Phys 225b: General Relativity 11 January 2016

WEEK 2: Dynamical Structure of General Relativity

Recommended Reading: Carroll Appendix D; Wald Chapter 10.2; MTW Chapter 21.4 and 21.5.

Consider a spacetime M with a three-dimensional surface Σ embedded in it. Let n^a denote the unit normal to this surface. Assume that $n^a n_a = -1$. Let $h_{ab} = g_{ab} + n_a n_b$ denote the metric on this surface. Let ∇_a denote the covariant derivative that is compatible with g_{ab}, and let D_a denote the covariant derivative that is compatible with h_{ab}. Let $K_{ab} = h_a{}^c \nabla_c n_b$ denote the extrinsic curvature of this surface, ${}^{(4)}R^a{}_{bcd}$ denote the four-dimensional Riemann curvature associated with ∇_a and ${}^{(3)}R^a{}_{bcd}$ the three-dimensional Riemann curvature associted with D_a.

#1 Derive an expression for the projections, $h_a{}^f h_b{}^g h_c{}^e h_k{}^d\,{}^{(4)}R^k{}_{efg}$, of the four-dimensional curvature in terms of the three-dimensional curvature associated with D_a, and the extrinsic curvature K_{ab}. In particular show that

$$h_a{}^f h_b{}^g h_c{}^e h_k{}^d\,{}^{(4)}R^k{}_{efg} = {}^{(3)}R^d{}_{cab} + K^d{}_a K_{cb} - K^d{}_b K_{ca}.$$

#2 Derive an expression for the projections, $h_a{}^f h_b{}^g h_c{}^e n^k\,{}^{(4)}R_{kefg}$, of the four-dimensional curvature in terms of the derivatives of the extrinsic curvature $D_c K_{ab}$. In particular show that

$$h_a{}^f h_b{}^g h_c{}^e n_k\,{}^{(4)}R^k{}_{efg} = D_b K_{ac} - D_a K_{bc}.$$

#3 Derive an expression for the projections, $n^f h_b{}^g h_c{}^e n^k\,{}^{(4)}R_{kefg}$, of the four-dimensional curvature in terms of the Lie derivative of the extrinsic curvature $\mathcal{L}_n K_{ab}$, the extrinsic curvature, the acceleration $a_b = n^a \nabla_a n_b$, and its derivative $D_a a_b$. In particular show that

$$h_b{}^g h_c{}^e n^f n_k\,{}^{(4)}R^k{}_{efg} = -\mathcal{L}_n K_{bc} + K_b{}^a K_{ac} + a_b a_c + D_b a_c.$$

#4 Derive expressions for the projections, $n^a n^{b(4)} G_{ab}$ and $n^a h^b{}_c{}^{(4)} G_{ab}$, of the four-dimensional Einstein curvature in terms of the extrinsic curvature and the three-dimensional curvature. In particular show that

$$n^a n^{b(4)} G_{ab} = \frac{1}{2}[{}^{(3)}R - K_{ab}K^{ab} + (K^a{}_a)^2]$$

and

$$n^a h^b{}_c{}^{(4)} G_{ab} = D_a K^a{}_c - D_c K^a{}_a.$$

#5 Derive an expression for the projections, $h_a{}^c h_b{}^{d(4)} R_{cd}$, of the four-dimensional Ricci curvature in terms of the Lie derivative of the extrinsic curvature $\mathcal{L}_n K_{ab}$, the extrinsic curvature, the acceleration $a_b = n^a \nabla_a n_b$, and its derivative $D_a a_b$. In particular show that

$$h_a{}^c h_b{}^{d(4)} R_{cd} = \mathcal{L}_n K_{bc} - a_b a_c - D_b a_c + {}^{(3)}R_{bc} - 2K_b{}^a K_{ac} + K_{bc} K^a{}_a.$$