WEEK 6: Static Stellar Models

Recommended Reading: J. B. Hartle, Phys. Rep. 46, 201 (1978).

#1 Assume that the stress energy tensor has the form of a perfect fluid: $T^{ab} = (\rho + p)u^a u^b + pg^{ab}$, where ρ is the energy density, p is the pressure, g_{ab} is the spacetime metric, and u^a (with $u^a u_a = -1$) is the four-velocity of the fluid.

a) Show that the conservation laws, $\nabla_a T^{ab} = 0$, for a perfect fluid can be written as:

$$u^a \nabla_a \rho + (\rho + p)\nabla_a u^a = 0,$$

$$(\rho + p)u^a \nabla_a u^b + (g^{ab} + u^a u^b)\nabla_b p = 0$$

in any spacetime.

b) Assume the spacetime is static and spherically symmetric with metric

$$ds^2 = -e^{2\Phi(r)}dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

Derive simplified expressions for the perfect fluid conservation laws in this spacetime.

#2 Show that the Einstein curvature tensor of the static spherical geometry given in Problem #2 b) can be written in the form:

$$G_{tt} = \frac{2e^{2\Phi}}{r^2}\frac{dm}{dr},$$

$$G_{rr} = \frac{2}{r}\frac{d\Phi}{dr} - \frac{2m}{r^2(r-2m)},$$

$$G_{\theta\theta} = r^2 e^{-\Phi}\sqrt{1 - \frac{2m}{r}}\frac{d}{dr}\left[\sqrt{1 - \frac{2m}{r}}e^{\Phi}\frac{d\Phi}{dr}\right] - r\frac{d}{dr}\left(\frac{m}{r}\right) + r\left(1 - \frac{2m}{r}\right)\frac{d\Phi}{dr}.$$

#3 Consider a static spherically symmetric spacetime with a perfect fluid source.

a) Show that Einstein's equation implies the follow equations for the functions $m(r)$ and $\Phi(r)$ that determine this spacetime metric:

$$\frac{dm}{dr} = 4\pi r^2 \rho,$$

$$\frac{d\Phi}{dr} = \frac{m + 4\pi r^3 p}{r(r-2m)}.$$

b) Use the result of Problem #1 b) to derive the following equation for the pressure,

$$\frac{dp}{dr} = -(\rho + p)\frac{m + 4\pi r^3 p}{r(r-2m)}.$$

#4 Consider a static spherically symmetric spacetime with perfect fluid source.

a) Solve the stellar structure equations analytically for the uniform density equation of state $\rho = \rho_o$. In particular find expressions for the functions $m(r)$ and $p(r)$ that involve as "constants of integration" only the total mass M and total radius R of the star.

b) Show explicitly that $M \leq \frac{4}{9}R$ for these stellar models.

#5 Show that

$$m(r) \leq \frac{2r}{9}\left[1 - 6\pi p(r) r^2 + \sqrt{1 + 6\pi p(r) r^2}\right]$$

in any static spherical star in which the density $\rho(r)$ is a decreasing function of r. Use this result to show that $M \leq \frac{4}{9}R$ for any stellar model.