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ON THE SECULAR INSTABILITIES OF THE MACLAURIN SPHEROIDS

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ABSTRACT

The combined effects of gravitational radiation reaction and of viscosity on the stability of the Maclaurin spheroids are discussed. Each of these dissipative effects is known to induce a secular instability in the Maclaurin sequence past the point of bifurcation of the Jacobi and the Dedekind sequences. We find, however, that when both effects are considered together, these instabilities tend to cancel each other. The sequence of stable Maclaurin spheroids therefore reaches past the bifurcation point to a new point determined by the ratio of the strengths of the viscous and the radiative forces.

Subject headings: gravitation — instabilities — relativity — rotation

The secular instability of the Maclaurin spheroids due to viscosity or gravitational radiation reaction (Chandrasekhar 1969, 1970*a*) is by now well understood; and similar instabilities of more general classes of stellar models are known to exist (Ostriker and Bodenheimer 1973; Miller 1973). The overall effect of such a secular instability is to place an upper limit on the ratio of the rotational kinetic energy, T, to the gravitational potential energy, W. This upper limit occurs at the point of bifurcation where the value of $|T/W|$ is given approximately by 0.14. After exhausting its nuclear fuel, a star may collapse to a state having $|T/W| > 0.14$, if it is rotating rapidly enough. In such a case it was thought that gravitational radiation reaction and viscosity would force the star to evolve through nonaxisymmetric configurations wherein T would be dissipated until $|T/W|$ approached 0.14.

We find that for Maclaurin spheroids this picture is not correct in the presence of *both* viscous and radiation reaction forces. Rather, we find that the secular instabilities caused by viscosity and by gravitational radiation tend to cancel each other. The particular point at which the secular instability actually sets in depends on X (see eq. [6]), the ratio of the strengths of the viscous and the gravitational forces. For a particular choice of X the stable portion of the Maclaurin sequence can be extended all the way to the point of the onset of dynamical instability, corresponding to $|T/W| = 0.274$.

The cancellation of the secular instabilities occurs because viscous dissipation and radiation reaction cause different modes to become unstable. In particular, the mode which is not unstable to a particular dissipative force is in fact stabilized by that force. Thus, for example, the mode which is unstable to radiation reaction is stabilized by viscosity. For a suitable choice of the ratio of the strengths of the dissipative forces, the stabilizing terms dominate in both modes.This results in stable Maclaurin spheroids past the point of bifurcation.

To show precisely how the cancellation of instabilities occurs, we must examine the perturbations of the Maclaurin spheroids. This subject has been discussed at length by Chandrasekhar (1969, 1970*b*); we adopt the notation of that work, and refer the reader thereto for the method of derivation of the equations which we employ.

We have examined all of the "second harmonic modes" of the Maclaurin spheroids, and find that only the toroidal modes have instabilities which are induced by the dissipative effects. The perturbation is described (in the corotating frame of the fluid) by a Lagrangian displacement of the form $\xi_i(\boldsymbol{x})e^{i\sigma t}$. The equations for the toroidal modes are expressed in terms of the quantity

$$V_{ij} = \int_V \rho(\xi_i x_j + \xi_j x_i) d^3x \,. \tag{1}$$

It is a straightforward matter to generalize the work of Chandrasekhar to obtain the equations governing the toroidal modes, with the inclusion of the effects of viscosity and gravitational radiation reaction. These equations are

$$[-\sigma^2 + 2(2B_{11} - \Omega^2) + 10i\sigma\nu/a_1{}^2 + 2DQ_1](V_{11} - V_{22}) - 4[i\sigma\Omega - \tfrac{1}{2}DQ_2]V_{12} = 0\,, \tag{2a}$$

and

$$[-\sigma^2 + 2(2B_{11} - \Omega^2) + 10i\sigma\nu/a_1{}^2 + 2DQ_1]V_{12} + [i\sigma\Omega - \tfrac{1}{2}DQ_2](V_{11} - V_{22}) = 0. \tag{2b}$$

In these equations we have used the symbols:

ν, viscosity;

$D = (\pi G\rho)^{3/2} GMa_1^2/5c^5$, gravitational radiation;

a_1, a_3, equatorial and polar radii of spheroid;

Ω, angular velocity of spheroid;

$$B_{11} = \int_0^\infty a_1^2 a_3 (a_1^2 + u)^{-3} (a_3^2 + u)^{-1/2} u du;$$

$$Q_1 = -2i\sigma(\sigma^2 + 12\Omega^2)(\Omega^2 - 2B_{11}) - \tfrac{3}{5} i\sigma^5 - 8i\sigma^3\Omega^2 + 16i\sigma\Omega^4;$$

$$Q_2 = 8\Omega(3\sigma^2 + 4\Omega^2)(\Omega^2 - 2B_{11}) + 8\sigma^4\Omega - 128\Omega^5/5 .$$

Equations (2) have nontrivial solutions if and only if the frequency, σ, satisfies the characteristic equation,

$$0 = \sigma^2 - 2\sigma\Omega - 2(2B_{11} - \Omega^2) - 10i\nu\sigma/a_1^2 + 4iD(2\Omega - \sigma)^3[2B_{11} - \Omega^2 + \tfrac{1}{10}(2\Omega - \sigma)(4\Omega + 3\sigma)] . \quad (3)$$

In the nondissipative limit (i.e., $D = \nu = 0$) the solutions to equation (3) for the characteristic frequencies of the toroidal modes are given by

$$\sigma_0^{(1)} = \Omega - (4B_{11} - \Omega^2)^{1/2} , \quad (4a)$$

and

$$\sigma_0^{(2)} = \Omega + (4B_{11} - \Omega^2)^{1/2} . \quad (4b)$$

When the effects of viscosity and gravitational radiation reaction are small (an assumption used in the derivation of eq. [2]), equation (3) may be solved approximately. Let $\sigma \approx \sigma_0 + \Delta\sigma$ represent the solution to equation (3) where $\Delta\sigma$ is considered to be small. It follows that

$$i\Delta\sigma = \frac{2D(2\Omega - \sigma_0)^5}{5(\sigma_0 - \Omega)} - \frac{5\nu\sigma_0}{a_1^2(\sigma_0 - \Omega)} . \quad (5)$$

Equation (5) can be written in a more convenient form by defining X, the ratio of the strengths of the dissipative forces:

$$X = 25\nu(1 - e_0^2)^{2/3}/[2a_1^2 D\Omega_0^4(\pi G\rho)^{-3/2}(1 - e^2)^{2/3}] . \quad (6)$$

The quantity Ω_0 is the angular velocity of the Maclaurin spheroid having $e_0 = 0.95289$, the point of dynamical instability ($\Omega_0^2/\pi G\rho = 0.44022$). When X is defined as in equation (6), it is a function only of the total mass, average radius, and average viscosity of the ellipsoid (see eq. [8]). Equation (5) can be rewritten as

$$i\Delta\sigma^{(1)} = -\frac{2D}{5(4B_{11} - \Omega^2)^{1/2}} \left[(\sigma_0^{(2)})^5 - X\Omega_0^4\sigma_0^{(1)} \frac{(1 - e^2)^{2/3}}{(1 - e_0^2)^{2/3}}\right] , \quad (7a)$$

and

$$i\Delta\sigma^{(2)} = -\frac{2D}{5(4B_{11} - \Omega^2)^{1/2}} \left[(\sigma_0^{(1)})^5 - X\Omega_0^4\sigma_0^{(2)} \frac{(1 - e^2)^{2/3}}{(1 - e_0^2)^{2/3}}\right] . \quad (7b)$$

These equations (7a, b) give expressions for the imaginary part of the characteristic frequencies of the toroidal modes. If either expression is positive, an instability occurs.

We have evaluated these equations (3) and (7) numerically for various values of the eccentricity ($e^2 = 1 - a_3^2/a_1^2$) of the spheroids and the ratio of the dissipative strengths, X. Figure 1 illustrates the results of these computations. This figure depicts the critical eccentricity (where instability first sets in) as a function of X. The entire region below the curve is stable. Note that for very large or very small values of X the critical eccentricity approaches 0.81267, the bifurcation point; this corresponds to the limiting cases with pure viscosity or pure radiation reaction. Also note that for $X = 1$, the region of stability is extended all the way to $e = 0.95289$, the point of dynamical instability. For values of $X < 1$ the spheroid is radiation-reaction dominated, and it is the $\sigma_0^{(2)}$ mode which becomes unstable; for $X > 1$, viscosity dominates and the $\sigma_0^{(1)}$ mode is unstable.

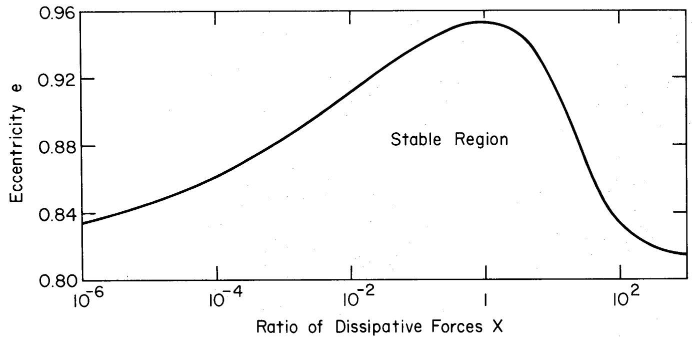

FIG. 1.—The eccentricity of the Maclaurin spheroid at the onset of secular instability is illustrated for given values of the ratio of the viscous force to the gravitational radiation reaction force.

To relate the scale of the parameter X to a more astrophysically relevant set of units, we note that equation (6) may be written in the form

$$X = 5.863 \times 10^{-3}\nu(\mathrm{cm^2\ s^{-1}})\left(\frac{R}{R_\odot}\right)^2\left(\frac{M_\odot}{M}\right)^3 . \tag{8}$$

The average radius of the ellipsoid is R and its mass is M. The maximum stabilizing effect occurs when $X = 1$. For a star of given mass and given average radius, we define the critical viscosity to be the one for which $X = 1$,

$$\nu_c = 170.6\left(\frac{R_\odot}{R}\right)^2\left(\frac{M}{M_\odot}\right)^3(\mathrm{cm^2\ s^{-1}}) . \tag{9}$$

Table 1 lists the estimated actual viscosity, ν, along with the critical viscosity ν_c for different types of stars. The actual viscosity of the Sun is estimated to be very close to the critical value; however, the time scales for both the viscous and the gravitational radiation induced evolution are so long in this case as to be ignorable. The compact objects (white dwarfs and neutron stars) are listed as having very small actual viscosities. Those estimates are based on the viscosity of a degenerate gas, and ignore the possibility of a crystalline structure which could increase the real values by many orders of magnitude.

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TABLE 1

CRITICAL VISCOSITIES FOR VARIOUS TYPES OF STARS

Type of Star	Estimated Actual Viscosity ($\mathrm{cm^2\ s^{-1}}$)	Critical Viscosity ($\mathrm{cm^2\ s^{-1}}$)
Sun*	$1 < \nu < 10^3$	$\nu_c = 170$
White dwarfs†	$10^{-1} < \nu < 10$	$10^4 < \nu_c < 10^8$
Neutron stars‡	$\nu \approx 1$	$10^7 < \nu_c < 10^{14}$

* Kopal 1968.
† Durisen 1973.
‡ Ruderman 1968.

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