Some properties of static general relativistic stellar models [a)]

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A number of properties of static general relativistic stellar models are presented which appear to be relevant to the ongoing search for a proof that all such models must have spherical symmetry. It is shown that any such model, having conformally flat spatial sections, must have spherical symmetry. A general procedure is described which allows one to construct the type of "divergence equals positive quantity" identities for static stellar models, which were used to prove that static black holes must have spherical symmetry. This procedure is used to produce a large new class of identities for the exterior vacuum regions of static stellar models and identities are constructed for the interior regions of uniform density models. These identities are used to prove that static uniform density stellar models must have spherical symmetry.

I. REVIEW OF STATIC STELLAR MODELS

In this paper we study the solutions of Einstein's field equations with matter corresponding to a perfect fluid which is in static equilibrium. It has been shown previously[1-6] that such a stellar model is a spacetime with a metric which can be represented by the line element

$$ds^2 = -V^2 dt^2 + g_{ab} dx^a dx^b . \quad (1)$$

The components of g_{ab} and the function V are independent of the coordinate t, and the tensor g_{ab} represents the positive definite metric on each $t =$ constant submanifold, each of which has the same topology as R^3. Einstein's equations for this system can be written in the form

$$\nabla_a \nabla^a V = 4\pi V(\rho + 3p) , \quad (2)$$

$$R_{ab} = V^{-1} \nabla_a \nabla_b V + 4\pi(\rho - p) g_{ab} . \quad (3)$$

The tensor R_{ab} represents the three-dimensional Ricci tensor of the metric g_{ab}, and ∇_a represents the three-dimensional covariant derivative compatible with g_{ab}. The functions ρ and p represent the mass density and pressure of the fluid, respectively. These are related by an equation of state; i.e., a given relationship of the form $\rho = \rho(p)$. The contracted Bianchi identities for the three-dimensional curvature, and Eq. (3) imply the equivalent of Euler's equation:

$$\nabla_a p = -V^{-1}(\rho + p)\nabla_a V . \quad (4)$$

In the discussion that follows, it will be helpful to define a number of additional quantities:

$$W = \nabla^a V \nabla_a V , \quad (5)$$

$$n^a = W^{-1/2} \nabla^a V , \quad (6)$$

$$\beta^{ab} = g^{ab} - n^a n^b , \quad (7)$$

$$H_{ab} = \beta_a{}^c \beta_b{}^d \nabla_c n_d , \quad (8)$$

$$\psi_{ab} = H_{ab} - \tfrac{1}{2}\beta_{ab} H . \quad (9)$$

These quantities describe the geometry of the $V =$ const. 2-surfaces. The unit vector field n^a is orthogonal to these surfaces; β_{ab} is the intrinsic metric; H_{ab} is the extrinsic curvature tensor and ψ_{ab} represents the trace-free part of H_{ab}.

The quantities which define the geometry of the $V =$ const. 2-surfaces are related to one another by splitting Einstein's equations into pieces tangent and orthogonal to these surfaces in the standard way. In the discussion that follows, we make use of two of the equations obtained by this splitting of Eqs. (2) and (3):

$$W^{-1} \nabla^a V \nabla_a W = -2W^{1/2} H + 8\pi V(\rho + 3p) , \quad (10)$$

$$\begin{aligned} W^{-1} \nabla^a V \nabla_a H &= -\tfrac{1}{2} W^{-1/2} H^2 + V^{-1} H - \beta^{ab} \nabla_a (\beta_b{}^c \nabla_c W^{-1/2}) \\ &\quad - W^{-1/2} \psi_{ab} \psi^{ab} - 8\pi W^{-1/2}(\rho + p) . \end{aligned} \quad (11)$$

The derivation of these equations can be found in the literature.[1-3,5]

It has long been suspected that no nonspherical, asymptotically flat solutions exist to Eqs. (2) and (3). This belief is motivated by an analogous theorem for Newtonian stellar models[7] and a similar result for the vacuum ($\rho = p = 0$) black hole solutions of Eqs. (2) and (3).[8-10] It has also been shown that stationary (nonstatic) general relativistic stellar models (made of dissipative fluids) must be axisymmetric.[11] Little progress has been made on the problem of static relativistic stellar models however. It has been shown that if

$$\beta^{ab} \nabla_b W = 0 \quad (12)$$

then the model must be spherical.[3] It has also been shown that no "almost" spherical static stellar models exist.[3,4]

For the remainder of this paper, we discuss some of the properties of these stellar models which appear to be relevant to the ongoing search for a proof that spherical symmetry is necessary. In Sec. II it is shown that if the 3-geometry described by g_{ab} is conformally flat, then the stellar model must be spherical. In Sec. III we describe a procedure which allows one to construct, for stellar models, the type of identities which were used[8-10] to prove that static black holes must have spherical symmetry. In Sec. IV we use the procedure described in Sec. III to construct identities applicable in the vacuum exterior regions of any static stellar model, and in the interior regions of models with uniform density. And finally in Sec. V, we use these identities to show that, in the special case of uniform density models, spherical symmetry is necessary.

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II. CONFORMAL FLATNESS AND SPHERICAL SYMMETRY

The conformal properties of a three-dimensional manifold are not described by the Weyl tensor (which vanishes identically) but by a certain third-rank tensor field,[12] defined by

$$R_{abc} = \nabla_c R_{ab} - \nabla_b R_{ac} + \tfrac{1}{4}(g_{ac}\nabla_b R - g_{ab}\nabla_c R)\,. \quad (13)$$

This expression can be evaluated in terms of the functions V, ρ, etc. by using Eq. (3) for R_{ab}. Making this substitution, and using the quantities defined in Egs. (5)–(9), it is straightforward to verfy the following;

$$R_{abc}R^{abc} = 8V^{-4}W^2\{\psi_{ab}\psi^{ab} + \tfrac{1}{8}W^{-2}\beta^{ab}\nabla_a W\nabla_b W\}\,. \quad (14)$$

It is interesting to note that the matter variables do not appear explicitly in this expression; this is precisely the same expression which was found to hold in a vacuum spacetime.[9] Expression (14), however, is true for any geometry satisfying Eqs. (2) and (3).

If the geometry g_{ab} were conformally flat, then the tensor R_{abc} must vanish.[12] From Eq. (14) it would follow that ψ_{ab} and $\beta^{ab}\nabla_b W$ must also vanish in this case. Consequently, it would follow from the standard arguments[3] that the stellar model would necessarily be spherical. Thus we have established the following lemma.

Lemma: *If the spatial geometry g_{ab} of a static general relativistic stellar model* [i.e., *a solution of Eqs.* (2) *and* (3)] *is conformally flat, then the stellar model necessarily has spherical symmetry.*

Another expression for the square of the conformal tensor, which will be useful in the following section, is the following:

$$\begin{aligned}\tfrac{1}{4}V^4W^{-1}R_{abc}R^{abc} &= \nabla_a\nabla^a W - V^{-1}\nabla^a V\nabla_a W - \tfrac{3}{4}W^{-1}\nabla^a W\nabla_a W\\ &\quad + 8\pi W(\rho+p) + 4\pi VW^{-1}(\rho+3p)\nabla^a V\nabla_a W\\ &\quad - 16\pi^2V^2(\rho+3p)^2 - 8\pi V\nabla^a V\nabla_a\rho\,. \end{aligned} \quad (15)$$

This expression is derived using essentially the same procedure as that described to derive Eq. (14); and we note that this expression agrees in the vacuum limit with an analogous expression derived previously.[9]

III. CONSTRUCTION OF DIVERGENCE IDENTITIES

The proof that static black holes must have spherical symmetry depends on constructing an identity which has the form of a divergence equaling a positive definite quantity (which vanishes if and only if the spacetime is spherical). One positive definite quantity, which might be suitable for such an identity, has been identified in the last section: $R_{abc}R^{abc}$. The existence of another suitable quantity is implied by Eq. (12). If the spacetime were spherical, then the vector $\nabla_a W$ must be proportional to $\nabla_a V$; we call the proportionality factor F. The function $F = F(V,W)$ can be determined explicitly (as shall be discussed in detail in Sec. IV) once the equation of state of the fluid in the stellar model is specified. Taking F as a known function, it follows that the quantity:

$$[\nabla_a W - F\nabla_a V][\nabla^a W - F\nabla^a V] \quad (16)$$

is positive and vanishes if and only if the model is spherical.

Having identified two suitable positive definite quantities, we are lead to ask when an identity of the following form can be found:

$$\begin{aligned}\nabla_a\{K_1(V,W)\nabla^a V + K_2(V,W)\nabla^a W\} &= \tfrac{1}{4}Q_1(V,W)V^4W^{-1}R_{abc}R^{abc}\\ &\quad + Q_2(V,W)W^{-1}|\nabla_a W - F\nabla_a V|^2\,, \end{aligned} \quad (17)$$

where K_1, K_2, Q_1 and Q_2 are functions of V and W with $Q_1 > 0$ and $Q_2 > 0$. (We note that every identity used to prove the spherical symmetry of black hole spacetimes has been of this form.[8-10,13]) On examining each side of Eq. (17) (using Eq. (15) to evaluate the first term on the right) we find, in addition to functions of V and W, terms linear in the three functions: $\nabla^a\nabla_a W$, $\nabla^a V\nabla_a W$, and $\nabla^a W\nabla_a W$. If we require that the coefficients of these functions on one side of the equation equal the corresponding coefficients on the other side, the following four constraints on the functions K_1, K_2, Q_1, and Q_2 are implied:

$$K_2 = Q_1\,, \quad (18)$$

$$Q_2 = W\frac{\partial Q_1}{\partial W} + \tfrac{3}{4}Q_1\,, \quad (19)$$

$$\begin{aligned}\frac{\partial Q_1}{\partial V} &= -2F\frac{\partial Q_1}{\partial W} - \frac{\partial K_1}{\partial W}\\ &\quad - \left[V^{-1} - 4\pi VW^{-1}(\rho+3p) + \frac{3}{2}W^{-1}F\right]Q_1, \end{aligned} \quad (20)$$

$$\begin{aligned}\frac{\partial K_1}{\partial V} &= F^2\frac{\partial Q_1}{\partial W} - 4\pi K_1VW^{-1}(\rho+3p)\\ &\quad - \Big[16\pi^2V^2W^{-1}(\rho+3p)^2\\ &\quad - \tfrac{3}{4}W^{-1}F^2 + 8\pi V\frac{d}{dV}(\rho+p)\Big]Q_1. \end{aligned} \quad (21)$$

Thus, if these partial differential equations can be solved for K_1, K_2, Q_1, and Q_2 (with $Q_1 > 0$ and $Q_2 > 0$), then an identity in the form of Eq. (17) will exist. Note that V and W play the role of independent variables in these equations. Each of the coefficients in these equations is a known function of V and W: F was assumed to be a known function, while ρ and p are explicit functions of V determined by integrating Eq. (4).

A large number of solutions clearly exist to Eqs. (18)–(21). Equations (20) and (21) form a linear system of equations for Q_1 and K_1. One can imagine solving these equations as a Cauchy initial value problem. On an initial surface, say $V = V_i$, we arbitrarily specify the functions $Q_1(V_i, W)$ and $K_1(V_i, W)$. Equations (20) and (21) allow us to compute the normal derivatives of these functions; consequently the equations can be integrated to find Q_1 and K_1 (at least for V sufficiently close to V_i). Thus a large number of solutions to these equations exist, each of which corresponds to an identity of the form in Eq. (17). In order to be useful as tools for proving the spherical symmetry of stars, we must limit the choice of functions to those for which $Q_1 > 0$ and $Q_2 > 0$. At present, it is not known whether or not there exist solutions with positive Q_1 and Q_2 in general. We see in the next section that in some special cases, however, positive solutions do exist.

IV. IDENTITIES FOR VACUUM AND UNIFORM DENSITY SPACES

In this section we use the procedure described above to derive identities for the special cases of vacuum spacetimes (the exterior regions of stars) and for the interior regions of uniform density stars. To do this we must explicitly integrate Eqs. (20) and (21) for these cases. Before these equations can be explicitly integrated, we need to discuss how the function F can be determined for a given equation of state. For the spherical solutions of Eqs. (2) and (3) the functions W and H (the trace of the 2-dimensional extrinsic curvature) depend only on V; let $W_0(V)$ and $H_0(V)$ denote those functions. Equations (10) and (11) imply that W_0 and H_0 satisfy the differential equations

$$\frac{dW_0}{dV} = -2W_0^{1/2}H_0 + 8\pi V(\rho + 3p), \tag{22}$$

$$\frac{dH_0}{dV} = -\tfrac{1}{2}W_0^{-1/2}H_0^2 + V^{-1}H_0 - 8\pi W_0^{-1/2}(\rho + p). \tag{23}$$

The function H_0 can be eliminated from these equations, to obtain a single equation for W_0:

$$\begin{aligned}\frac{d^2W_0}{dV^2} &= V^{-1}\frac{dW_0}{dV} + \tfrac{3}{4}W_0^{-1}\left(\frac{dW_0}{dV}\right)^2 - 8\pi VW_0^{-1}\frac{dW_0}{dV}(\rho + 3p)\\ &\quad + 16\pi^2W_0^{-1}V^2(\rho + 3p)^2 + 8\pi V\frac{d}{dV}(\rho + p).\end{aligned} \tag{24}$$

Given an equation of state, $\rho = \rho(p)$, Eq. (4) can be integrated to determine the functions $\rho(V)$ and $p(V)$. Using these functions, Eq. (24) can be integrated to determine $W_0(V)$: the function to which W would be equal if the solution were spherical. Given this function, W_0, it is easy now to find the function F. In fact, F can be chosen in an infinite number of ways. One obvious choice is $F = dW_0/dV$, but $F = dW_0/dV + (W_0 - W)^n$ or $F = W^nW_0^{-n}\,dW_0/dV$ would do just as well. Thus for each equation of state which we specify there exist an infinite number of different choices of the function F; and for each F there exist an infinite number of identities in the form of Eq. (17).

Let us now explicitly utilize the procedure, which is outlined above, to obtain identities that are relevant to the study of stellar models. We begin with the simplest case: identities which describe the vacuum exterior regions of any stellar model. The first step is to solve Eq. (24) for W_0:

$$W_0 = \frac{1}{16M^2}(1 - V^2)^4, \tag{25}$$

and thus

$$\frac{dW_0}{dV} = -\frac{V}{2M^2}(1 - V^2)^3. \tag{26}$$

In these expressions, the constant M represents the asymptotically defined mass of the star. These solutions now allow us to choose the function F in any number of different ways. We select two different choices, each of which allows us to explicitly solve Eqs. (20) and (21) in a straightforward manner.

For our first choice we let

$$F = \frac{W}{W_0}\frac{dW_0}{dV} = -8VW(1 - V^2)^{-1}. \tag{27}$$

This expression is substituted in Eqs. (20) and (21), and the equations are integrated. The general solution of these equations is given by:

$$Q_1 = V^{-1}(1 - V^2)^{-2}\{A(x) - 8M^2(1 - V^2)^{-1}B'(x)\}, \tag{28}$$

$$K_1 = B(x) + 8VW(1 - V^2)^{-1}Q_1, \tag{29}$$

where $x = W/W_0$; A and B are arbitrary functions, and $B'(x) = dB/dx$. It is clearly possible to choose A and B in such a way that $Q_1 > 0$ and $Q_2 > 0$; so that these identities are potentially useful in our search for a proof of spherical symmetry. This class of identities contains, as special cases, every identity that has been constructed to prove the spherical symmetry of black holes. For example, Robinson's identities[10] are given by

$$A(x) = -c, \tag{30}$$

$$B(x) = -(c + d)x/8M^2, \tag{31}$$

where c and d are arbitrary constants. The two identities originally discovered by Israel[8] (or in this notation by Müller zum Hagen, *et al.*[9]) are obtained by setting

$$A(x) = 0, \tag{32}$$

$$B(x) = -4M^{-1/2}x^{1/4}, \tag{33}$$

for one identity, and

$$A(x) = -2M^{3/2}x^{-3/4}, \tag{34}$$

$$B(x) = -M^{-1/2}x^{1/4}, \tag{35}$$

for the other. We see that the technique described here yields a considerable degree of generalization over previously known identities.

Another choice of the function F, for the vacuum case, is

$$F = \left(\frac{W}{W_0}\right)^{3/4}\frac{dW_0}{dV} = -4M^{-1/2}VW^{3/4}. \tag{36}$$

This choice of F also allows Eqs. (20) and (21) to be integrated in general in a straightforward manner. The general solutions for Q_1 and K_1 are:

$$Q_1 = V^{-1}W^{3/4}C(y) - \tfrac{1}{20}M^{3/2}V^{-1}W^{-1/2}D'(y), \tag{37}$$

$$K_1 = D(y) + 4M^{-1/2}VW^{3/4}Q_1, \tag{38}$$

where $y = W^{1/4} - W_0^{1/4}$; $C(y)$ and $D(y)$ are arbitrary functions of y. Clearly it is possible to select the functions C and D so that $Q_1 > 0$ and $Q_2 > 0$. Consequently this represents another large class of divergence identities which may be useful in the study of the spherical symmetry of static stars.

Let us move on now to a consideration of the non-vacuum interior regions of static stellar models. The first problem one encounters is solving Eq. (24) for an arbitrary equation of state. I have not determined how this can be accomplished in general, yet. For the special case of uniform density stars, however, the solution can be found. We begin by integrating Eq. (4) for this case, to find that

$$p = \rho V^{-1}(V_s - V), \tag{39}$$

where ρ is the constant density of the star, and V_s is the value of the potential V at the suface of the star. Using this expression for p, it is easy to verify that

$$\frac{dW_0}{dV} = \tfrac{8}{3}\pi V(\rho + 3p) = \tfrac{8}{3}\pi\rho(3V_s - 2V) \tag{40}$$

is the first integral of Eq. (24). We also find that with the choice

$$F = \frac{dW_0}{dV} = \tfrac{8}{3}\pi V(\rho + 3p), \tag{41}$$

Eqs. (20) and (21) can be integrated in a straightforward fashion, with the result:

$$Q_1 = V^{-1}E(W - W_0), \tag{42}$$

$$K_1 = -\tfrac{8}{3}\pi(\rho + 3p)E(W - W_0), \tag{43}$$

where E is an arbitrary function of $W - W_0$. A more general integral of these equations exists, which involves an additional arbitrary function of $W - W_0$. While it is straightforward to obtain the more general solution, it is rather lengthy and complicated and it will not be needed in the proof that uniform density stars mush have spherical symmetry.

V. UNIFORM DENSITY STARS MUST BE SHPERICAL

In this final section we will show how the particular identities derived in the last section can be used to prove that spherical symmetry is necessary in the special case of uniform density stellar models. This discussion is a somewhat more detailed version of the proof given in Ref. 14. Before proceeding directly to the proof it is necessary to discuss in more detail the smoothness assumptions and boundary conditions for the solutions of Eqs. (2) and (3) which are appropriate for stellar models. We assume that V and g_{ab} are C^3 except at the boundary ($V = V_s$) between the interior and exterior of the star. This assumption guarantees that suitable coordinates exist so that V and g_{ab} are analytical functions.[15,16] At the suface of the star, the differentiability is reduced however. The exact differentiability can be inferred by requiring that Eqs. (2) and (3) [and consequently Eqs. (10) and (11)] are satisfied even at the surface of the star. Equation (11) implies that the extrinsic curvature H must be continuous at the surface[17]; while Eq. (10) shows that $\nabla_a W$ will have a discontinuity in the direction of the normal to the surface if the density function has a discontinuity there. The magnitude of this discontinuity is given by

$$\lim_{V\to V_s^+} W^{-1}\nabla^a V \nabla_a W - \lim_{V\to V_s^-} W^{-1}\nabla^a V\nabla_a W = -8\pi V_s \lim_{p\to 0+} \rho. \tag{44}$$

In the case of uniform density stars the density function must have a discontinuity at the surface, while other equations of state may not have this discontinuity. To make use of the formalism derived above, we must also take care to properly match the function W_0 across the surface of the star. To this end we will choose the mass constant of Eq. (25) and the constant obtained from integrating Eq. (40) so that W_0 is continuous while its first derivative satisfies the discontinuity equation:

$$\lim_{V\to V_s^+} W^{-1}\nabla^a V\nabla_a W_0 - \lim_{V\to V_s^-} W^{-1}\nabla^a V\nabla_a W_0 = -8\pi V_s \lim_{p\to 0+} \rho. \tag{45}$$

This leads to the following function W_0:

$$W_0 = \tfrac{2}{3}\pi\rho(1 - V^2)^4(1 - V_s^2)^{-3} \quad \text{for } V > V_s, \tag{46}$$

$$W_0 = \tfrac{8}{3}\pi\rho V(3V_s - V) + \tfrac{2}{3}\pi\rho(1 - 9V_s^2) \quad \text{for } V < V_s. \tag{47}$$

Note that by choosing W_0 in this way, the gradient $\nabla_a(W - W_0)$ is continuous even at the surface of the star.

We are now prepared to proceed with the proof of the following theorem:

Theorem: A static asymptotically flat general relativistic stellar model, which is made of uniform (positive) density fluid, is necessarily spherically symmetric.

The goal of the first step in the proof is to use the identities derived in Sec. IV to establish that the function $W - W_0$ must attain its maximum value on the surface of the star. Integrate Eq. (17) over the exterior region of the star using Eqs. (18), (19), (28), and (29) with $B = 0$. The divergence on the left-hand side is converted to a boundary integral at the surface of the star and at infinity. The surface integral at infinity vanishes if A is bounded. Therefore the following relationship is true:

$$-\int_{V=V_s^+} ({}^2g)^{1/2}V^{-1}W^{-1/2}(1 - V^2)^{-2}A(W/W_0)\nabla^a V \times [\nabla_a W - F\nabla_a V]d^2x = I, \tag{48}$$

where

$$I = \int ({}^3g)^{1/2}\{\tfrac{1}{4}Q_1 V^4 W^{-1}R_{abc}R^{abc} + Q_2 W^{-1}|\nabla_a W - F\nabla_a V|^2\}d^3x. \tag{49}$$

Let us choose the function $A(U)$ so that it vanishes for $U < U_0$ and smoothly increases to positive values for $U > U_0$. In this case Q_1 and Q_2 are nonnegative functions. If the maxiumum value that W/W_0 assumed in the exterior of the star were larger than the maximum value which it assumed on the surface of the star, one could choose the constant U_0 to lie somewhere between these values. In this case the boundary integral on the left of Eq. (48) would vanish. Since the volume integral on the right would vanish in this situation only if the star were spherical (see Sec. II), we conclude that W/W_0 attains its maximum value (relative to the exterior region) on the surface of the star, or that the star is spherical. Thus W/W_0 attains its maximum value (relative to the exterior region) on the surface of the star, since this also occurs in the spherical case. Since W_0 also attains its maximum on the surface of the star, it follows that $W - W_0$ also attains its maximum value (relative to the exterior region) on the surface of the star. Next integrate Eq. (17) using Eqs. (18), (19), (42), and (43) over the interior of the star. By appropriately choosing the function E, in an argument analogous to that described above for the exterior, it is straightforward to show that $W - W_0$ attains its maximum value (relative to the interior region) on the surface of the star. Thus the absolute maximum value of $W - W_0$ occurs somewhere on the surface of the star. Also, since the gradient $\nabla_a(W-W_0)$ was shown to be continuous, it must vanish at this maximum point.

The next step in the proof is to show that the function $W-W_0$ is in fact constant in the interior of the star. This is accomlished using a maximum principle for elliptic differential equations.[18] In the interior of the star, Eqs. (17), (18), (19), (42), and (43) with $E=1$ imply that

$$\nabla^a[V^{-1}\nabla_a(W-W_0)]\geqslant 0. \tag{50}$$

The maximum principle for this type of differential equation states (roughly, see Ref. 18 for a precise statement) that if $W-W_0$ satisfies Eq. (50) and has a maximum at a boundary point and if the outward normal derivative of $W-W_0$ is not positive at this maximum point, then $W-W_0$ must be constant. Since the gradient of $W-W_0$ vanishes at the maximum point, $W-W_0$ must be constant. From this it follows (again using Eqs. (17)–(19), (42), and (43) with $E=1$) that $R_{abc}=0$ in the interior of the star.

The final step is to show that W/W_0 is constant in the exterior of the star. Chose $A=1$ and $B=0$ in Eqs. (17)–(19), (28), and (29) to find that

$$\nabla_a[V^{-1}(1-V^2)^{-2}W_0\nabla^a(W/W_0)]\geqslant 0 \tag{51}$$

in the exterior region. We know that W/W_0 attains its maximum (relative to the exterior region) on the boundary of the star. To employ the maximum principle, we must compute the outward directed (that is out of the exterior region) normal derivative of W/W_0. We find

$$\begin{aligned} d(W/W_0)/dn &= -\lim_{V\to V_s^+} W^{-1/2}\nabla^a V\,\nabla_a(W/W_0) \\ &= \lim_{V\to V_s^+} W^{1/2}W_0^{-2}(W-W_0)dW_0/dV. \end{aligned} \tag{52}$$

At the maximum point $W-W_0\geqslant 0$ since this quantity vanishes at infinity. Therefore $d(W/W_0)/dn\leqslant 0$ at the maximum point since $dW_0/dV<0$ there [see Eq. (46)]. The maximum principle therefore guarantees that W/W_0 is constant in the exterior of the star. It follows from Eqs. (17)–(19), (42), and (43) that $R_{abc}=0$ in the exterior of the star also. Consequently (see Sec. II) the star must be spherical.

The argument given above has implicitly assumed that only a single star was present. The argument can be easily generalized to eliminate the possibility of mutiple static uniform density stars. Even if multiple static stars existed, the argument using Eqs. (48) and (49) would still imply that the maximum of W/W_0 would occur on the surface of one of the stars. If one chooses this maximal star to supply the parameters ρ and V_s for Eqs. (46) and (47), the argument given above will go through exactly as before, with the conclusion that the spacetime is spherical, and consequently only one star is present.

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[1]A. Avez, R. Acad. Sci. Paris **257**, 3316 (1963).
[2]A. Avez, Ann. Inst. Henri Poincaré **A 1**, 291 (1964).
[3]H.P. Künzle, Commun. Math. Phys. **20**, 85 (1971).
[4]H.P. Künzle and J.R. Savage, preprint (1979).
[5]L. Lindblom, *Fundamental Properties of Equilibrium Stellar Models*, (Ph.D. Dissertation, University of Maryland, College Park, Maryland, 1978), pp. 90–115.
[6]L. Lindblom and D. Brill, in *Essays in General Relativity: A Festschrift for Abraham Taub* ed. by F. Tipler (Academic Press, New York, 1980).
[7]L. Lindblom, J. Math. Phys. **18**, 2352 (1977).
[8]W. Israel, Phys. Rev. **164**, 1776 (1967).
[9]H. Müller zum Hagen, D.C. Robinson, and H.J. Seifert, Gen. Rel. Grav. **4**, 53 (1973).
[10]D.C. Robinson, Gen. Rel. Grav. **8**, 695 (1977).
[11]L. Lindblom, Astrophys.J. **208**, 873 (1976).
[12]L.P. Eisenhart, *Riemannian Geometry* (Princeton U.P., Princeton, New Jersey, 1925), pp. 89–92.
[13]H.Müller zum Hagen, D.C. Robinson, and H.J. Seifert, Gen. Rel. Grav. **5**, 61 (1974).
[14]L. Lindblom (to be published) (1980).
[15]H. Müller zum Hagen, Proc. Cambridge Philos. Soc. **67**, 415 (1970).
[16]L. Lindblom, J. Math. Phys. **17**, 1810 (1976).
[17]W. Israel, Nuovo Cimento B **44**, 1 (1966); B **48**, 463 (1967).
[18]L. Bers, F. John, and M. Schecter, *Lectures in Applied Mathematics: Partial Differential Equations* Vol. III (Interscience, New York, 1964), p. 150.