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ON THE SECULAR INSTABILITY FROM THERMAL CONDUCTIVITY IN ROTATING STARS[1]

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ABSTRACT

A criterion to test the secular stability of rigidly rotating, isothermal Newtonian stellar models is formulated, which includes the dissipative effects of viscosity and thermal conductivity. This criterion reveals a previously unnoticed fact: thermal conductivity can cause secular instability in rotating stellar models. Any stellar model (except for models having degenerate thermodynamic properties such as an incompressible fluid) which is unstable if viscosity is present would also be unstable if the heat conduction coefficient were nonzero. Furthermore, it is shown that the time scale associated with the heat conduction instability can be shorter than the viscous instability time scale.

Subject headings: instabilities — stars: rotation

I. INTRODUCTION

In this paper it is shown that thermal conductivity can cause secular instability in rigidly rotating, isothermal, Newtonian stellar models. This effect occurs (heuristically) as follows. When a star is perturbed, the temperature of the fluid will fluctuate from its uniform equilibrium value. The resulting temperature gradients give rise to thermal conduction which dissipates energy from the original perturbation. Therefore, if near the equilibrium figure there exist configurations of the fluid which have lower energy, the heat conduction will provide a means to dissipate the excess energy and allow the star to change into the lower energy state. This is called a secular instability.

Secular instabilities of rotating stars have been understood in the past primarily from the study of this phenomenon in the Maclaurin spheroids. In this way Roberts and Stewartson (1963) demonstrated that viscosity could cause secular instability in rotating stars, and similarly Chandrasekhar (1970) showed that gravitational radiation can also cause secular instability. The effect which is explored in this paper, secular instability from thermal conductivity, is not present in the Maclaurin spheroids for the following reason. The initial equilibrium Maclaurin spheroid has uniform density, and must have uniform temperature if it is to be in thermal equilibrium (with a nonzero thermal conductivity). It follows from the first law of thermodynamics and the nonconstancy of the pressure in the star that the temperature must be a function only of the density (and therefore independent of the entropy density of the fluid). Consequently the standard perturbations of the Maclaurin spheroid (which have vanishing density fluctuations) will have vanishing temperature fluctuations. Therefore, the temperature in the perturbed Maclaurin spheroid remains uniform, and as a result no energy dissipation from thermal conductivity will occur. Since the effect of interest here does not occur in the simple analytical Maclaurin spheroids (because of the peculiar degenerate thermodynamic properties of an incompressible fluid), it is necessary to consider a more general class of stellar models. A general formalism for studying secular instability in uniformly rotating stars has been developed by Friedman and Schutz (1978*a*, *b*). This formalism is extended here to include the effects of dissipation from thermal conductivity.

Section II reviews the equations of motion of a fluid stellar model, including the effects of viscosity and thermal conductivity. The first-order Lagrangian perturbations of these equations are given; and a generalization of the canonical energy of the perturbation in a rotating frame (Friedman and Schutz 1978*a*, *b*) is defined. In § III the time derivative of the generalized canonical energy is shown to be negative. This fact is used to formulate a criterion by which the secular stability of rigidly rotating isothermal Newtonian stellar models can be tested. This criterion is not strictly equivalent to the one formulated by Friedman and Schutz (1978*b*) to test for secular instability in the presence of viscosity alone. It is shown, however, that any model which is unstable according to the Friedman and Schutz criterion will also be unstable according to the more general criterion presented here. In § IV some of the implications of this new type of secular instability are discussed. The characteristic time scale associated with the thermal conductivity type instability is compared to the time scale of the viscosity type instability. It is shown that the thermal conductivity time scale can be much shorter than the viscosity time scale; consequently in some cases the physically relevant instability will be the one from thermal conductivity. In particular, the thermal conductivity time scale is much shorter than the viscosity time scale in the core of a newly formed neutron star as long as neutrino transport provides the dissipating mechanism.

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II. THE FLUID EQUATIONS

In this section the equations governing the evolution of a viscous, heat-conducting, Newtonian self-gravitating fluid are reviewed. The equations which describe the perturbations to the stationary equilibrium solutions of these fluid equations are recalled. An energy functional appropriate for the study of secular instability in these models is introduced.

The stellar models considered here are those described by the solutions of the Newtonian equations of a viscous heat-conducting fluid. The four fundamental quantities which describe the state of the fluid are the fluid velocity v^i, the mass density ρ, the entropy per unit mass s, and the gravitational potential φ. These quantities satisfy the following equations of motion:

$$\rho[\partial_t v^i + v^j\nabla_j v^i] = -\nabla^i p + \rho\nabla^i\varphi + \nabla_j(\eta\sigma^{ij}) + \nabla^i(\zeta\theta)\,, \tag{1}$$

$$\rho T[\partial_t s + v^i\nabla_i s] = \nabla^i[\kappa\nabla_i T] + \zeta\theta^2 + \tfrac{1}{2}\eta\sigma_{ij}\sigma^{ij}\,, \tag{2}$$

$$\partial_t\rho + \nabla_i(\rho v^i) = 0\,, \tag{3}$$

$$\nabla^i\nabla_i\varphi = -4\pi G\rho\,. \tag{4}$$

To specify the additional thermodynamic properties of the fluid, the above equations must be supplemented by an equation of state. The internal energy density of the fluid ϵ is assumed to be a given function of ρ and s:

$$\epsilon = \epsilon(\rho, s)\,. \tag{5}$$

The temperature T and pressure p are then defined as follows:

$$T = \frac{1}{\rho}\left(\frac{\partial\epsilon}{\partial s}\right)_\rho\,, \tag{6}$$

$$p = \rho\left(\frac{\partial\epsilon}{\partial\rho}\right)_s - \epsilon\,. \tag{7}$$

Other kinematic properties of the fluid are described by the shear tensor σ^{ij} and the expansion θ:

$$\sigma^{ij} = \nabla^i v^j + \nabla^j v^i - \tfrac{2}{3}g^{ij}\theta\,, \tag{8}$$

$$\theta = \nabla_i v^i\,. \tag{9}$$

Finally, the dissipative processes in the fluid are governed by the coefficients of viscosity, η and ζ, and the heat conduction coefficient κ. These are assumed to be positive functions of ρ and s.

The unperturbed equilibrium solutions to equations (1)–(4) are assumed to be stationary:

$$\partial_t v^i = \partial_t\rho = \partial_t s = 0\,. \tag{10}$$

It follows that the solutions must also be axisymmetric, rigidly rotating, and isothermal (see Lindblom 1978, § 3); therefore,

$$\sigma^{ij} = \theta = v^i\nabla_i\rho = v^i\nabla_i s = \nabla_i T = 0\,. \tag{11}$$

The first-order perturbations of these equations are discussed here primarily in the Lagrangian framework, in a notation which is the same as that of Friedman and Schutz (1978*a*). Eulerian variations of the fluid quantities are denoted by δ, while Lagrangian variations, Δ, are defined by $\Delta = \delta + \mathscr{L}_\xi$. ($\mathscr{L}_\xi$ denotes the Lie derivative along the vector field ξ^i.) The Lagrangian displacement vector ξ^i is chosen to relate fluid elements in the unperturbed star with those in the perturbed configuration. It follows from this definition of ξ^i, and the first-order variation of equation (3), that the variations in v^i and ρ are related to ξ^i by

$$\delta v^i = \partial_t\xi^i + v^j\nabla_j\xi^i - \xi^j\nabla_j v^i\,, \tag{12}$$

$$\delta\rho = -\nabla_i(\rho\xi^i)\,. \tag{13}$$

The variations in the gravitational potential φ are related to ξ^i through the variation of equation (4):

$$\nabla^i\nabla_i\delta\varphi = 4\pi G\nabla_i(\rho\xi^i)\,. \tag{14}$$

The independent variables of the perturbation therefore are the Lagrangian displacement ξ^i, and the variation in the entropy per unit mass Δs.

The evolution of a perturbation of the star is determined by the evolution of the independent functions ξ^i and Δs. The time development of these quantities are given by the variations of equations (1) and (2):

$$A_{ij}\partial_t^2\xi^j + B_{ij}\partial_t\xi^j + C_{ij}\xi^j = F_i \,, \tag{15}$$

$$\rho T[\partial_t(\Delta s) + v^i\nabla_i(\Delta s)] = \nabla^i\{\kappa\nabla_i(\delta T)\}\,. \tag{16}$$

The operators A_{ij}, B_{ij}, and C_{ij} are defined (for an arbitrary vector field ξ^i) as follows:

$$A_{ij}\xi^j = \rho\xi_i \,, \tag{17}$$

$$B_{ij}\xi^j = 2\rho v^k\nabla_k\xi_i \,, \tag{18}$$

$$C_{ij}\xi^j = \rho v^k\nabla_k(v^j\nabla_j\xi_i) - \nabla_i\left[\rho\left(\frac{\partial p}{\partial\rho}\right)_s\nabla_j\xi^j\right] + \nabla_i p\nabla_j\xi^j - \nabla_j p\nabla_i\xi^j - \rho\nabla_i\delta\varphi - \rho\xi^j\nabla_i\nabla_j\varphi\,. \tag{19}$$

In equation (19) the quantity $\delta\varphi$ is taken as the implicit function of ξ^i given by equation (14). The driving force F_i in equation (15) is given by the following expression:

$$F_i = -\nabla_i\left[\left(\frac{\partial p}{\partial s}\right)_\rho\Delta s\right] + \nabla^j(\eta\delta\sigma_{ij}) + \nabla_i(\zeta\delta\theta)\,. \tag{20}$$

The variations in the shear and expansion are related to the Eulerian variations in the velocity (and thus to ξ^i by eq. [12]) as follows:

$$\delta\sigma^{ij} = \nabla^i\delta v^j + \nabla^j\delta v^i - \tfrac{2}{3}g^{ij}\delta\theta\,, \tag{21}$$

$$\delta\theta = \nabla_i\delta v^i\,. \tag{22}$$

The secular stability of these rotating stellar models has been studied by Friedman and Schutz (1978*b*) by considering the properties of a certain functional of the perturbed stellar model, which is related to the energy of the perturbation. This energy functional is defined by

$$\tilde{E} = \tfrac{1}{2}\int\{\partial_t\xi^i A_{ij}\partial_t\xi^j + \xi^i C_{ij}\xi^j + 2\mathscr{L}_v\xi^i[A_{ij}\partial_t\xi^j + \tfrac{1}{2}B_{ij}\xi^j]\}d^3x\,. \tag{23}$$

The time derivative of this expression is given by

$$\frac{d\tilde{E}}{dt} = \int \delta v^i F_i d^3x\,. \tag{24}$$

This energy functional is not appropriate for the study of secular instability when the coefficient of thermal conductivity κ is nonzero, however, because of the following qualitative difference between a heat-conducting and a non–heat-conducting fluid. From equation (16) it follows that initially adiabatic perturbations ($\Delta s = 0$) will remain adiabatic (to first order) only if the heat conduction coefficient vanishes. Therefore, when heat conductivity is present, it is no longer possible to ignore the nonadiabatic contributions to the motion of the fluid. It is not surprising, therefore, to find that an appropriate energy functional contains nonadiabatic contributions. A generalized energy functional E,

$$E = \tilde{E} + \int \rho\Delta s\left[\left(\frac{\partial T}{\partial\rho}\right)_s\Delta\rho + \frac{1}{2}\left(\frac{\partial T}{\partial s}\right)_\rho\Delta s\right]d^3x\,, \tag{25}$$

is shown in the next section to be a useful tool for the study of the secular instability of rigidly rotating, isothermal stellar models which have nonvanishing coefficients of viscosity and heat conduction.

III. A CRITERION TO TEST FOR SECULAR INSTABILITY

The generalized energy functional E is used in this section to define a criterion to test for secular instability. The time derivative of E is computed in Appendix A, with the result:

$$\frac{dE}{dt} = -\int\left\{\tfrac{1}{2}\eta\delta\sigma_{ij}\delta\sigma^{ij} + \zeta(\delta\theta)^2 + \frac{\kappa}{T}\nabla_i(\delta T)\nabla^i(\delta T)\right\}d^3x\,. \tag{26}$$

Therefore E is a decreasing function of time. Note that this expression for the time derivative depends only on Eulerian variations of the fluid variables. It is also shown in Appendix B that E itself can be expressed purely in terms of Eulerian variations:

$$E = \frac{1}{2}\int\left\{\rho\delta v^i\delta v_i + \frac{1}{\rho}\left(\frac{\partial\rho}{\partial p}\right)_s(\delta p)^2 + \frac{1}{\rho}\left(\frac{\partial\rho}{\partial s}\right)_p\frac{dp}{ds}(\delta s)^2 - \frac{1}{4\pi G}\nabla_i\delta\varphi\nabla^i\delta\varphi\right\}d^3x\,. \tag{27}$$

These expressions show that E and its time derivative are invariant under the "trivial" gauge transformations (see Friedman and Schutz 1978*a*, *b*) and therefore it is not necessary to restrict the choice of initial data for the perturbation (ξ^i, $\partial_t\xi^i$, Δs) to some canonical set of data.

While it is not appropriate to restrict the class of perturbations (ξ^i, $\partial_t\xi^i$, Δs) to those which are locally adiabatic, $\Delta s = 0$, it is appropriate to consider only those perturbations which leave the total entropy of the star:

$$S = \int \rho s d^3x\,, \tag{28}$$

invariant (to first order). The first-order variation of the total entropy δS is given by the expression

$$\delta S = \int \rho \Delta s d^3 x\,. \tag{29}$$

It is easy to verify (using eq. [16]) that

$$\frac{d(\delta S)}{dt} = 0 \tag{30}$$

(to first order). Therefore perturbations which have $\delta S = 0$ initially will maintain $\delta S = 0$ throughout their evolution. Perturbations for which $\delta S = 0$ will be called quasi-adiabatic.

The criterion to test the secular stability of isothermal rigidly rotating stellar models can now be stated. If the functional E is positive for all quasi-adiabatic choices of initial data (ξ^i, $\partial_t\xi^i$, Δs), then the stellar model is stable, since the energy functional must decrease and therefore only a finite amount of energy can be dissipated. On the other hand, if E is negative for some choice of initial data, then the perturbation can grow without bound while decreasing the energy E infinitely. Therefore the stellar model will be unstable or marginally unstable if there exist quasi-adiabatic initial data (ξ^i, $\partial_t\xi^i$, Δs) which make the energy functional E negative.

For adiabatic perturbations, $\Delta s = 0$, the energy functional E reduces to $\tilde{E}$, which was used by Friedman and Schutz (1978*b*) to formulate a criterion to test secular stability in the presence of viscosity alone. If there exist initial data (ξ^i, $\partial_t\xi^i$) which make $\tilde{E}$ negative, then clearly these same data with $\Delta s = 0$ will make E negative. Thus any stellar model which is secularly unstable when viscosity is present (according to the criterion of Friedman and Schutz) would also be unstable if the fluid had nonzero heat conductivity. The converse is not obviously true, however. The nonadiabatic contributions to E do not have definite sign, and therefore could give rise to instabilities which have no analog in the adiabatic case.

Another simple class of perturbations are those for which $\xi^i = \partial_t\xi^i = 0$. For these perturbations $\Delta\rho = 0$, so the energy functional reduces to

$$E = \frac{1}{2}\int \rho \left(\frac{\partial T}{\partial s}\right)_\rho (\Delta s)^2 d^3x\,. \tag{31}$$

Thus one recovers the well-known condition for thermodynamic stability, $(\partial T/\partial s)_\rho > 0$.

IV. DISCUSSION

In the last section it was shown that thermal conductivity could give rise to secular instability in rigidly rotating isothermal stellar models. To determine whether or not this instability is relevant in any real astrophysical context, one must determine the time scale with which the instability grows. The only reliable way of doing this would be to actually solve the perturbation equations for some unstable mode, and compute explicitly the growth rate of the perturbation. This is not an easy task, however, especially for the heat conduction instability. The only simple analytic equilibrium rotating stellar models are the Maclaurin spheroids, and it was shown in § I that the heat conduction instability does not occur in these models. The only simple calculation which can be done, therefore, involves writing down the characteristic time scales of the physical processes involved. These time scales can be derived by dimensional arguments based on the equations of motion (see, e.g., Lindblom and Detweiler 1979). The time scale associated with the viscous instability is given by

$$\tau_v = \frac{3M}{4\pi R\eta}\,, \tag{32}$$

where M is the total mass of the star, R is the average radius, and η is the average viscosity. The thermal conductivity instability time scale should be comparable to the characteristic cooling time scale which can be derived from the thermal diffusion equation. This time scale is given by

$$\tau_c = \frac{3Mc_p}{4\pi R\kappa}\,, \tag{33}$$

where c_p is the average specific heat (per unit mass) and κ is the average heat conduction coefficient. These time scales can be used to estimate the growth times of unstable modes (assuming the mode has a low spherical-harmonic index l).

A rapidly rotating neutron star produced by the catastrophic gravitational collapse of a stellar core is one place that secular instabilities could have a decisive role in determining the evolution of the star. In this situation, the dissipative mechanism will be primarily due to radiation transport by the neutrinos produced during the collapse. The viscosity and heat conduction coefficients for this process are given by (see, e.g., Lindblom and Detweiler 1979)

$$\eta = \tfrac{7}{30} a T^4 \lambda c^{-1}, \tag{34}$$

$$\kappa = \tfrac{7}{6} a T^3 \lambda c, \tag{35}$$

where a and c are respectively the Stefan-Boltzmann constant and the speed of light, T is the temperature of the star, and λ is the mean free path of the neutrinos within the star. The specific heat for a hot neutron gas is given by the formula

$$c_p = 5k/2m, \tag{36}$$

where k is the gas constant and m is the mass of the neutron.

Consider the ratio of the viscous instability time scale to the heat-conduction instability time scale for the situation described above:

$$\tau_v/\tau_c = 2mc^2/kT. \tag{37}$$

Since the mass of the neutron ($mc^2 = 1$ GeV) is much larger than the typical postcollapse temperature ($kT = 20$ MeV), it follows that the heat-conduction instability will grow at a faster rate in this situation than the viscous instability. Substituting typical neutron star parameters into equation (33), one estimates that the heat-conducting instability of a sufficiently rapidly rotating neutron star would grow with a time scale of about 100 s (see, e.g., Sawyer and Soni 1979).

APPENDIX A

THE TIME DERIVATIVE OF E

In this appendix it is shown that the time derivative of the energy functional E (defined by eq. [25]) is negative. Begin with equation (25) and use equation (24) to evaluate the time derivative of $\tilde{E}$:

$$\frac{dE}{dt} = \int \delta v^i F_i d^3x + \int \rho\left[\left(\frac{\partial T}{\partial \rho}\right)_s \Delta\rho\,\partial_t(\Delta s) + \left(\frac{\partial T}{\partial \rho}\right)_s \Delta s\,\partial_t(\Delta\rho) + \left(\frac{\partial T}{\partial s}\right)_\rho \Delta s\,\partial_t(\Delta s)\right] d^3x. \tag{A1}$$

The time derivative of Δs is evaluated using equation (16) while the time derivative of $\Delta\rho$ is given by the first order variation of equation (3):

$$\partial_t(\Delta\rho) + v^i\nabla_i(\Delta\rho) = -\rho\nabla_i\delta v^i. \tag{A2}$$

By using these expressions for the time derivatives of $\Delta\rho$ and Δs, using the properties of the equilibrium model (eqs. [10]–[11]), and setting the integrals of divergences to zero, the expression for the time derivative of E can be converted to the following:

$$\frac{dE}{dt} = -\int [\tfrac{1}{2}\eta\delta\sigma_{ij}\delta\sigma^{ij} + \zeta(\delta\theta)^2] d^3x + \int \frac{1}{T}\left[\left(\frac{\partial T}{\partial \rho}\right)_s \Delta\rho + \left(\frac{\partial T}{\partial s}\right)_\rho \Delta s\right] \nabla_i[\kappa\nabla^i\delta T] d^3x$$
$$+ \int \left[\left(\frac{\partial p}{\partial s}\right)_\rho - \rho^2\left(\frac{\partial T}{\partial \rho}\right)_s\right] \Delta s\nabla_i\delta v^i d^3x. \tag{A3}$$

The third integral on the right-hand side of equation (A3) vanishes because of the Maxwell relation:

$$\left(\frac{\partial p}{\partial s}\right)_\rho = \rho^2\left(\frac{\partial T}{\partial \rho}\right)_s. \tag{A4}$$

Since the equilibrium stellar model is taken to be isothermal, the Lagrangian and Eulerian variations of the temperature are equal; thus

$$\delta T = \Delta T = \left(\frac{\partial T}{\partial \rho}\right)_s \Delta\rho + \left(\frac{\partial T}{\partial s}\right)_\rho \Delta s. \tag{A5}$$

Using this expression in equation (A3), we arrive at the final expression for the time derivative of E:

$$\frac{dE}{dt} = -\int \left[\tfrac{1}{2}\eta\delta\sigma_{ij}\delta\sigma^{ij} + \zeta(\delta\theta)^2 + \frac{\kappa}{T}\nabla_i(\delta T)\nabla^i(\delta T)\right]d^3x\,. \tag{A6}$$

The viscosity coefficients, η and ζ, and the heat conduction coefficient κ are nonnegative functions; consequently the right-hand side of equation (A6) is negative. Therefore, the energy functional E is a decreasing function of time.

The energy functional E does not represent the total energy of the perturbation in the rotating frame of the star (even for adiabatic perturbations as implied by Friedman and Schutz 1978*b*). If it did represent the total energy, then its time derivative would be zero since energy is conserved. When dissipative processes are present, energy is converted from kinetic to thermodynamic forms; however, the total energy is conserved (see, e.g., Landau and Lifshitz 1959, § 49). In particular, E does not contain some of the thermodynamic energy terms. For example, E does not contain the term $\int \rho T\Delta s d^3x$, and this term has a nonzero time derivative (at second order) even in the purely viscous case considered by Friedman and Schutz (1978*b*). The fact that E does not represent the total energy of the perturbation does not decrease its value as a tool for the study of secular instability, however.

APPENDIX B

THE GAUGE INVARIANCE OF E

The purpose of this appendix is to demonstrate that the energy functional E defined by equation (25) is invariant under the "trivial" gauge transformations (see Friedman and Schutz 1978*a*). A "trivial" transformation of the perturbation functions ξ^i and Δs is one which changes the values of the Lagrangian variations of the fluid parameters (Δv^i, $\Delta\rho$, Δs) while leaving the physical Eulerian variations (δv^i, $\delta\rho$, δs) unchanged. It is shown here that E can be written as a functional involving only the Eulerian variations. It follows, therefore, that E is invariant under the "trivial" transformations. The derivation sketched here (briefly) is a straightforward generalization of the demonstration by Friedman and Schutz (1978*a*) of the gauge invariance of $\tilde{E}$ when the perturbation is adiabatic.

Consider first the expression for $\tilde{E}$ in equation (23). Use the expressions for the operators A_{ij} and B_{ij} from equations (17)–(18) and the definition of δv^i to express $\tilde{E}$ as

$$\tilde{E} = \tfrac{1}{2}\int [\rho\delta v^i\delta v_i + \rho(v^j\nabla_j\xi^i)(v^k\nabla_k\xi_i) + \rho\xi^i\xi^j\nabla_i(v^k\nabla_k v_j) + \xi^i C_{ij}\xi^j]d^3x\,. \tag{B1}$$

The expression for C_{ij} in equation (19) can now be substituted, as well as the expression for the acceleration from the equilibrium version of equation (1):

$$v^j\nabla_j v^i = -\frac{1}{\rho}\nabla^i p + \nabla^i\varphi\,. \tag{B2}$$

After several integrations by parts, and the use of equations (13)–(14), it follows that

$$\tilde{E} = \frac{1}{2}\int\left\{\rho\delta v^i\delta v_i + \frac{1}{\rho}\delta p\delta\rho - \frac{1}{\rho}\Delta\rho\xi^j\left[\nabla_j p - \left(\frac{\partial p}{\partial\rho}\right)_s\nabla_j\rho\right] - \frac{1}{4\pi G}\nabla_i\delta\varphi\nabla^i\delta\varphi - \frac{1}{\rho}\left(\frac{\partial p}{\partial s}\right)_\rho\delta\rho\Delta s\right\}d^3x\,. \tag{B3}$$

To simplify this expression further, the fact that the fluid is barotropic in its equilibrium configuration is used (see, e.g., Lindblom 1978, § 3). From this fact it follows that the level surfaces of the entropy, the pressure, and the mass density all coincide. Consequently there exist functional relationships of the form

$$p = p(s) \qquad \text{and} \qquad \rho = \rho(s)\,. \tag{B4}$$

This makes it possible to convert gradients of the density function into gradients of the entropy

$$\nabla_i\rho = \frac{d\rho}{ds}\nabla_i s\,, \tag{B5}$$

where $d\rho/ds$ is the derivative of the expression in equation (B4). Using this fact, the third term on the right-hand side of equation (B3) can be converted to

$$-\frac{1}{\rho}\Delta\rho\xi^j\left[\nabla_j p - \left(\frac{\partial p}{\partial\rho}\right)_s\nabla_j\rho\right] = \frac{1}{\rho}\left(\frac{\partial p}{\partial s}\right)_\rho\delta s\left[\delta\rho + \frac{d\rho}{ds}(\Delta s - \delta s)\right] - \frac{1}{\rho}\left(\frac{\partial p}{\partial s}\right)_\rho\Delta\rho\Delta s\,. \tag{B6}$$

This expression is substituted back into equation (B3), and the additional terms from equation (25) are now added to form E. To simplify the resulting expression, one more obscure thermodynamic identity is needed. This is obtained by expressing the gradient of the temperature using equation (B5):

$$\nabla_i T = \left[\left(\frac{\partial T}{\partial s}\right)_\rho + \frac{d\rho}{ds}\left(\frac{\partial T}{\partial \rho}\right)_s\right]\nabla_i s\,. \tag{B7}$$

Since the star is isothermal but not isentropic, it follows that

$$\left(\frac{\partial T}{\partial s}\right)_\rho + \frac{d\rho}{ds}\left(\frac{\partial T}{\partial \rho}\right)_s = 0\,. \tag{B8}$$

In this way the expression for E can be converted to the following form:

$$E = \frac{1}{2}\int\left\{\rho\delta v^i\delta v_i + \frac{1}{\rho}\,\delta\rho\delta p + \frac{1}{\rho}\left(\frac{\partial p}{\partial s}\right)_\rho \delta s\delta\rho - \frac{1}{\rho}\left(\frac{\partial p}{\partial s}\right)_\rho \frac{d\rho}{ds}(\delta s)^2 - \frac{1}{4\pi G}\,\nabla_i\delta\varphi\nabla^i\delta\varphi\right\}d^3x\,. \tag{B9}$$

This expression depends only on the Eulerian variations of the fluid variables, and consequently is invariant under the "trivial" gauge transformations. While this expression is sufficient to demonstrate the gauge invariance of E, a more appealing form may be obtained by repeated use of the identity

$$0 = \left(\frac{\partial p}{\partial s}\right)_\rho + \left(\frac{\partial \rho}{\partial s}\right)_p\left(\frac{\partial p}{\partial \rho}\right)_s\,. \tag{B10}$$

The resulting expression for E is

$$E = \frac{1}{2}\int\left\{\rho\delta v^i\delta v_i + \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_s(\delta p)^2 + \frac{1}{\rho}\left(\frac{\partial \rho}{\partial s}\right)_p \frac{dp}{ds}(\delta s)^2 - \frac{1}{4\pi G}\,\nabla_i\delta\varphi\nabla^i\delta\varphi\right\}d^3x\,. \tag{B11}$$

This expression is precisely the same as that derived by Friedman and Schutz (1978*a*), except that this energy functional allows nonadiabatic perturbations while theirs does not. Furthermore, their arguments that $dp/ds > 0$ based on local stability apply here also.

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