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Criticism of some non-conservative gravitational theories†

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Abstract. We show that the 'non-conservative' gravitational theories of the type considered by Rastall, Smalley and Malin do have conservation laws and are formally equivalent to general relativity. The supposed difference' between these theories and general relativity is shown to lie entirely in the question of whether the stress–energy tensor of matter fields is conserved in special relativity (flat space–time). If one chooses to interpret these theories as non-conservative, then the coefficient λ in these theories, which measures the degree to which stress–energy is not conserved, can be constrained to values $|\lambda| \leqslant 10^{-15}$ by considering the propagation of sound in a fluid.

It is of fundamental interest to question the extent to which the stress–energy tensor of matter is conserved (i.e. the extent to which it is divergence-free). Some attempts to address this question have been put forward by Rastall (1972), Smalley (1974a, b, 1975, 1976, 1978), Smalley and Prestage (1976) and Malin (1975). They have suggested certain model theories of gravitation which appear to be non-conservative in regions of large space–time curvature, thereby giving a framework in which possible non-conservative effects might be investigated.

For simplicity we consider here the form of these theories given by Rastall. Somewhat more general theories have been suggested by Smalley, and our remarks apply to them also. The theories which we shall consider are defined by the field equations

$$R_{\mu\nu} + (\lambda - \tfrac{1}{2})g_{\mu\nu}R = \kappa T_{\mu\nu}. \tag{1}$$

The tensors $R_{\mu\nu}$ and $T_{\mu\nu}$ are to be interpreted in these equations as the Ricci curvature tensor and the standard stress–energy tensor of matter, respectively; κ and λ are constants.

We first note (following Rastall) that the theory with $\lambda = \frac{1}{4}$ can be trivially excluded from consideration. If $\lambda = \frac{1}{4}$, the contraction of (1) then implies that

$$T^{\alpha}{}_{\alpha} = 0 \tag{2}$$

which is certainly not true for standard stress–energy tensors of a general matter field.

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If $\lambda \neq \frac{1}{4}$, then, as noted by Rastall, these theories can all be recast into a conservative form by defining a new stress–energy tensor

$$S_{\mu\nu} = T_{\mu\nu} + \frac{\lambda}{1-4\lambda} T^{\alpha}{}_{\alpha} g_{\mu\nu}. \tag{3}$$

The field equations then take the form

$$G_{\mu\nu} = \kappa S_{\mu\nu} \tag{4}$$

and the Bianchi identities require that

$$\nabla_{\nu} S_{\mu}{}^{\nu} = 0. \tag{5}$$

Thus, we see by (4) that these theories are in fact formally identical to general relativity. We should make the point here that the transformation from (1) to (3) is not a 'units transformation' in the sense of Dicke (1962) and Bekenstein (1977). The metric has not been conformally transformed; the initial representation (1) uses exactly the same measuring rods and clocks as the conservative representation (3)–(5).

We also now see that these theories are necessarily conservative in the most fundamental sense of the word; there exists a divergence-free tensor ($S_{\mu\nu}$) which is constructed solely from the matter fields.

The question which remains is how to construct the stress–energy tensor out of the matter fields of the real world. Since $S_{\mu\nu}$ is always exactly conserved, and the definition of $S_{\mu\nu}$ in terms of $T_{\mu\nu}$, (3), is independent of space–time curvature, this remaining question is really a question about the conservation of stress–energy in special relativity, and not an alternative theory of gravity at all. Massless fields (such as the electromagnetic field) are traceless classically, and hence $T_{\mu\nu} = S_{\mu\nu}$ for such fields. The most commonly treated source for the gravitational field for which $T_{\mu\nu} \neq S_{\mu\nu}$ is a perfect fluid. The stress–energy tensor of a perfect fluid with energy density ε, pressure p and four-velocity u^{μ} is

$$t_{\mu\nu} = (\varepsilon + p) u_{\mu} u_{\nu} + p g_{\mu\nu}. \tag{6}$$

If $t_{\mu\nu}$ is identified with $S_{\mu\nu}$, then the ordinary conservative laws of hydrodynamics hold. If, on the other hand, $t_{\mu\nu}$ is identified with the non-conserved tensor $T_{\mu\nu}$, then $S_{\mu\nu}$ is still the conserved stress–energy tensor for a perfect fluid, but now

$$S_{\mu\nu} = (\varepsilon' + p') u_{\mu} u_{\nu} + p' g_{\mu\nu} \tag{7}$$

where

$$\varepsilon' = (1-4\lambda)^{-1}[(1-3\lambda)\varepsilon - 3\lambda p] \tag{8}$$

$$p' = (1-4\lambda)^{-1}[(1-\lambda)p - \lambda\varepsilon]. \tag{9}$$

Since in many cases we measure physical quantities indirectly by assuming that certain equations of motion are satisfied (e.g. $\nabla_{\nu} T^{\mu\nu} = 0$), it is clear that ε' and p' are almost certainly what we measure in the laboratory. Again in this case we see that theories of the Rastall type are devoid of any differences from general relativity, since ε, p and the non-conserved tensor $T_{\mu\nu}$ may now be completely ignored. After all, in a conserved system it is always possible to construct non-conserved quantities; using such quantities to define the dynamical equations of the system merely makes the equations of motion more complicated.

Finally, what if we insist that $t_{\mu\nu}$ be identified with $T_{\mu\nu}$ and insist that ε and p are the energy density and pressure as measured in the laboratory? This is the only case in which the experimental predictions of the Rastall-type theories differ from special and general relativity; and with this interpretation they predict very non-conservative behaviour for laboratory perfect fluids unless $\lambda \lll 1$. The Bianchi identities imply the following equations of motion for the physical variables ε and p:

$$u^\mu \nabla_\mu[(1-3\lambda)\varepsilon - 3\lambda p] + (1-4\lambda)(\varepsilon+p)\nabla_\mu u^\mu = 0 \tag{10}$$

$$(1-4\lambda)(\varepsilon+p)u^\mu\nabla_\mu u^\nu = -(\delta^\mu{}_\nu + u^\mu u_\nu)\nabla_\mu[(1-\lambda)p - \lambda\varepsilon]. \tag{11}$$

Smalley (1975) and Lindblom and Nester (1975) have shown that acceptable values of the constant λ can be constrained by considering the Newtonian limit of (10) and (11). In the Newtonian limit ($u^\mu = (1, v^i)$, $v^i \ll 1$, $p \ll \varepsilon$, and negligible self-gravitational effects) (10) and (11) become

$$\frac{\partial}{\partial t}[(1-3\lambda)\varepsilon - 3\lambda p] + (1-4\lambda)\varepsilon\nabla_i v^i = 0 \tag{12}$$

$$(1-4\lambda)\varepsilon\left(\frac{\partial v^i}{\partial t} + v^j\nabla_j v^i\right) = -\nabla^i[(1-\lambda)p - \lambda\varepsilon]. \tag{13}$$

It has been previously pointed out that the non-conservation of mass predicted by (12) constrains λ to very small values; however, a more sensitive test can be found by considering the propagation of density fluctuations (i.e. sound waves) predicted by (12) and (13). Following the standard derivations of the propagation of sound waves (Landau and Lifshitz 1959) it is straightforward to show that small density fluctuations, $\delta\varepsilon$, propagate according to the equation

$$\frac{\partial^2}{\partial t^2}(\delta\varepsilon) - v_s^2\nabla^i\nabla_i(\delta\varepsilon) = 0 \tag{14}$$

$$v_s^2 = \left((1-\lambda)\frac{\partial p}{\partial \varepsilon} - \lambda\right)\bigg/\left[1 - 3\lambda\left(1 + \frac{\partial p}{\partial \varepsilon}\right)\right]. \tag{15}$$

One can test the viability of these theories therefore by comparing the experimentally measured velocities of sound with measured thermodynamic properties of the fluid, $\partial p/\partial\varepsilon$, in (15). This equation is a particularly sensitive measure of λ because, in units where the speed of light c has not been set to unity and for $\lambda \ll 1$, (15) becomes

$$v_s^2 = \partial p/\partial\varepsilon - \lambda c^2. \tag{16}$$

By considering fluid systems which have low sound velocities and well-understood thermodynamic properties (e.g. monatomic gases at low temperatures), one can constrain λ to values considerably below v_s^2/c^2. Using data (Van Itterbeek 1955) on gaseous helium near 4 K, one can show that $|\lambda|$ can be constrained to values smaller than 10^{-15}. For this system $v_s \simeq 10^4\ \mathrm{cm\ s^{-1}}$, and the data considered above confirm the standard relationship $v_s^2 = \partial p/\partial\varepsilon$ to within the random experimental errors, which are at the 1% level for their data. Consequently one estimates $|\lambda| \leqslant 0.01\ v_s^2/c^2 \approx 10^{-15}$. A better system and/or better data may well exist.

While no explicit experimental constraints on the value of λ have previously been published, Smalley (1978) has argued that λ must be small, i.e. $\lambda \leqslant O(v^2)$ in a post-Newtonian-type expansion. The constraint presented here is significantly more stringent than would be inferred from Smalley's constraint using typical orbital velocities for the solar system $O(v^2) \simeq 10^{-7}$ or even the value $O(v^2) \simeq 10^{-9}$ for Earth satellites.

In conclusion, we see that it is quite impossible to use theories of this type as a framework for investigating non-conservation of stress–energy in strong gravitational fields. We have shown that these theories are in fact identical to general relativity modulo special relativistic questions of how one constructs the stress–energy tensor out of the matter fields. Either the theory is completely identical with ordinary general relativity, or, if one chooses the *apparently* non-conservative construction of the stress–energy tensor, its predictions differ from those of general relativity by at most 1 part in 10^{15}. In any case, these theories are general relativity and are conservative, if one measures the right quantities. A non-conservative theory is one which does not possess conservation laws for stress–energy, whereas theories of the Rastall–Smalley–Malin type merely construct non-conserved quantities within a conservative theory. Such quantities are always constructable, but their existence does not negate the fact that the underlying theory is conservative.

References

Bekenstein J D 1977 *Phys. Rev.* D **15** 1458
Dicke R H 1962 *Phys. Rev.* **125** 2163
Landau L D and Lifshitz E M 1959 *Fluid Mechanics* (Oxford: Pergamon) § 63
Lindblom L and Nester J 1975 *Phys. Rev.* D **12** 626
Malin S 1975 *Phys. Rev.* D **11** 707
Rastall P 1972 *Phys. Rev.* D **6** 3357
Smalley L L 1974a *Phys. Rev.* D **9** 1635
—— 1974b *Lett. Nuovo Cimento* **10** 231
—— 1975 *Phys. Rev.* D **12** 376
—— 1976 *Phys. Lett.* **57A** 300
—— 1978 *Found. Phys.* **8** 59
Smalley L L and Prestage J 1976 *Nuovo Cimento* B **35** 224
Van Itterbeek A 1955 in *Progress in Low Temperature Physics* vol 1, ed C J Corter (New York: Interscience) p 363