Phys 225b: General Relativity 9 March 2016

Final Problem Set

This problem set is the take-home final exam that determines 50% of the grade in Phys 225b. It must be turned in (to me personally, under my office door, or to my mail box on the 4th floor of the SERF building), no later than the official scheduled final exam time for this class: 2:30pm on Wednesday 16 March 2016.

Complete 4 of the following 5 problems:

#1 The generalized harmonic gauge condition sets $g_{ab}\nabla^c\nabla_c x^b = -\Gamma_a = H_a$, where H_a is a prescribed function of the spacetime coordinates and the metric g_{ab} (but not its derivative). Thus the vacuum Einstein equation in a generalized harmonic gauge is obtained by replacing Γ_a by $-H_a$ in the vacuum Einstein equations. This gauge condition can be considered to be a constraint on the allowed values of the metric and its first derivatives:

$$\mathcal{C}_a = \Gamma_a + H_a = 0.$$

Show that the vacuum Einstein equation in a generalized harmonic gauge can be written as

$$R_{ab} = \nabla_{(a}\mathcal{C}_{b)}.$$

Use the contracted Bianchi identity to show that the constraint $\mathcal{C}_a$ must satisfy the constraint evolution equation:

$$0 = \nabla^b\nabla_b\mathcal{C}_a + R_{ab}\mathcal{C}^b = \nabla^b\nabla_a\mathcal{C}_a + \mathcal{C}^b\nabla_{(a}\mathcal{C}_{b)}.$$

#2 Consider a spacetime M with a three-dimensional surface Σ embedded in it. Let n^a denote the unit normal to this surface. Assume that $n^a n_a = -1$. Let $h_{ab} = g_{ab} + n_a n_b$ denote the metric on this surface. Let ∇_a denote the covariant derivative that is compatible with g_{ab}, and let D_a denote the covariant derivative that is compatible with h_{ab}. Let $K_{ab} = h_a{}^c\nabla_c n_b$ denote the extrinsic curvature of this surface, ${}^{(4)}R^a{}_{bcd}$ denote the four-dimensional Riemann curvature associated with ∇_a and ${}^{(3)}R^a{}_{bcd}$ the three-dimensional Riemann curvature associated with D_a.

Derive expressions for the projections, $n^a n^{b(4)}G_{ab}$ and $n^a h^b{}_c{}^{(4)}G_{ab}$, of the four-dimensional Einstein curvature in terms of the extrinsic curvature and the three-dimensional curvature. In particular show that

$$n^a n^{b(4)}G_{ab} = \frac{1}{2}[{}^{(3)}R - K_{ab}K^{ab} + (K^a{}_a)^2]$$

and

$$n^a h^b{}_c{}^{(4)}G_{ab} = D_a K^a{}_c - D_c K^a{}_a.$$

#3 Assume the three-metric g_{ab} satisfies the equations $R_{ab} = V^{-1}D_aD_bV$ and $R = 0$, where R_{ab} is the three-dimensional Ricci tensor. Define the Bach tensor, $R_{abc} = 2D_{[c}R_{b]a} - \frac{1}{2}g_{a[b}D_{c]}R$, and let $n_a = W^{-1/2}D_aV$ denote the unit normal to the constant V two-surfaces. Let $\beta_{ab} = g_{ab} - n_a n_b$ denote the metric, and $H_{ab} = \beta_a{}^c\beta_b{}^d D_c n_d$ the extrinsic curvature of these constant V two-surfaces. Show that

$$R_{abc}R^{abc} = 8V^{-4}\Big(W^2\psi^{ab}\psi_{ab} + \tfrac{1}{8}\beta^{ab}D_aW D_bW\Big),$$

where $\psi_{ab} = H_{ab} - \frac{1}{2}\beta_{ab}H^c{}_c$.

#4 Show that

$$m(r) \le \frac{2r}{9}\left[1 - 6\pi p(r)r^2 + \sqrt{1 + 6\pi p(r)r^2}\right]$$

in any static spherical star in which the density $\rho(r)$ is a decreasing function of r. Use this result to show that $M \le \frac{4}{9}R$ for any stellar model.

#5 Let the spacetime metric g_{ab} be written in the form $g_{ab} = g^{(B)}_{ab} + h_{ab}$ where $g^{(B)}_{ab}$ is a fixed (not necessarily flat) background metric tensor. Show that the Ricci tensor R_{ab} of g_{ab} is given by

$$R_{ab} = R^{(B)}_{ab} - \frac{1}{2}g^{cd}_{(B)}\Big(\nabla_c\nabla_d h_{ab} + \nabla_a\nabla_b h_{cd} - \nabla_c\nabla_a h_{bd} - \nabla_c\nabla_b h_{ad}\Big) + \mathcal{O}(h^2),$$

where ∇_a is the covariant derivative compatible with $g^{(B)}_{ab}$.