal propagation of sound waves. The sufficient condition for stability is that the magnitude of the dissipative contributions to the energy be smaller than the perfect fluid contributions, for all possible short length scale perturbations:

$$\int_\Sigma \left\{(\rho + p)\gamma_{bc}\,\delta u^b \delta u^c + \frac{1}{\rho + p}\left(\frac{\partial \rho}{\partial s}\right)_p \frac{dp}{ds}(\delta s)^2 + (\rho + p)^{-1}\left(\frac{\partial \rho}{\partial p}\right)_s \left[\delta p + (\rho + p)\left(\frac{\partial p}{\partial \rho}\right)_s v_b \delta u^b\right]^2 \right.$$
$$\left. + (\rho + p)\left[1 - \left(\frac{\partial p}{\partial \rho}\right)_s v_b v^b\right](v_c\,\delta u^c)^2 (v_d v^d)^{-1}\right\} k^a d\Sigma_a$$
$$> \left| \int_\Sigma \left\{\delta q_b \left[\delta u^b + v^b\left(\frac{\delta T}{T}\right)\right] - 4\eta\,\delta\sigma_{bc} v^b \delta u^c - 2\zeta\,\delta\theta\, v_b\,\delta u^b\right\} k^a d\Sigma_a \right|\,. \tag{82}$$

We have not been able to determine the consequences of this condition on the full class of short length scale perturbations. In Appendix B we investigate this condition on a class of plane wave–like perturbations. We find that there always exist sufficiently short wavelength perturbations which violate this condition. This does not imply that real fluids will exhibit short wavelength instabilities. Real fluid materials do not satisfy the fluid equations of motion for arbitrarily short wavelength motions. Wavelengths shorter than the average interparticle separation $n^{-1/3}$ or the mean free path between collisions are not physical. Let λ_c represent the shortest acceptable wavelength. In Appendix B we show that the energy functional will be positive for the physically relevant plane wave–like perturbations as long as the thermal conductivity is sufficiently small:

$$\kappa < \frac{2\lambda_c c^2}{v_s T}\left[\frac{\alpha}{n c_p} - (\rho c^2 + p)^{-1}\right]^{-1}\,, \tag{83}$$

where c is the speed of light, v_s the adiabatic sound speed, α the thermal expansion coefficient, and c_p the specific heat at constant pressure. No analogous constraints on the viscosity coefficients arise from these perturbations. We show in the appendix that this inequality is trivially satisfied for normal materials.

The secular instability which appears to occur when equation (83) is violated may be a manifestation of the unphysical superluminal propagation of thermal fluctuations in the simple dissipation theory used here. (We thank B. Schutz for pointing this out to us.) Our finding that equation (83) is always satisfied for normal materials seems to be consistent with this viewpoint and with Weymann's (1967) claim that the simple dissipation theory has no such unphysical effects within the domain of applicability of the hydrodynamic equations themselves. To investigate whether a secular instability actually does occur when equation (83) is violated, it would seem necessary to invoke a more physical theory of dissipation such as that developed by Israel (1976). Such a calculation is beyond the scope of the present work.

In summary, we have shown that rotating stars composed of dissipative fluids will be stable to all short length scale perturbations as long as (*a*) they satisfy the relativistic Schwarzschild condition (eq. [78]); (*b*) they have subluminal adiabatic sound speeds, $v_s < c$; and (*c*) the dissipation coefficients are sufficiently small (in particular the thermal conductivity must satisfy eq. [83]).

This analysis reveals that the generic gravitational radiation secular instability (discovered by Friedman 1978 in his analysis of perfect fluid perturbations of relativistic stellar models) is not present in stars composed of dissipative fluids. This results from the fact that the perturbations found to exhibit the gravitational radiation secular instability by Friedman belong to the short length scale class. When dissipation is present in the fluid, we have shown that these perturbations are stable subject to the three thermodynamic constraints described above. Thus, the generic gravitational radiation secular instability is not present in dissipative fluids.

To make this argument a bit more precise, let us analyze in more detail the difference between the dissipative and nondissipative cases. For nondissipative fluids the energy E is no longer monotonically decreasing because of the ambiguity in the sign of the gravitational energy flux. Consequently, Friedman based his stability analysis on a different energy functional E_c. The energy E_c is analogous to our $\tilde{E}$ (eq. [37]), with k^a replaced by the globally time-like Killing vector field τ^a. The energy E_c is monotonically decreasing for dissipation-free fluids. Friedman showed that the energy E_c could be made negative for the sinusoidal perturbations described by equations (71)–(74) if the integer m was chosen large enough. In particular, he showed that if $|m\Omega| > C_1$ for some constant C_1 that the energy E_c would be negative. The constant C_1 depends on the structure of the equilibrium stellar model, and C_1/Ω is not bounded as the angular velocity Ω goes to zero (along a continuous sequence of equilibrium models).

When dissipation is present in the fluid, the energy E_c is no longer monotonically decreasing. The expression for its time derivative contains dissipative terms of ambiguous sign. It is appropriate, however, to use the energy E to analyze the stability of such perturbations, as we have shown in § V. We showed, in particular, that it was appropriate to use the energy E to analyze stability whenever $|m| > C$ for some constant C, which depends on the structure of the background star and the values of the dissipation coefficients. Whenever $C_1 > |\Omega| C$, it follows that all short length scale perturbations (including all those found unstable by Friedman's analysis) are stable, subject only to the three thermodynamic conditions described above. This inequality will always be satisfied by stars with sufficiently small angular velocities, Ω. Thus, the gravitational radiation secular instability is not generic in rotating stars. It will not exist in sufficiently slowly rotating stars. The stabilization of the gravitational radiation secular instability by the fluid dissipation mechanisms is analogous to the effects of these processes on the modes of the Maclaurin spheroids found by Lindblom and Detweiler (1977) and Comins (1979*a*, *b*).

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APPENDIX A

GAUGE INVARIANCE OF THE ENERGY FUNCTIONAL

The computation required to show that the energy functional defined in equation (39) is equivalent to the manifestly gauge invariant expression given in equation (47) is fairly straightforward but very lengthy and tedious. We could not present the details of that calculation without at least doubling the length of this paper. We will present instead a brief sketch of the calculation. We hope that this outline will provide enough information to guide anyone interested in reproducing the computation and some intermediate steps to mark his progress.

The expression for the energy in equation (39) contains three types of terms: (*a*) perfect fluid perturbations; (*b*) viscosity, thermal conductivity, and nonadiabatic terms; and (*c*) purely gravitational terms. We find that it is helpful to deal with each type of term separately. Let us begin with the perfect fluid contributions to the energy. These terms are given by

$$E^a_{\mathrm{PF}} = U^{abcd}(\mathscr{L}_k \xi_b \nabla_c \xi_d - \xi_b \nabla_c \mathscr{L}_k \xi_d) + V^{cdab}(h_{cd} \mathscr{L}_k \xi_b - \xi_b \mathscr{L}_k h_{cd}) . \tag{A1}$$

The tensors U^{abcd} and V^{cdab} are defined in equations (31)–(32). The basic idea of the calculation is to replace the Lagrangian perturbation quantities (ξ^a and $\mathscr{L}_k \xi^a$) by Eulerian quantities, and possibly some divergence terms. Perhaps the most helpful expressions for this purpose are

$$\lambda q^a{}_b \mathscr{L}_k \xi^b = \delta u^a - \tfrac{1}{2} u^a u^b u^c h_{bc} , \tag{A2}$$

and

$$q^a{}_b \nabla_a (n\xi^b) = -\delta n - \tfrac{1}{2} n q^{ab} h_{ab} . \tag{A3}$$

The expression can be simplified using the equilibrium fluid equations:

$$u^a \nabla_a u^b = -(\rho + p)^{-1} \nabla^b p = -T^{-1} \nabla^b T . \tag{A4}$$

This equation implies that the fluid is barotropic; thus, the spatial gradients of the thermodynamic quantities are all parallel. We denote by $d\rho/ds$ the proportionality factor in

$$\nabla_a \rho = \frac{d\rho}{ds} \nabla_a s \, . \tag{A5}$$

We can also use the perturbed fluid equations to help simplify the expression:

$$\delta u^a \nabla_a n + u^a \nabla_a \delta n + n \nabla_a \delta u^a + \tfrac{1}{2} n u^a \nabla_a h^b{}_b = 0 \, , \tag{A6}$$

and

$$(\rho + p)(\delta u^a \nabla_a u^b + u^a \nabla_a \delta u^b + u^a u^c \delta \Gamma^b{}_{ac}) + (\delta\rho + \delta p) u^a \nabla_a u^b$$
$$= -q^{ab} \nabla_a \delta p + (h^{ab} - \delta u^a u^b) \nabla_a p - q^b{}_c \nabla_a (u^a \delta q^c + u^c \delta q^a - 2\eta \delta\sigma^{ac} - \zeta q^{ac} \delta\theta) \, . \tag{A7}$$

The perturbed Christoffel symbol is given by

$$\delta \Gamma^b{}_{ac} = \tfrac{1}{2}(\nabla_c h^b{}_a + \nabla_a h^b{}_c - \nabla^b h_{ac}) \, . \tag{A8}$$

These equations, along with numerous straightforward applications of the fact that k^a is a Killing vector field, result in the expression

$$\lambda E^a_{\rm PF} = (\rho + p)(u^a q^b{}_c \delta u_b \delta u^c + 2 q^a{}_b \delta u^b u_c \delta u^c + \tfrac{1}{2} u^a u_b \delta u^b q^{cd} h_{cd}) + 2 q^a{}_b \delta u^b \delta p + u^a u_b \delta u^b \delta\rho + \tfrac{1}{2} u^a \delta p q^{bc} h_{bc}$$
$$+ (\rho + p)^{-1} u^a \left[\delta p \delta \rho + \left(\frac{\partial p}{\partial s} \right)_\rho \delta s \left(\delta\rho - \frac{d\rho}{ds} \delta s \right) \right] + \lambda D^a + \lambda \tilde{E}^a_{\rm PF} \, , \tag{A9}$$

where D^a is a divergence given by

$$D^a = 2\nabla_b \{ p \mathscr{L}_k \xi^{[a} \xi^{b]} + \lambda^{-1}(\rho + p) u^{[a} q^{b]}{}_c \xi^c u_d \delta u^d + \lambda^{-1} \delta p u^{[a} q^{b]}{}_c \xi^c \} \, , \tag{A10}$$

and $\tilde{E}^a_{\rm PF}$ are dissipative terms which are given by

$$\lambda \tilde{E}^a_{\rm PF} = -\left[n T u^a u_b + \left(\frac{\partial p}{\partial s} \right)_n q^a{}_b \right] \delta u^b \Delta s - u^a (\rho + p)^{-1} n T \delta p \Delta s - u^a (\rho + p)^{-1} \left(\frac{\partial p}{\partial s} \right)_\rho \left(\delta\rho - \frac{d\rho}{ds} \delta s \right) \Delta s$$
$$- q^a{}_b \xi^b n^{-1} \left(\frac{\partial p}{\partial s} \right)_n \nabla_c \left(\frac{\delta q^c}{T} \right) + u^a q_{bd} \xi^d \nabla_c (u^b \delta q^c + u^c \delta q^b - 2\eta \delta\sigma^{bc} - \zeta q^{cb} \delta\theta) \, . \tag{A11}$$

This expression for $E^a_{\rm PF}$ may not have an elegant appearance; however, it is gauge invariant except for the divergence D^a and the dissipative contributions $\tilde{E}^a_{\rm PF}$.

Next we consider the dissipative contributions to the energy in equation (39):

$$\lambda E^a_D = u^a n \Delta s \left[\left(\frac{\partial T}{\partial n} \right)_s \Delta n + \frac{1}{2} \left(\frac{\partial T}{\partial s} \right)_n \Delta s \right] + T^{-1} \Delta T \delta q^a + u_b \Delta u^b (\delta q^a + u^a n T \Delta s)$$
$$+ \lambda \mathscr{L}_k \xi_b \delta_D T^{ab} + \tfrac{1}{4} u^a h_{bc} \delta_D T^{bc} - \tfrac{1}{2} u^a \xi_b \nabla_c (\delta_D T^{bc}) \, . \tag{A12}$$

This expression must be combined with $\tilde{E}^a_{\rm PF}$, the dissipative terms which appeared in the manipulations involving the perfect fluid portion of the energy. To simplify the resulting expression one uses the perturbed entropy equation,

$$n u^a \nabla_a \Delta s + \nabla_a (\delta q^a / T) = 0 \, , \tag{A13}$$

along with assorted thermodynamic trivia from § II. The resulting expression for the dissipative terms is given by

$$\lambda E^a_D + \lambda \tilde{E}^a_{\rm PF} = (\tfrac{1}{2} u^a h^b{}_c + 2\delta^a{}_c \delta u^b)(u^c \delta q_b + u_b \delta q^c - 2\eta \delta\sigma_b{}^c - \zeta q_b{}^c \delta\theta)$$
$$+ 2\delta q^a \frac{\delta T}{T} - 2\lambda \nabla_b \left\{ \lambda^{-1} \left(\frac{\partial p}{\partial s} \right)_n \Delta s u^{[a} q^{b]}{}_c \xi^c \right\} . \tag{A14}$$

This expression is also gauge invariant, except for the last term which is a divergence.

Finally, the purely gravitational terms in the energy must be considered:

$$64\pi E^a_{\rm GW} = -\epsilon^{aceg} \epsilon^{bdf}{}_g (\mathscr{L}_k h_{cd} \nabla_b h_{ef} - h_{cd} \nabla_b \mathscr{L}_k h_{ef}) + 2\nabla_b (k^{[a} \epsilon^{b]ceg} \epsilon^{ldf}{}_g h_{cd} \nabla_l h_{ef}) \, . \tag{A15}$$

To simplify this expression we use the perturbed Einstein equations (17) and (24) and the fact that k^a is a Killing vector field to find

$$32\pi E^a_{\rm GW} = -2\epsilon^{aceg} \epsilon^{bdf}{}_g \mathscr{L}_k h_{cd} \nabla_b h_{ef} + \lambda^{-1} u^a \epsilon^{becg} \epsilon^{ldf}{}_g \nabla_b h_{cd} \nabla_l h_{ef} - 2\lambda^{-1} u^a (8\pi \delta T^{bc} - G^{bcde} h_{de}) h_{bc} \, . \tag{A16}$$

The total energy is the sum of the three types of contributions discussed above. From equations (A9), (A14), and (A16), this sum is given by

$$\begin{aligned}\lambda(E^a_{\rm PF} + E^a_D + E^a_{\rm GW}) = {} & (\rho + p)u^a q^b{}_c \delta u_b \delta u^c + 2(\rho + p)q^a{}_b \delta u^b u_c \delta u^c + \tfrac{1}{2}(\rho + p)u^a u_b \delta u^b q^{cd} h_{cd} + 2q^a{}_b \delta u^b \delta p + u^a u_b \delta u^b \delta \rho \\ & + \tfrac{1}{2}u^a \delta p q^{bc} h_{bc} + (\rho + p)^{-1} u^a \left[\delta p \delta \rho + \left(\frac{\partial p}{\partial s}\right)_\rho \delta s \left(\delta \rho - \frac{d\rho}{ds}\delta s\right)\right] \\ & + (\tfrac{1}{2}u^a h^b{}_c + 2\delta^a{}_c \delta u^b)(u^c \delta q_b + u_b \delta q^c - 2\eta \delta \sigma_b{}^c - \zeta q_b{}^c \delta\theta) \\ & + 2\delta q^a \frac{\delta T}{T} - (16\pi)^{-1} \lambda \epsilon^{aceg} \epsilon^{bdf}{}_g \mathscr{L}_k h_{cd} \nabla_b h_{ef} + (32\pi)^{-1} u^a \epsilon^{becg} \epsilon^{ldf}{}_g \nabla_b h_{cd} \nabla_l h_{ef} - \tfrac{1}{2}u^a \delta T^{bc} h_{bc} \\ & + (16\pi)^{-1} u^a G^{bcde} h_{bc} h_{de} + \lambda \tilde{D}^a \,, \end{aligned} \tag{A17}$$

where $\tilde{D}^a$ is a divergence defined by

$$\tilde{D}^a = D^a - 2\nabla_b \left\{\lambda^{-1} \left(\frac{\partial p}{\partial s}\right)_n \Delta s u^{[a} q^{b]}{}_c \xi^c\right\} . \tag{A18}$$

This expression is manifestly gauge invariant, except for the divergence term $\tilde{D}^a$. It follows that the integrated energy E will be gauge invariant as long as the boundary of the surface is outside the support of the fluid. (The integral of $\tilde{D}^a$ vanishes in this case.) To transform the expression

$$E(\Sigma) = \frac{1}{2}\int_\Sigma (E^a_{\rm PF} + E^a_D + E^a_{\rm GW}) d\Sigma_a \tag{A19}$$

using equation (A17) into the form which is given in equation (47), it is necessary to perform a few thermodynamic manipulations and use the definitions of $\delta T^a{}_b$, δu^b, etc.

APPENDIX B

THE ENERGY FOR PLANE WAVE–LIKE PERTURBATIONS

In this appendix we evaluate the energy functional for a rotating dissipative fluid, equation (76). We obtain an expression for the energy which is valid for very short wavelength plane wave–like perturbations. This expression reveals that all "mathematical" fluids will be unstable to sufficiently short length scale secular instabilities induced by the thermal conductivity. Real gases and liquids satisfy the fluid equations of motion only for sufficiently long wavelength perturbations: longer than the mean free path of particles in the fluid, for example. Using such a minimum acceptable wavelength, the expression for the energy functional gives an *upper bound* to the thermal conductivity κ which is allowed in a stable fluid.

Consider a point x in the fluid, and construct a smooth spacelike surface which passes through x and whose normal at x is u^a. We will consider a small subset of this surface containing the point x: small enough so that all of the fluid variables (n, s, u^a) of the unperturbed star are well approximated throughout the region by their values at x. We consider perturbations of the fluid which have the form of plane waves in the neighborhood of x:

$$\xi^b = \alpha^b \cos(l^a x_a) + \beta^b \sin(l^a x_a) \,, \tag{B1}$$

$$\mathscr{L}_k \xi^b = \gamma^b \cos(l^a x_a) \,, \tag{B2}$$

where α^a, β^a, and γ^a are covariantly constant vector fields at x which are orthogonal to the fluid velocity u^a (and hence are tangent to the surface). The coordinates x^a are local Cartesian-like coordinates at x. The vector l^a is the wavevector of the perturbations; it is covariantly constant and orthogonal to u^a at x. For short wavelength perturbations l^a will be large, and gradients of perturbation quantities will be large compared to the corresponding gradients of the unperturbed equilibrium quantities. We will also limit our consideration here to perturbations having adiabatic initial values, $\Delta s = 0$, which for these perturbations also satisfy $\delta s = 0$.

We now use these plane wave–like perturbations to obtain a simple expression for the energy functional, equation (76). In the expression for the energy we encounter terms whose spatial dependence is given by $\sin^2(l^a x_a)$, $\cos^2(l^a x_a)$, or by $\sin(l^a x_a)\cos(l^b x_b)$. When integrated over a number of wavelengths, the first two dependences are well approximated by their average value 1/2. The third type of term will have zero average value, and, consequently, these terms can be neglected. With these considerations in mind it is straightforward to obtain the following averaged expression for the nondissipative contributions to the energy:

$$(\rho + p)\delta u^a \delta u_a + (\rho + p)^{-1} \left(\frac{\partial \rho}{\partial p}\right)_s (\delta p)^2 = \tfrac{1}{2}\lambda^2 (\rho + p)\gamma^a \gamma_a + \tfrac{1}{2}n \left(\frac{\partial p}{\partial n}\right)_s [(\alpha_a l^a)^2 + (\beta^a l_a)^2] \,. \tag{B3}$$

Note that with our choice of surface the vector v^a (see eq. [75]) vanishes at x and, consequently, can be neglected in a small neighborhood of x.

There are two types of terms in the energy functional involving the dissipation coefficients. The first type of term is proportional to the vector v^a, which vanishes for the choice of spacelike slice taken here. (We could, in principle, examine other slices; however, ambiguities in the initial value problem for these dissipative fluids on arbitrary slices prevent us from presenting that more general analysis here.) The second type of term involves only $\delta u_a \delta q^a$. The expression for the perturbed thermal current δq^a is very complicated; however, a simple expression may be derived when the perturbed acceleration of the fluid is due primarily to nondissipative effects. In this approximation the thermal current is given by

$$\delta q^a = -\kappa q^{ab}\left[\nabla_b \delta T - \frac{T}{\rho + p}\nabla_b \delta p\right]. \tag{B4}$$

The dissipative contribution to the energy functional in equation (76) for our plane wave–like solutions is given therefore by

$$\delta u_a \delta q^a = -\tfrac{1}{2}\lambda n\kappa\left[\left(\frac{\partial T}{\partial n}\right)_s - \frac{T}{\rho + p}\left(\frac{\partial p}{\partial n}\right)_s\right](\alpha^a l_a)(\gamma^b l_b). \tag{B5}$$

The total averaged energy for these plane wave–like perturbations, therefore, is given by

$$E = \tfrac{1}{2}\lambda^2(\rho + p)\gamma^a\gamma_a + \tfrac{1}{2}n\left(\frac{\partial p}{\partial n}\right)_s[(\alpha^a l_a)^2 + (\beta^a l_a)^2] - \tfrac{1}{2}\lambda n\kappa\left[\left(\frac{\partial T}{\partial n}\right)_s - \frac{T}{\rho + p}\left(\frac{\partial p}{\partial n}\right)_s\right](\alpha^a l_a)(\gamma^b l_b). \tag{B6}$$

In this expression we are free to adjust the values of the vectors α^a, β^a, and γ^a since these comprise the initial data for the fluid perturbations.

To investigate the possibility of instability of these perturbations, we must determine if the energy in equation (B6) can be negative for any choices of α^a, β^a, and γ^a. The energy can be minimized by choosing these functions as follows:

$$\beta^a l_a = 0, \tag{B7}$$

$$n\left(\frac{\partial p}{\partial n}\right)_s(\alpha^a l_a) = \tfrac{1}{2}\lambda n\kappa\left[\left(\frac{\partial T}{\partial n}\right)_s - \frac{T}{\rho + p}\left(\frac{\partial p}{\partial n}\right)_s\right](\gamma^a l_a). \tag{B8}$$

With these choices the energy has the form

$$E = \tfrac{1}{2}\lambda^2(\rho + p)\gamma^a\gamma_a - \tfrac{1}{8}\lambda^2 n\kappa^2\left(\frac{\partial p}{\partial n}\right)_s^{-1}\left[\left(\frac{\partial T}{\partial n}\right)_s - \frac{T}{\rho + p}\left(\frac{\partial p}{\partial n}\right)_s\right]^2(\gamma^a l_a)^2. \tag{B9}$$

This energy may be further minimized by taking the direction of wave propagation l^a to be parallel to γ^a. The energy finally reduces then to the following simple expression:

$$E = \tfrac{1}{2}\lambda^2(\rho + p)(\gamma^a\gamma_a)\left\{1 - \tfrac{1}{4}\kappa^2\left(\frac{\partial p}{\partial \rho}\right)_s\left[\left(\frac{\partial T}{\partial p}\right)_s - \frac{T}{\rho + p}\right]^2(l^b l_b)\right\}. \tag{B10}$$

This expression can clearly be made negative by choosing perturbations with sufficiently short wavelength (that is by making $l^a l_a$ sufficiently large). We infer from this that solutions to the perturbed fluid equations will exhibit thermal conductivity induced secular instabilities if they have sufficiently short wavelengths. This does *not* imply that real fluid materials will exhibit such instabilities, however. Real materials do not behave like solutions to the perturbed fluid equations for arbitrarily short wavelength perturbations. By requiring that the wavelength be larger than some minimum wavelength cutoff, λ_c (for example, the average interparticle separation $n^{-1/3}$, or the mean free path between collisions in the fluid), we arrive at an upper limit for $l^a l_a$:

$$l^a l_a < \lambda_c^{-2}. \tag{B11}$$

This constraint leads to the following lower bound on the energy functional for physically meaningful perturbations:

$$E > \tfrac{1}{2}\lambda^2(\rho + p)(\gamma^a\gamma_a)\left\{1 - \tfrac{1}{4}\kappa^2\lambda_c^{-2}\left(\frac{\partial p}{\partial \rho}\right)_s\left[\left(\frac{\partial T}{\partial p}\right)_s - \frac{T}{\rho + p}\right]^2\right\}. \tag{B12}$$

This expression will be positive as long as the thermal conductivity κ is sufficiently small. Therefore, our analysis has led us to the following thermodynamic condition which must be satisfied by any real stable fluid system:

$$\kappa^2 < 4\lambda_c^2\left(\frac{\partial p}{\partial \rho}\right)_s^{-1}\left[\left(\frac{\partial T}{\partial p}\right)_s - \frac{T}{\rho + p}\right]^{-2}. \tag{B13}$$

The thermodynamic derivatives in this expression can be replaced by the adiabatic sound speed v_s, the thermal expansion coefficient α, and the specific heat at constant pressure c_p. These functions are defined by

$$v_s^2 = \left(\frac{\partial p}{\partial \rho}\right)_s , \tag{B14}$$

$$\alpha = -\frac{1}{n}\left(\frac{\partial n}{\partial T}\right)_p , \tag{B15}$$

$$c_p = T\left(\frac{\partial s}{\partial T}\right)_p . \tag{B16}$$

It is straightforward to show that

$$\left(\frac{\partial T}{\partial p}\right)_s = \frac{T\alpha}{nc_p} . \tag{B17}$$

Using these functions, and units where the speed of light c has not been set equal to one, the upper limit for κ becomes

$$\kappa < \frac{2\lambda_c c^2}{v_s T}\left(\frac{\alpha}{nc_p} - \frac{1}{\rho c^2 + p}\right)^{-1} . \tag{B18}$$

For normal laboratory fluids

$$\frac{nc_p}{\alpha} \ll \rho c^2 + p . \tag{B19}$$

Consequently, the constraint on κ simplifies to

$$\kappa < 2\lambda_c T^{-1} \frac{c^2}{v_s} \frac{nc_p}{\alpha} . \tag{B20}$$

This new constraint on κ is trivially satisfied for normal materials. We can show that any (classical) ideal gas must satisfy inequality (B20). The thermal conductivity for such a gas is given by (see Huang 1963)

$$\kappa \approx \lambda_c nkv_s , \tag{B21}$$

where λ_c is the mean free path between collisions in the gas and k is Boltzmann's constant. The thermal expansion coefficient and specific heat for a monatomic ideal gas are given by (see Callen 1960):

$$\alpha = \frac{1}{T} , \tag{B22}$$

$$c_p = \tfrac{5}{2}k . \tag{B23}$$

The constraint on κ for this system reduces to the inequality

$$(v_s/c)^2 < 5 \tag{B24}$$

which is trivially satisfied. One might expect a liquid metal with its large thermal conductivity to be a more severe test. For mercury at 0° C (see Vargaftik 1975) the thermal conductivity κ has the value 8.178 (W/m K). The right-hand side of equation (B20) with $\lambda_c = n^{-1/3}$ has the value 1.388×10^{12} (W/m K). Thus, the constraint on κ is trivially satisfied for mercury. Furthermore, since the thermal conductivity in superconductors is only a few hundred times the conductivity of the best normal conductors, no violations of this condition are expected from such materials.

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