

WEEK 4: Black Hole Uniqueness**Recommended Reading: W. Israel, Phys. Rev. 164, 1776 (1967)**

- #1 The Gauss-Bonnet theorem states that the integral of the two-dimensional scalar curvature over the entire volume of any compact two-dimensional manifold is

$$\int R dV = 8\pi(1 - q_n),$$

where q_n is the number of “topological handles” in the manifold. Verify that this expression is correct by explicitly performing the integral for specific (simple) choices of the metric on a sphere and a torus.

- #2 Define the Bach tensor: $R_{abc} = 2D_{[c}R_{b]a} - \frac{1}{2}g_{a[b}D_{c]}R$. Let g_{ab} be a conformally flat, $g_{ab} = \Phi^4\delta_{ab}$ (where δ_{ab} is a flat metric), three-metric. Show that $R_{abc} = 0$.

- #3 Assume the three-metric g_{ab} satisfies the equations $R_{ab} = V^{-1}D_a D_b V$ and $R = 0$, where R_{ab} is the three-dimensional Ricci tensor. Show that $\frac{1}{4}V^4 W^{-1} R_{abc} R^{abc} = D^a D_a W - V^{-1} D^a V D_a W - \frac{3}{4} W^{-1} D^a W D_a W$, where $W = D^a V D_a V$.

- #4 Assume the conditions of Problem #3, and show that

$$D_a \left(F V^{-1} D^a W + G W D^a V \right) = \frac{1}{4} F V^3 W^{-1} R_{abc} R^{abc} + \frac{3}{4} F V^{-1} W^{-1} Y_a Y^a,$$

where $F = (cV^2 + d)/(1 - V^2)^3$, $G = (6d - 2c + 8cV^2)/(1 - V^2)^4$, $Y_a = D_a W + 8WV(1 - V^2)^{-1} D_a V$, and c and d are arbitrary constants.

- #5 Assume the conditions of Problem #3, and let $n_a = W^{-1/2} D_a V$ denote the unit normal to the constant V two-surfaces. Let $\beta_{ab} = g_{ab} - n_a n_b$ denote the metric, and $H_{ab} = \beta_a^c \beta_b^d D_c n_d$ the extrinsic curvature of these constant V two-surfaces. Show that

$$R_{abc} R^{abc} = 8V^{-4} \left(W^2 \psi^{ab} \psi_{ab} + \frac{1}{8} \beta^{ab} D_a W D_b W \right),$$

where $\psi_{ab} = H_{ab} - \frac{1}{2} \beta_{ab} H^c_c$.