Phys 225b: General Relativity Winter 2016

WEEK 1: Dynamical Structure of General Relativity

Recommended Reading: Lindblom, et al. *Classical and Quantum Gravity*, **23**, S447-S462 (2006).

#1 Let ψ denote a scalar field on a spacetime with metric g_{ab}. Let ∇_a denote the covariant derivative compatible with g_{ab} and let $\Gamma^a{}_{bc}$ denote the connection associted with this covariant derivative. Assume that ψ is the solution to the scalar wave equation

$$\nabla^a\nabla_a\psi = \sigma,$$

where σ is a prescribed source function. Show that this equation can be written in the form

$$\nabla^a\nabla_a\psi = g^{ab}\partial_a\partial_b\psi - g^{ab}\Gamma^c{}_{ab}\partial_c\psi = \sigma,$$

where ∂_a denotes the partial derivative. At any point in space and time, show that there exist coordinates such that the scalar wave equation at that point has the form

$$\nabla^a\nabla_a\psi = -\partial_t^2\psi + \partial_x^2\psi + \partial_y^2\psi + \partial_z^2\psi = \sigma.$$

#2 Show that the evolution equation for a vacuum electromagnetic field A_a is given by

$$\nabla^b\nabla_b A_a = \nabla_a H$$

when using the generalized Lorentz gauge condition $\nabla^a A_a = H$. This gauge condition restrictes the allowed values of the first derivatives of A_a, and therefore must be imposed as a constraint on physical initial data. Let $\mathcal{C} = \nabla^a A_a - H$ represent this constraint. Show that this constraint must satisfy the equation

$$\nabla^a\nabla_a\mathcal{C} = 0$$

in flat spacetime as a consequence of the vacuum electromagnetic field equations. Solutions to this constraint evolution system will remain zero for all time if and only if initial data for A_a are chosen so that $\mathcal{C}$ and its time derivative $\partial_t\mathcal{C}$ vanish initially.

#3 Assume that one of the functions used as a spacetime coordinate, x^a for some particular value of ^{a}, satisfies the following scalar wave equation,

$$\nabla^b\nabla_b x^a = \sigma.$$

Using a coordinate system that includes x^a as one of the spacetime coordinates, show that this equation can be re-written in the form

$$\nabla^b\nabla_b x^a = -g^{bc}\Gamma^a_{bc} = \sigma.$$

#4 Show that the spacetime Ricci tensor may be written in terms of the spacetime metric g_{ab} and the connection associted with this metric $\Gamma_{abc} = g_{ad}\Gamma^d{}_{bc}$ in the following way:

$$R_{ab} = -\frac{1}{2}g^{cd}\partial_c\partial_d g_{ab} + \nabla_{(a}\Gamma_{b)} + g^{cd}g^{ef}(\partial_e g_{ca}\partial_f g_{db} - \Gamma_{ace}\Gamma_{bdf}),$$

where $\Gamma_a = g^{bc}\Gamma_{abc}$ and $\nabla_a\Gamma_b = \partial_a\Gamma_b - \Gamma^c{}_{ab}\Gamma_c$.

#5 The generalized harmonic gauge condition sets $g_{ab}\nabla^c\nabla_c x^b = -\Gamma_a = H_a$, where H_a is a prescribed function of the spacetmime coordinates and the metric g_{ab} (but not its derivative). This gauge condition is a constraint on the allowed initial values of the metric and its first derivatives:

$$\mathcal{C}_a = \Gamma_a + H_a = 0.$$

Show that the vacuum Einstein equation in a generalized harmonic gauge can be written as

$$R_{ab} = \nabla_{(a}\mathcal{C}_{b)}.$$

Use the contracted Bianchi identity to show that the constraint $\mathcal{C}_a$ must satisfy the constraint evolution equation:

$$0 = \nabla^b\nabla_a\mathcal{C}_a + R_{ab}\mathcal{C}^b = \nabla^b\nabla_a\mathcal{C}_a + \mathcal{C}^b\nabla_{(a}\mathcal{C}_{b)}.$$