Some properties of static general relativistic stellar models. II [a]

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We expand here the study of the type of identity which has been successfully used to prove the necessity of spherical symmetry in static black holes and uniform density stellar models. We show how the system of partial differential equations, whose solutions correspond to these identities, can be decoupled and partially integrated for fluids with an arbitrary equation of state. The problem of finding such identities is reduced thereby to the problem of finding the solutions to a single ordinary differential equation, plus quadratures.

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I. INTRODUCTION

At the present time, the program to determine from general relativity theory its predictions, concerning the properties of the equilibrium states of astrophysical objects (stars and black holes), is far from complete. One aspect of that program, to confirm our intuitive belief that static (i.e., stationary and nonrotating) stellar models and black holes must necessarily have spherical symmetry, has been partially completed. It has been shown that isolated static black holes[1-3] and uniform density stellar models[4] must have spherical symmetry. The crucial element in the arguments which lead to those results is the existence of certain identities satisfied by the static solutions of Einstein's equations. In this paper we discuss identites of this type which apply in the interior regions of stellar models having arbitrary equations of state. We show how the procedure which was developed to derive these identities[4] can be significantly simplified. In particular, we show how the system of partial differential equations, whose solutions correspond to these identities, can be decoupled and partially integrated. The problem of finding identities for fluids with arbitrary equations of state is reduced thereby to finding the solutions of a single ordinary differential equation, plus some quadratures. Since identities of this type have played such a crucial role in the understanding of static black holes and uniform density stellar models, it seems likely that the simplifications presented here will bring us a step closer to the complete proof that all static stellar models must be spherical.

II. REVIEW

In the first paper of this series (Ref. 4; hereafter referred to as Paper I) we described how useful identities for static stellar models could be obtained by solving an appropriate system of partial differential equations. We review now the results of that work.

A static stellar model is a metric,

$$ds^2 = -V^2 dt^2 + g_{ab}dx^a dx^b, \tag{1}$$

(where the scalar V and 3-metric g_{ab} are independent of the coordinate t) which satisfies Einstein's equations:

$$\nabla^a\nabla_a V = 4\pi V(\rho + 3p), \tag{2}$$

$$R_{ab} = V^{-1}\nabla_a\nabla_b V + 4\pi(\rho - p)g_{ab}. \tag{3}$$

In these equations R_{ab} and ∇_a are the Ricci curvature and covariant derivative of g_{ab}, while ρ and p are the density and pressure of the fluid. The density and pressure are assumed to be related by a given equation of state $\rho = \rho(p)$, and as a consequence of Eqs. (2) and (3) must also satisfy Euler's equation:

$$\nabla_a p = -V^{-1}(\rho + p)\nabla_a V. \tag{4}$$

In Paper I it was shown that identities of the form

$$\nabla_a[K_1(V, W)\nabla^a V + K_2(V, W)\nabla^a W] = \tfrac{1}{4}K_2(V, W)V^4W^{-1}R_{abc}R^{abc} + \left(\frac{\partial K_2(V, W)}{\partial W} + \frac{3}{4}W^{-1}K_2(V, W)\right)|\nabla_a W - F\nabla_a V|^2 \tag{5}$$

will exist if the functions $K_1(V, W)$ and $K_2(V, W)$ satisfy the following system of partial differential equations:

$$\frac{\partial K_1}{\partial V} = F^2\frac{\partial K_2}{\partial W} - 4\pi K_1 VW^{-1}(\rho + 3p) - \left(16\pi^2V^2W^{-1}(\rho + 3p)^2 - \tfrac{3}{4}W^{-1}F^2 + 8\pi V\frac{d}{dV}(\rho + p)\right)K_2, \tag{6}$$

$$\frac{\partial K_2}{\partial V} = -2F\frac{\partial K_2}{\partial W} - \frac{\partial K_1}{\partial W} - [V^{-1} - 4\pi VW^{-1}(\rho + 3p) + \tfrac{3}{2}W^{-1}F]K_2. \tag{7}$$

In the above equations W is defined as $W = \nabla^a V\nabla_a V$; $F = F(V, W)$ is a function chosen so that a spherical model would satisfy $\nabla_a W = F\nabla_a V$; $R_{abc}R^{abc}$ is the square of the three-dimensional conformal tensor which vanishes if and only if the stellar model is spherical (see Paper I); and ρ and p are taken to be the known functions of V determined by the equation of state and Eq. (4).

Another useful function, introduced in Paper I, is

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$W_0 = W_0(V)$: the function to which W would be equal if the model were spherical. This function satisfies the second order differential equation

$$\frac{d^2W_0}{dV^2} = \frac{1}{V}\frac{dW_0}{dV} + \frac{3}{4}\frac{1}{W_0}\left(\frac{dW_0}{dV}\right)^2 - 8\pi\frac{V}{W_0}\frac{dW_0}{dV}(\rho + 3p) + 16\pi^2\frac{V^2}{W_0}(\rho + 3p)^2 + 8\pi V\frac{d}{dV}(\rho + p). \tag{8}$$

For a detailed discussion of these results, the reader should consult Paper I.

III. A SIMPLIFIED PROCEDURE FOR FINDING IDENTITIES

In this section we show how the procedure for finding identities in the form of Eq. (5) can be simplified by decoupling and partially integrating the system of partial differential equations (6), (7). The first step in this procedure is to change dependent variables to the set: $A(V, W) = K_1(V, W) + F(V, W)K_2(V, W)$ and $B(V, W) = K_2(V, W)$. We next take appropriate linear combinations of Eqs. (6), (7), expressed in these new variables, to obtain the following system:

$$\frac{\partial A}{\partial V} + F\frac{\partial A}{\partial W} + 4\pi V W^{-1}(\rho + 3p)A = \left(\frac{\partial F}{\partial V} + F\frac{\partial F}{\partial W} - \frac{F}{V} - \frac{3}{4}\frac{F^2}{W} + 8\pi\frac{V}{W}F(\rho + 3p) - 16\pi^2\frac{V^2}{W}(\rho + 3p)^2 - 8\pi V\frac{d}{dV}(\rho + p)\right)B, \tag{9}$$

$$\frac{\partial B}{\partial V} + F\frac{\partial B}{\partial W} - \left(\frac{\partial F}{\partial W} - \frac{1}{V} + 4\pi\frac{V}{W}(\rho + 3p) - \frac{3}{2}\frac{F}{W}\right) \times B + \frac{\partial A}{\partial W} = 0. \tag{10}$$

The next step is to take advantage of the arbitrariness in the choice of the function F. The only restriction which F must satisfy is that, when the star is spherical (i.e., when $W = W_0$), F must satisfy $\nabla_a W = F\nabla_a V$; this is equivalent to requiring that $F(V, W_0) = dW_0/dV$. At this point, however, we are free to choose how F will behave for $W \neq W_0$. To make that choice specific, let us require that F satisfy the following partial differential equation:

$$\frac{\partial F}{\partial V} + F\frac{\partial F}{\partial W} = \frac{F}{V} + \frac{3}{4}\frac{F^2}{W} - 8\pi\frac{V}{W}F(\rho + 3p) + 16\pi^2\frac{V^2}{W}(\rho + 3p)^2 + 8\pi V\frac{d}{dV}(\rho + p). \tag{11}$$

Since Eq. (11) reduces to Eq. (8) on the surface $W = W_0$ when $F = dW_0/dV$, it follows that Eq. (11) admits solutions which satisfy our criterion $F(V, W_0) = dW_0/dV$. With this choice of F, the system of equations (9), (10) decouples. The coefficient of B in Eq. (9) vanishes, so that this equation can be used to determine A. Given the solution for A, Eq. (10) can be integrated to determine B.

To proceed further, it is necessary to change independent variables from the set (V, W) to a more appropriate set, which we call (X, Y). The new variables $X = X(V, W)$ and $Y = Y(V, W)$ are defined by the following equations:

$$X(V, W) = V, \tag{12}$$

$$\left(\frac{\partial}{\partial V} + F(V, W)\frac{\partial}{\partial W}\right)Y(V, W) = 0. \tag{13}$$

For the discussion that follows, it will be most helpful to treat V and W as the functions of X and Y given by the inverse of the transformation defined by Eqs. (12) and (13). The differentials of these variables are related by

$$\frac{\partial}{\partial X} = \frac{\partial}{\partial V} + F\frac{\partial}{\partial W}, \tag{14}$$

$$\frac{\partial}{\partial Y} = \frac{\partial W}{\partial Y}\frac{\partial}{\partial W}. \tag{15}$$

We note that it follows from Eq. (14) that $F = \partial W/\partial X$. With this choice of variables, the equations for A and B become

$$\frac{\partial A}{\partial X} + 4\pi\frac{X}{W}(\rho + 3p)A = 0, \tag{16}$$

$$\frac{\partial B}{\partial X} - \left[\left(\frac{\partial W}{\partial Y}\right)^{-1}\frac{\partial^2 W}{\partial X\partial Y} - \frac{1}{X} + 4\pi\frac{X}{W}(\rho + 3p) - \frac{3}{2}\frac{1}{W}\frac{\partial W}{\partial X}\right]B + \left(\frac{\partial W}{\partial Y}\right)^{-1}\frac{\partial A}{\partial Y} = 0. \tag{17}$$

It is straightforward now to completely reduce Eqs. (16) and (17) to quadratures. We define the function

$$\Psi(X, Y) = \exp\left\{-4\pi\int^X \frac{X'[\rho(X') + 3p(X')]dX'}{W(X', Y)}\right\}. \tag{18}$$

If follows that

$$A(X, Y) = a(Y)\Psi(X, Y), \tag{19}$$

$$B(X, Y) = \left[b(Y) - \int^X X'W^{3/2}\left(\frac{\partial W}{\partial Y}\right)^{-2}\frac{\partial A}{\partial Y}\Psi(X', Y)dX'\right]\frac{\partial W}{\partial Y}\Big/[XW^{3/2}\Psi], \tag{20}$$

where $a(Y)$ and $b(Y)$ are completely arbitrary C^1 functions of Y. These quadratures [Eqs. (18)–(20)] can be performed as soon as one determines the function $W(X, Y)$. Since $F = \partial W/\partial X$, as we have seen, the real problem lies in determining the function F from Eq. (11). This problem can be expedited by changing to the variables (X, Y) discussed above, and making the substitution $F = \partial W/\partial X$ in Eq. (11):

$$\frac{\partial^2 W}{\partial X^2} - \frac{1}{X}\frac{\partial W}{\partial X} - \frac{3}{4}\frac{1}{W}\left(\frac{\partial W}{\partial X}\right)^2 + 8\pi\frac{X}{W}\frac{\partial W}{\partial X}(\rho + 3p) - 16\pi^2\frac{X^2}{W}(\rho + 3p)^2 - 8\pi X\frac{d}{dX}(\rho + p) = 0. \tag{21}$$

We note that only the independent variable X appears explicitly in Eq. (21). Therefore, the equation to determine $W(X, Y)$ is an *ordinary* differential equation. In fact this is the same differential equation used to determine W_0, Eq. (8). The only difference between W_0 and W then is one of boundary conditions. The boundary conditions on W_0 are uniquely determined by the asymptotic flatness conditions and appropriate smoothness conditions at the surface of the star (see Paper I). On the other hand, $W(X, Y)$ is a one-parameter family of solutions to Eq. (21). Since Eq. (21) is a second order equation, there are in principle two free parameters in the most general solution. Therefore, there is considerable freedom still available in the choice of the function $W(X, Y)$.

To summarize, we have succeeded in reducing the problem of finding identities from that of solving the system of partial differential equations (6), (7) to that of solving the single ordinary differential equation (21) plus the quadratures in Eqs. (18)–(20).

IV. EXAMPLES: NEW VACUUM IDENTITIES

To illustrate the generality and usefulness of the simplifications which have been presented here, we show now how every previously discussed identity for vacuum and uniform density spaces is a special case of the results presented here. Also, we explicitly write down the most general vacuum identity of this type. These vacuum identities are significantly more general than those presented elsewhere[1-4] and may possibly provide the way to show that no multiple static vacuum black hole solutions exist.

Turning to the uniform density case first, we find that a one-parameter family of solutions to Eq. (21) is given by

$$W = W_0(X) + Y, \tag{22}$$

$$W_0 = \tfrac{8}{3}\pi\rho X(3X_s - X) + \tfrac{2}{3}\pi\rho(1 - 9X_s^2), \tag{23}$$

where ρ is the density of the star and X_s is the value of X at the surface of the star. Given these integrals of Eq. (21), it is straightforward to evaluate the quadratures in Eqs. (18)–(20). In particular, we find

$$A(X, Y) = a(Y)W^{-3/2}, \tag{24}$$

$$B(X, Y) = X^{-1}\left[b(Y) - \int^X X'\left(\frac{da}{dY}W - \frac{3}{2}a\right)W^{-5/2}dX'\right]. \tag{25}$$

The integral indicated here is simple to perform [using Eqs. (22), (23)], but the result is lengthy and unenlightening. The uniform density identity presented in Paper I is the special case of the above with $a(Y) = 0$.

For vacuum spaces, it is possible to explicitly evaluate the most general integral of Eq. (21). In particular, we find that the general solution is given by

$$W(X, Y) = [g(Y)X^2 + h(Y)]^4, \tag{26}$$

where g and h are arbitrary functions of Y. It is also straightforward now to perform the quadratures in Eqs. (18)–(20). If $dg/dY \equiv g' \neq 0$, we find

$$A(X, Y) = a(Y), \tag{27}$$

$$B(X, Y) = \frac{4(g'X^2 + h')}{XW^{3/4}}\left[b(Y) + \frac{a'}{32g'(g'X^2 + h')}\right], \tag{28}$$

where $a' = da/dY$, etc. When $g' = 0$, we get Eq. (27) and

$$B(X, Y) = \frac{4h'}{XW^{3/4}}\left[b(Y) - \frac{a'X^2}{32h'^2}\right]. \tag{29}$$

Specific examples of these vacuum identities have been given by Israel,[1] Robinson,[3] and in Paper I. One example given in Paper I is $g(Y) = -h(Y) = -\frac{1}{2}M^{-1/2}Y^{1/4}$, where M is the constant which is the asymptotically defined mass of the star (or black hole). The identities of Israel and Robinson are special cases of this example, as was shown in Paper I. Another example given in Paper I (except for typographical errors) is $g(Y) = -\frac{1}{2}M^{-1/2}$ and $h(Y) = \frac{1}{2}M^{-1/2} + Y$. Clearly a large number of other possibilities also exist for other choices of g and h. Perhaps some of these identities can be used to eliminate the possibility of multiple static vacuum black holes.

Note added in proof: An error has been discovered in Sec. V of Paper I. The argument given there, that uniform density static stellar models must be spherical, is not correct. One can only conclude, from an argument such as that presented there, that any nonspherical uniform density model must have $W/W_0 < 1$ everywhere (including spatial infinity). This implies, for example that the constants of a nonspherical model must satisfy the inequality $32\pi\rho M^2 > 3(1 - V_s^2)^3$.

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