Mirror planes in Newtonian stars with stratified flows[a)]

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(Received 7 June 1977)

This paper shows that a certain class of Newtonian stellar models must possess a plane of mirror symmetry. A corollary of this result is that static Newtonian stars must be spherical. The new features of the results given here are that: (a) The assumptions about the velocity distribution of the fluid are weaker than previous treatments and (b) the method of proof given here does not depend as strongly on the linearity of the gravitational field equations as the previously published treatments. Therefore, this proof may serve as a model for a general relativistic generalization of the mirror plane theorem.

1. INTRODUCTION

An interesting feature of equilibrium stellar models is that extra symmetries are acquired from the field equations and the boundary conditions by the stationary equilibrium configurations. Perhaps the oldest known result of this type is Lichtenstein[1,2] and Wavre's[3] proof that rotating Newtonian stellar models must have a plane of mirror symmetry which is perpendicular to the rotation axis of the star. A related theorem by Carleman[4] and Lichtenstein[2] shows that static Newtonian stellar models must be spherical. A few results are also known for general relativistic models: static black holes are spherical[5]; stationary black holes are axisymmetric[6]; and stationary viscous stars are axisymmetric.[7]

This paper will present a new type of proof of the mirror plane theorem for Newtonian stellar models. The assumptions on which the present proof is based are somewhat weaker than those used previously. It had been assumed that the fluid motion in the star was purely azimuthal; here we assume that there is a Cartesian coordinate system in which the z component of the velocity vanishes. (Thus, the velocity field of the fluid will be called stratified.) This weaker assumption allows us to consider somewhat more complex velocity distributions such as those in the Dedekind ellipsoids.[8] Furthermore, we do not make any assumption about stationarity here. Thus, we are able to prove the existence of mirror symmetry for objects which are nonaxisymmetric and rotate with respect to the inertial frame of reference (e.g., the Jacobi and Riemann S ellipsoids[8]).

The method of proof employed in the present work may also be of some interest. This proof is based on the maximum principles (see the Appendix) which must be satisfied by the solutions to certain elliptic differential equations. This proof depends in a less crucial way on the linearity of the gravitational field equations than the Green's function approach taken by Lichtenstein[2] and Wavre.[3] Therefore, the present type of proof is more likely to form the basis for a general relativistic generalization of this theorem than the previous approaches to this problem.

[a)]This research was supported by the National Science Foundation Grant GP-25548.

We now give a qualitative outline of the proof which is given in detail in the following sections. Section 2 makes explicit the physical and mathematical assumptions on which the proof of this theorem is based. The purpose of Sec. 3 is to construct the plane which is shown to be a mirror plane in Sec. 4. We begin by considering the set of chords which are parallel to the z axis, and which have both endpoints on the same level surface of the gravitational potential function. Lemma 2 is used to show that every point is the endpoint of some such chord. Next we consider the set of midpoints of those chords. For this purpose we define a function m_ϕ, which maps the endpoints of chords into their midpoints. In Lemma 3 we show that there is a chord whose midpoints z component, z_m, is larger than or equal to the z component of the midpoint of any other chord. We will decompose each of the functions into even and odd parts with respect to reflection about the plane $z=z_m$; and we will show that this plane is a mirror plane of the star. In Lemma 4 we derive the important fact that the odd part of the mass density, ϵ^-, is negative for all z exceeding z_m. In Sec. 4 we prove the main theorem. We show that the odd part of the gravitational potential, ϕ^-, must satisfy the differential equation $\nabla_i\nabla^i\phi^- = -4\pi G\epsilon^- \geq 0$ for all $z \geq z_m$; this follows from Lemma 4. In addition we argue that ϕ^- must have a maximum in the half space $z > z_m$. The maximum principles for this type of differential equation are then invoked to show that in fact $\phi^- = 0$ everywhere. It follows that the odd parts of the mass density and pressure must vanish also. Thus the star must have a plane of mirror symmetry.

2. NEWTONIAN STELLAR MODELS

We will be considering the properties of stratified Newtonian stellar models. These models are completely defined by the following functions of the Cartesian coordinates x, y, z:

ϕ—gravitational potential,

ϵ—mass density,

p—pressure,

v_i—velocity of fluid,

a_i—acceleration of fluid.

These functions are assumed to satisfy the usual differential equations which describe a Newtonian fluid stellar model:

$$\nabla_i \nabla^i \phi = -4\pi G\epsilon, \tag{1}$$

$$\epsilon a_i = -\nabla_i p + \epsilon \nabla_i \phi, \tag{2}$$

$$a_i = \partial v_i/\partial t + v^j \nabla_j v_i. \tag{3}$$

In addition we make the following assumptions:

(a) The z component of the velocity of the fluid vanishes (this defines our meaning of stratified).

(b) The density is a function of the pressure: $\epsilon(p)$ with $d\epsilon/dp \geq 0$, $\epsilon \geq 0$, and $p \geq 0$.

(c) The density has compact support at every instant of time.

(d) The gravitational potential, ϕ, vanishes as $x^2 + y^2 + z^2 \to \infty$.

(e) The gravitational potential is C^3 except at the boundary of the star, where it is C^1 with respect to normal derivatives and C^2 with respect to tangential derivatives.

(f) The magnitudes of the functions ϵ, p, and ϕ are bounded.

The following lemma shows that if the velocity field of the star is stratified, then the Euler's equation (2) can be written in an important simplified form.

Lemma 1: The Euler's equation for a Newtonian stellar model which satisifes assumptions (a) and (b) may be written in the form

$$\nabla_i p = \epsilon \nabla_i \psi, \tag{4}$$

where $\psi = \phi - \Upsilon$ and Υ is some function which is independent of z.

Proof: Equation (2) may be written in the form $a_i = -\epsilon^{-1}\nabla_i p + \nabla_i \phi$. When assumption (b) is satisfied, the right-hand side is a gradient, thus $a_i = \nabla_i \Upsilon \equiv -\epsilon^{-1}\nabla_i p + \nabla_i \phi$. This can be re-arranged into the form of Eq. (4). Also since $v_z = 0$ by assumption (a), it follows that $a_z = 0$ by Eq. (3). Therefore $a_z = \partial\Upsilon/\partial z = 0$. ∎

We note that for the special case of an azimuthal velocity field, $v^\varphi = \Omega$, the potential Υ takes the familiar form of the centrifugal potential, $\nabla_i \Upsilon = -\frac{1}{2}\Omega^2 \nabla_i (x^2 + y^2)$.

3. PRELIMINARY LEMMAS

To construct the plane, $z = \text{const}$, which we show in Sec. 4 is a plane of mirror symmetry of the stellar model, we need to classify the points in the star, based on the nearby behavior of the gravitational potential ϕ.

Definition: A point (x, y, z) will be called normal if $\partial\phi/\partial z(x, y, z) \neq 0$; and a point will be called special if $\partial\phi/\partial z(x, y, z) = 0$.

Lemma 2: Let ϕ be the gravitational potential of a stratified Newtonian stellar model satisfying assumptions (a) through (f). For every normal point (x, y, z) there exists a unique associated point $(x, y, \bar{z})$ which has the property $\phi(x, y, z) = \phi(x, y, \bar{z})$ and $\phi(x, y, z) < \phi(x, y, z')$ for all z' between z and $\bar{z}$. (A special point is said to be associated with itself.)

Proof: Let us first show that $\phi(x, y, z) > 0$ everywhere. If there is a point with $\phi(x, y, z) \leq 0$, then we could find some point, say (x', y', z'), with $\phi(x', y', z') \leq \phi(x, y, z)$ for all points (x, y, z). By Eq. (1) and assumption (b) we have $\nabla_i \nabla^i \phi \leq 0$. Using Theorem 2A (see the Appendix) one can show that if the point (x', y', z') exists, then $\phi = 0$ everywhere. If the point (x', y', z') lies on the boundary of the star, a slightly different argument using Theorem 1A gives the same result, $\phi = 0$. Thus we can conclude that ϕ must be positive everywhere.

We next consider the normal point (x, y, z). One can start at (x, y, z) and proceed along the line $(x, y) = \text{const}$ in the direction of increasing ϕ. When one reaches points having sufficiently large values of $x^2 + y^2 + z^2$, the potential ϕ will become arbitrarily small. This guarantees that a point, say $(x, y, \hat{z})$, will be reached along the line at which $\phi(x, y, z) = \phi(x, y, \hat{z})$. If one takes the first such point reached along the line, say $(x, y, \bar{z})$, then $\phi(x, y, z') > \phi(x, y, z)$ for all z' between z and $\bar{z}$. Thus $(x, y, \bar{z})$ is associated with (x, y, z) and the lemma is proved. ∎

To assist in the construction of the plane which is shown to be a symmetry plane of the model in Sec. 4, we will consider the following function.

Definition: The function m_ϕ maps points (x, y, z) from the support of the mass density function into some subset of R^3. We define

$$m_\phi(x, y, z) \equiv (x, y, \tfrac{1}{2}[z + \bar{z}]), \tag{5}$$

where $(x, y, \bar{z})$ is the point associated with (x, y, z).

The next lemma will derive an important property of the function m_ϕ.

Lemma 3: There exists a point (x_0, y_0, z_0) in the domain of m_ϕ, whose image $(x_0, y_0, z_m) \equiv m_\phi(x_0, y_0, z_0)$ is a least upper bound of the z component of the range of m_ϕ; i.e., for every point (x, y, z) in the range of m_ϕ, $z_m \geq z$.

Proof: Let us first argue that the z components of the range of m_ϕ are bounded. We can consider the total potential ψ, defined in Lemma 1. The function m_ψ, constructed using ψ rather than ϕ, is identical to the function m_ϕ because $\phi - \psi = \Upsilon$ is independent of z. By Eq. (4) the level surfaces of ψ coincide with the level surfaces of the functions ϵ and p. Therefore the points which are associated with normal points within the support of the density will also lie within the support of the density. Thus, the range of m_ϕ must be bounded since the domain is bounded by assumption (c). Since the range of m_ϕ is bounded, the z component of the range must also be bounded and therefore must have a least upper bound, say z_m.

We will now show that z_m is the z component of some element in the range of m_ϕ. In any case, there must be a sequence of numbers ζ_n each of which is the z component of some element of the range of m_ϕ, and $\lim \zeta_n = z_m$. There must also be a corresponding sequence of points ξ_n in the domain of m_ϕ whose images have ζ_n as z components: $m_\phi(\xi_n)=(x_n, y_n, \zeta_n)$. The domain of m_ϕ is compact, therefore, there is a subsequence ξ_n' of ξ_n which converges to a point in the domain, say $\lim\xi_n' =(x_0, y_0, z_0)$. It follows that $\lim m_\phi(\xi_n')=(x_0, y_0, z_m)$. The prime will henceforth be dropped from the name of the sequence of points ξ_n'. If m_ϕ were a continuous function, it would follow that $m_\phi(x_0, y_0, z_0)=(x_0, y_0, z_m)$ and the proof would be complete. m_ϕ is not necessarily continuous however.

Let us first consider the case where there is a subsequence ξ_n'' of ξ_n which are all special points. At each of these points we have $\partial\phi/\partial z(\xi_n'')=0$; and since $\partial\phi/\partial z$ is a continuous function, $(\partial\phi/\partial z)(z_0, y_0, z_0)=0$. For special points $m_\phi(x, y, z)=(x, y, z)$, therefore $\lim m_\phi(\xi_n'')=(x_0, y_0, z_0)=(x_0, y_0, z_m)$. Therefore (x_0, y_0, z_m) must be an element of the domain of m_ϕ with the property $m_\phi(x_0, y_0, z_m)=(x_0, y_0, z_m)$. Thus we have shown that the lemma follows if there exists a subsequence ξ_n'' of special points.

The other case we need to consider is when ξ_n are all normal points when n becomes sufficiently large. To each of the normal points ξ_n (with z component ω_n) there is an associated point $\bar{\xi}_n$ (with z component $\bar{\omega}_n$). We also know that $\lim\omega_n=z_0$ and $\lim\frac{1}{2}(\omega_n+\bar{\omega}_n)=z_m$, thus $\lim\bar{\omega}_m =2z_m-z_0$. There are three possibilities: $z_0=z_m$, $z_0 >z_m$, and $z_0<z_m$. We will consider first the case where $z_0=z_m$. The chord connecting each pair of points ξ_n to $\bar{\xi}_n$ in our sequence must contain a point ξ_n'', where $\partial\phi/\partial z(\xi_n'')=0$. Thus, the sequence ξ_n'' are all special points. Furthermore $\lim\xi_n''=\lim\xi_n=\lim\bar{\xi}_n=(x_0, y_0, z_0)$. Thus, we have a sequence of special points whose limit point is (x_0, y_0, z_0). We have shown above that the lemma follows in this case. We next consider the case where $z_0>z_m$; then (x_0, y_0, z_0) must be a normal point with associated point $(x_0, y_0, \bar{z}_0)$. It follows that $\bar{z}_0\leq 2z_m-z_0$ because z_m is the least upper bound. Since ϕ is a continuous function $\lim\phi(\xi_n)=\phi(x_0, y_0, z_0)=\lim\phi(\bar{\xi}_n) =\phi(x_0, y_0, 2z_m-z_0)$. Therefore the point $(x_0, y_0, 2z_m-z_0)$ must be the point associated with (x_0, y_0, z_0) and as a result $m_\phi(x_0, y_0, z_0)=(x_0, y_0, z_m)$ and the lemma follows. The last possibility is that $z_0<z_m$. In this case the sequence of associated points $\bar{\xi}_n$ must converge to $(x_0, y_0, 2z_m-z_0)$ and $2z_m-z_0>z_m$. The same argument as the one given for the case $z_0>z_m$ shows that (x_0, y_0, z_0) is the point associated with $(x_0, y_0, 2z_m-z_0)$. In this case $m_\phi(x_0, y_0, 2z_m-z_0)=(x_0, y_0, z_m)$ and the lemma follows. ∎

We can now derive a very important inequality for the old part of the density function, when it is taken with respect to the plane $z=z_m$.

Lemma 4: Let ϵ be the mass density of a stratified Newtonian stellar model satisfying assumptions (a) through (f). Then,

$$\epsilon^-(x, y, z)\equiv\tfrac{1}{2}\epsilon(x, y, z)-\tfrac{1}{2}\epsilon(x, y, 2z_m-z)\leq 0 \ \forall\ z\geq z_m.$$

Proof: Consider a point (x, y, z) with $z>z_m$. If (x, y, z) is not in the support of ϵ, then $\epsilon^-(x, y, z) =-\frac{1}{2}\epsilon(x, y, 2z_m-z)\leq 0$ by assumption (b). Next suppose that (x, y, z) is in the support of ϵ. Since z_m is the least upper bound of the midpoints, (x, y, z) must be a normal point and the associated point $(x, y, \bar{z})$ must satisfy $\bar{z}\leq 2z_m-z\leq z$. Lemma 2 implies $\phi(x, y, 2z_m-z) \geq\phi(x, y, z)$ so that $\phi^-(x, y, z)=\frac{1}{2}\phi(x, y, z) -\frac{1}{2}\phi(x, y, 2z_m-z)\leq 0$. The total potential ψ, defined in Lemma 1 satisfies $\psi^-=\phi^-$, because Υ is independent of z; consequently $\psi^-(x, y, z)\leq 0$. From Eq. (4) it follows that the level surfaces of ϵ, p, and ψ all coincide. This fact and the requirement that $\epsilon\geq 0$, $p\geq 0$ and $d\epsilon/dp\geq 0$ from assumption (b) imply that $\epsilon^-(x, y, z)\leq 0$ for all $z\geq z_m$. ∎

4. THE MAIN THEOREM

We can now prove that these stratified Newtonian stellar models have a plane of mirror symmetry.

Theorem: Consider a stratified Newtonian stellar model which satisfies assumptions (a) through (f). There exists a plane $z=z_m$, such that the odd parts of the functions ϕ, ϵ and p vanish when taken with respect to the plane $z=z_m$. Thus, the star has a plane of mirror symmetry for these functions

Proof: From Lemma 3 we know that there is a point (x_0, y_0, z_0) such that $m_\phi(x_0, y_0, z_0)=(x_0, y_0, z_m)$. We will consider two separate cases. In the first case (x_0, y_0, z_0) is assumed to be a normal point, in the second case it is assumed to be a special point.

Case 1: Associated with the point (x_0, y_0, z_0) is the point $(x_0, y_0, \bar{z}_0)$ with $\bar{z}_0=2z_m-z_0$. Since $\phi^-(x_0, y_0, z_0) =\frac{1}{2}\phi(x_0, y_0, z_0)-\frac{1}{2}\phi(x_0, y_0, \bar{z}_0)=0$, there exists a point [either (x_0, y_0, z_0) or $(x_0, y_0, \bar{z}_0)$] say (x_0, y_0, z_0) with $z_0>z_m$, where ϕ^- vanishes. The function ϕ^- vanishes on the boundary of the half space $z>z_m$. In the interior of this region ϕ^- is bounded due to assumption (f); therefore there must exist a point $(\hat{x}, \hat{y}, \hat{z})$ in this half space where ϕ^- is maximal. The odd part of Eq. (1) is given by $\nabla_i\nabla^i\phi^-=-4\pi G\epsilon^-$. From Lemma 4 we have $\nabla_i\nabla^i\phi^-\geq 0$ for all $z>z_m$. This inequality, the existence of a point where ϕ^- is maximal and Theorem 2A (see the Appendix) guarantee that $\phi^-=0$ everywhere. That $\epsilon^-=p^-=0$ follows trivially.

The argument given above is not strictly correct for the case where the maximum of ϕ^- lies on the boundary of the star. The density ϵ need not be continuous at the surface of the star, and consequently the potential ϕ need not be sufficiently differentiable there to apply Theorem 2A. Consider now the case where the maximum of ϕ^-, $(\hat{x}, \hat{y}, \hat{z})$ lies on the boundary of the star. Find an open ball B which has $(\hat{x}, \hat{y}, \hat{z})$ as a point on its boundary, which is tangent to the surface of the star at $(\hat{x}, \hat{y}, \hat{z})$ and which is sufficiently small that all of the points of B lie in the exterior of the star. Within B, ϕ^- will be C^3, and ϕ^- is C^1 at (x, y, z). Furthermore $\phi^-\leq\phi^-(\hat{x}, \hat{y}, \hat{z})$ at all points in B and $\nabla_i\phi^-(\hat{x}, \hat{y}, \hat{z})=0$, since ϕ^- is a maximum at $(\hat{x}, \hat{y}, \hat{z})$. From Theorem 1A if follows that ϕ^- has the constant value $\phi^-(\hat{x}, \hat{y}, \hat{z})$ everywhere in B and consequently everywhere. This constant value must be zero since ϕ^- vanishes on the boundary of the half-space $z>z_m$.

Case 2: We now consider the case where (x_0, y_0, z_0) is a special point. We have shown that $\phi^-\leq 0$ and $\epsilon^-\leq 0$ for all $z\geq z_m$. Similarly $\phi^-\geq 0$ and $\epsilon^-\geq 0$ for all $z\leq z_m$. It follows that there is a neighborhood U of the plane $z=z_m$ in which the following inequalities must hold: $\partial\phi^-/\partial z\leq 0$, $\partial\epsilon^-/\partial z\leq 0$. From Eq. (1) it follows that

$\nabla_i\nabla^i(\partial\phi^-/\partial z) = -4\pi G\partial\epsilon^-/\partial z$, hence $\nabla_i\nabla^i(\partial\phi^-/\partial z) \geq 0$ in U. At a special point $\partial\phi/\partial z = 0 = \partial\phi^+/\partial z + \partial\phi^-/\partial z$, but at $z = z_m$, $\partial\phi^+/\partial z$ vanishes, therefore $\partial\phi^-/\partial z(x_0, y_0, z_m) = 0 \geq \partial\phi^-/\partial z$ for all points in U. By Theorem 2A it follows that $\partial\phi^-/\partial z = 0$ everywhere in U, and consequently $\phi^- = 0$ everywhere in U, and as a result $\phi^- = 0$ everywhere.

As in Case 1 special consideration must be given to the case that (x_0, y_0, z_m) is on the boundary of the star. From assumption (e) we know that ϕ must be at least C^1 in the normal direction, and C^2 in the tangential direction at the surface of the star. Therefore Theorem 2A cannot be applied and Theorem 1A must be used. Since (x_0, y_0, z_m) is a special point, it follows that $\partial\phi/\partial z = = \partial\psi/\partial z = 0$. There $\partial/\partial z$ is a tangential derivative to the surface at this point; it follows that $\partial\phi^-/\partial z$ is C^1 at (x_0, y_0, z_m). We have argued that $\partial\phi^-/\partial z \leq 0$ in the set U. Thus, $\partial\phi^-/\partial z$ will be a maximum at (x_0, y_0, z_m) so that $\nabla_i(\partial\phi^-/\partial z) = 0$ there also. Construct an open ball B which contains (x_0, y_0, z_m) as one of its boundary points, which is tangent to the surface of the star at (x_0, y_0, z_m), and which is sufficiently small that B lies completely within U and completely within the exterior of the star. Within B, $\nabla_i\nabla^i(\partial\phi^-/\partial z) = 0$ and $\partial\phi^-/\partial z$ is C^2. Thus by Theorem 1A, $\partial\phi^-/\partial z = 0$ in B, and therefore $\phi^- = 0$ in B (the plane $z = z_m$ intersects the center of B). It follows that $\phi^- = 0$ everywhere since it vanishes at an interior point of the half space $z > z_m$. ∎

5. DISCUSSION

In the special case of static stellar models ($v_i = 0$) there is no orientation picked out by the velocity stratification. Therefore, the Theorem proved in the last section shows that a mirror plane must exist for any choice of orientation. As a result, one can show that the star must be spherical.[2,4] We also note that the mirror plane theorem in the last section is in a sense incomplete. We have shown that the functions ϵ, p, and ϕ must all have mirror symmetry. However, it appears that no simple analogous result exists for the velocity field of the fluid, v^i. For example, consider a stationary axisymmetric star with azimuthal velocity field. An infinite number of related stellar models may be constructed by keeping the functions ϵ, p, and ϕ fixed while defining a new velocity field $v'^i = hv^i$, where h is an arbitrary function which is independent of azimuthal angle and $h^2 = 1$. Note that h may be discontinuous, so that parts of the fluid may rotate one direction while other parts rotate the other way. These related stellar models need not have simple mirror symmetry in the velocity field. A final point to note is that asumption (a), that the velocity field is stratified, is only used to prove Lemma 1. This assumption could be replaced by the weaker (but physically less transparent) assumption $0 = a_z = \partial v_z/\partial t + v^j\nabla_j v_z$.

ACKNOWLEDGMENT

I would like to thank D. Brill, W. Hiscock, J. Isenberg, and P. Yasskin for helpful suggestions and comments.

APPENDIX

We reproduce here the version of the maximum principle on which the proof of the mirror plane theorem is based. Reference 9 may be consulted for discussions of these results, and also for stronger versions of the theorems than are needed here.

Theorem 1A: Let B be an open ball, and x_0 a point on its boundary. Assume that f is a C^2 function everywhere in B, and C^0 in the closure of B. Let $\nabla_i\nabla^i f \geq 0$ and $f \leq f(x_0)$ everywhere in B. Then the outward normal derivative $df/dn > 0$ at x_0, or $f = f(x_0)$ everywhere in B.

Theorem 2A: Assume that f is a C^2 function everywhere in a bounded open neighborhood U, and that $\nabla_i\nabla^i f \geq 0$ everywhere in U. If there is a point x_0 in U such that $f(x_0) \geq f(x)$ for all x in U, then $f(x_0) = f(x)$ for all x in U.

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