BOUNDS ON THE MASS AND MOMENT OF INERTIA OF NON-ROTATING NEUTRON STARS

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Abstract:

This article reviews the problem of placing bounds on the mass and moment of inertia of non-rotating neutron stars assuming that the properties of the constituent matter are known below a fiducial density ρ_0 while restricted only by minimal general assumptions above this density. We chiefly consider bounds on perfect fluid stars in Einstein's general relativity for which the energy density, ρ, is positive and for which the matter is microscopically stable ($p \geqslant 0$, $\mathrm{d}p/\mathrm{d}\rho \geqslant 0$). The effect of the additional restriction $(\mathrm{d}p/\mathrm{d}\rho)^{1/2} \leqslant 1$ on the bounds on the mass is also discussed as well as work indicating the effects of rotation, non-perfect fluid matter, and other theories of gravity.

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0. Introduction

The mass and the moment of inertia are the two gross structural parameters of neutron stars which are most accessible to observation. It is the mass which controls the gravitational interaction of the star with other systems such as a binary companion. It is the moment of inertia which controls the energy stored in rotation and thereby the energy available to the pulsar emission mechanism. Rough observational determinations have been made for the masses of some neutron star members of binary X-ray systems (for reviews see [1]) and may in time be determined for the binary pulsar [2]. Observational estimates of the moment of inertia have been obtained for the Crab pulsar [3]. Determining the possible ranges and correlations of neutron star masses and moments of inertia is therefore an important theoretical question.

In the case of the mass the question is particularly important because the theoretically predicted non-rotating neutron star mass range plays a central role in the observational identification of black holes. There is a maximum mass a non-rotating neutron star can have. There is no upper limit on the mass a black hole can have. If, therefore, one can find a dense, highly compact object, can plausibly argue that its rotation is slow, and can deduce that its mass is greater than the maximum allowed to non-rotating neutron stars, then one has an excellent candidate for a black hole. Arguments of this kind have already been used to support the identification of the X-ray source in Cygnus with a black hole [4].

Viewed as the primary tool for distinguishing observationally between neutron stars and black holes the maximum non-rotating neutron star mass becomes one of the most important observational predictions of relativistic gravitational theory coupled with a theory of matter at the endpoint of thermonuclear evolution. As such it deserves a firm theoretical foundation.

The prediction of the allowed ranges for neutron star masses and moments of inertia would be a simple problem if the equation of state of matter at the endpoint of thermonuclear evolution were known at all densities. Then the equations of relativistic spherical stellar structure could be integrated to calculate the mass, radius, moment of inertia, and all other structural parameters of all possible spherical equilibrium configurations constructed from matter obeying this equation of state. Unfortunately, the equation of state of matter at the endpoint of thermonuclear evolution has not been satisfactorily predicted at densities much above nuclear densities (for reviews see [5–6]) and this is just the range which is crucial for the properties of neutron stars.

In view of the uncertainties in the equation of state and of the utility of the theoretical prediction of non-rotating neutron star masses and moments of inertia, it becomes an important theoretical problem to place *bounds* on these numbers making use of those properties of the matter which can be accurately predicted but making only minimal general assumptions elsewhere. An upper bound on the maximum mass of non-rotating neutron stars is of special interest since this plays so important a role in the observational identification of black holes. This article is a review of work deriving bounds on the mass and moment of inertia.

At the outset it should be clear that no information on the mass and moment of inertia can be obtained from relativistic gravitational theories alone. The interesting classical theories of gravity contain no constants having the dimension of length or mass. Some information on the properties of the matter is needed, and the more restrictive this information is, the tighter will be the bounds on the mass and moment of inertia. In this article we will neither attempt to review nor to take any stand on the issues connected with the equation of state at the endpoint of thermonuclear evolution. Rather, we shall develop the theory of bounds on the mass and moment of inertia in a general way independent of any particular equation of state. Although illustrative examples will be given, it will essentially be left to the reader to assess the current state of superdense matter calculations and to use those results he or she considers well founded to obtain actual numbers for the bounds. This approach clearly separates issues of relativistic stellar structure from those of the physics of superdense matter. This is important both because these issues *can* be separated and because the approach allows for different and evolving points of view to be taken on the reliability of superdense matter calculations within the context of an agreed upon theory of relativistic stellar structure.

This review will be concerned chiefly with optimum bounds – bounds for which there is at least one configuration consistent with the assumptions on the matter for which the bounding inequality becomes an equality. Such bounds can therefore not be improved without more restrictive assumptions on the properties of the matter. While optimum bounds are often more difficult to establish than simple inequalities, it seems appropriate to concentrate on them because they are the best one can do.

We will be concerned solely with bounds to the parameters of possible equilibrium configurations without entering into questions of stability. Typical model calculations of the non-rotating endstates of stellar evolution give a maximum mass occurring at or very near to a point where the sequence of stars is changing stability. Therefore typically the maximum mass for all equilibrium configurations is also the maximum mass for all stable stars. We know of no general reason, however, why this should be so. Establishing bounds on the parameters of the stable equilibria is an interesting but largely open question.

For almost all of this review we shall concentrate on bounds within the context of Einstein's general relativity. We do this not only because Einstein's theory is in many ways the most compelling theory of relativistic gravity but also because it is within the framework of general relativity that the subject of bounds on the structural parameters of neutron stars has been most extensively and completely developed. Bounds in other theories of gravity will, however, be discussed in section 6.

The question of bounding neutron star masses and moments of inertia is not the same as asking for the best estimates of these parameters. For these the reader should refer to reviews of the calculations of the equation of state [5–6] and of the neutron star models which can be calculated from them [6–7]. The bounds on the mass and moment of inertia are in a sense the least which can be said about these parameters making use only of the properties of matter in regimes where it is thought to be well understood and of minimal general assumptions elsewhere. While there exist a number of pedagogically interesting ways of estimating the maximum mass, we will concentrate on those bounds which follow logically and precisely from the assumptions on the matter and from a relativistic theory of stellar structure (or for which there is at least a program by which they might do so). This seems appropriate in view of the observational significance of these numbers.

This article is not the place to give a detailed review of the history of the issues connected with the maximum mass of neutron stars. However, in the following few paragraphs, we shall sketch some of the highlights of the development of *bounds* on this number as distinct from the history of its actual computation using model equations of state. (For some recent and varied discussion of the effect of various model equations of state on the maximum mass see [6–10].) Consideration in the following brief summary is restricted to work on perfect fluid non-rotating neutron stars in general relativity since the work relaxing these restrictions is mostly recent and is reviewed in section 6.

Chandrasekhar's 1931 and subsequent [11] analysis of white dwarf structure demonstrated the existence of a maximum mass for equilibrium configurations constructed from a free Fermi gas within the framework of Newtonian gravitational theory. Subsequently in 1932 Landau [12] also estimated this number in an investigation into the possible stellar equilibrium configurations in which no energy was generated. Thus, the idea of a maximum mass entered in a central way at the very outset of the study of the endstates of stellar evolution. Remarkably, the idea that there should exist a maximum mass for the non-rotating endstates of stellar evolution was not accepted and even opposed by some of the most eminent astrophysicists of the period, among them A.S. Eddington. The reason was precisely because it was realized that the existence of a maximum mass implied that the endstate of some stars would be a state of perpetual gravitational collapse. This conclusion was felt to be sufficiently absurd as to compel a re-examination of the microscopic theory of relativistic degeneracy from which it followed. (For a fascinating review of this controversy, so curious in the light of today, see [13].)

Shortly after the discovery of the neutron in 1932, the idea that a condensed core of neutrons is another possible endstate of stellar evolution was proposed by Baade and Zwicky [14] and by Landau [15]. (See also the interesting historical remarks in [6].) It is in the more detailed attempts to calculate the properties of this object that relativistic gravity is applied for the first time to the endstates of stellar evolution, notably in the work of Oppenheimer and Volkoff [16], and of Zwicky [17]. Both of these works estimate the maximum mass of neutron stars but use rather different approximations to the matter. In Oppenheimer and Volkoff there appears for the first time the idea that a maximum mass exists for a wide *range* of equations of state and that a bound on this number can be obtained by calculating the maximum mass with the stiffest possible equation of state consistent with fundamental physical principles. Oppenheimer and Volkoff argued that the limiting equation of state was $p = \rho/3$ above a density of 10^{15} g/cm^3, and that of a free neutron gas below this density. They obtained a bound† of about $1M_\odot$. Actually relativistic gravity does not play a central role in this conclusion since it was shown by Chandrasekhar [18] that a maximum mass exists for stars with cores constructed from equations of state of the form $p \propto \rho$ even in Newtonian gravitational theory.

The role relativity can play in establishing a neutron star maximum mass was brought out clearly in the paper of Zwicky [17]. He applied to neutron stars the long known property of Schwarzschild's [19] relativistic solution for incompressible matter that, for a given density, there is a maximum mass which can be supported against gravity. Taking nuclear density ($\sim 10^{14}$ g/cm^3) for the density of the incompressible matter he found a maximum mass of about $13M_\odot$. He obtained this number, however, not from the mass at which the pressure becomes infinite at the center (although he noted this fact) but rather from the mass at which the sphere is at its Schwarzschild radius which he regarded as a more secure limit. He thereby obtained a value about 20% higher than would result from a similar argument today.

The ideas that there exists a maximum mass, that a bound could be placed on it by calculating with the stiffest possible equation of state, and that relativistic gravity would play an important role in placing such a bound were thus all present at a very early state in the development of the subject. It remained for later workers chiefly to refine and clarify these ideas and to put their derivation from the fundamental theory on a careful footing.

On the theoretical side, the modern development of bounds on neutron star masses owes much to the systematic calculations, in the middle fifties, by J.A. Wheeler and co-workers of the endstates of stellar evolution [20–22]. The issue is reviewed and very clearly stated in Wheeler's 1964 article

†Interestingly enough they argued that if p could be made much greater than ρ, then arbitrarily high masses could be supported. Unfortunately their conclusions were based on incompressible matter solutions which were not spatially flat at the origin.

[22]: "Evidently no equation of state – even the most extreme and non-physical case of incompressibility – offers any escape from the conclusion that there is a *limit to the mass of any stable static collection of cold catalyzed matter*".

On the observational side, there is no mistaking the stimulus produced by the discovery of the pulsar neutron stars in 1967 [23].

The first work in the direction of *deriving* an optimum upper bound on the mass of non-rotating neutron stars in a careful manner was first published in 1974 by Rhoades and Ruffini [24]. These authors used a variational technique. Essentially they searched for an extremum of the total mass of a non-rotating configuration with respect to variations of the equation of state above a fiducial density ρ_0 subject to the constraints that the configuration be in equilibrium, that the equation of state have positive energy density ($\rho > 0$), be microscopically stable ($p \geqslant 0$, $\mathrm{d}p/\mathrm{d}\rho \geqslant 0$), and have a hydrodynamic velocity of sound less than the velocity of light $(\mathrm{d}p/\mathrm{d}\rho)^{1/2} \leqslant c$. These restrictions were discussed earlier in detail by Harrison, Thorne, Wakano and Wheeler [21]. Using a fiducial density of $\rho_0 = 4.6 \times 10^{14}$ g/cm^3 and the Harrison–Wakano–Wheeler equation of state [20] below this density, Rhoades and Ruffini found a bound of $3.2 M_\odot$. Their derivation of this bound was incomplete, however, because they did not fully examine the solutions of the extremum conditions which arose from their variational problem. An examination of these solutions by Chitre and Hartle [25] nonetheless supported the conclusion that the value of the bound was that obtained by Rhoades and Ruffini, although a complete analytic solution to this problem is not yet in hand.

Before the publication of the paper by Rhoades and Ruffini, Nauenberg and Chapline in 1973 [9] had calculated non-optimum bounds for the maximum non-rotating neutron star mass by making use of the uniform density trial functions in the variational principle for relativistic stellar structure. While the values they obtained are upper bounds to the maximum mass calculated from typical equations of state and are close to the numbers obtained by Rhoades and Ruffini using similar assumptions, we as yet do not have a proof that the stable variational trial functions they used give an upper bound to the maximum mass for every equation of state satisfying their restrictions. An arbitrary trial configuration cannot be expected to give such an upper bound since a stable equilibrium is only a local minimum of the total mass and no equilibrium configuration is an absolute minimum. This can be verified by considering, for example, a sequence of uniform density trial configurations, each far from equilibrium, each having the same total baryon number in which the radius approaches the Schwarzschild radius. The total mass approaches zero along such a sequence so that no finite positive value of the mass can ever be an absolute minimum of the mass in the variational sense.

The problem of finding an optimum bound on the mass of non-rotating neutron stars when only positive energy and microscopic stability are assumed, with no assumption on the largest value of $(\mathrm{d}p/\mathrm{d}\rho)^{1/2}$, was considered by Sabbadini and Hartle in 1973 [26] and subsequently [27]. Consideration of this problem was prompted by debate over whether $(\mathrm{d}p/\mathrm{d}\rho)^{1/2} \leqslant c$ is a reasonable restriction on zero temperature matter at the endpoint of thermonuclear evolution – a question which is still unsettled today (see section 1.1 for discussion). The bound on the mass in this case arises from essentially the same general relativistic effect which gives rise to the maximum mass allowable to a sphere of given uniform density. For a ρ_0 of 5×10^{14} g/cm^3 and a reasonable equation of state below this density they found a bound of $5 M_\odot$. Essentially the same result (if adjusted for a different choice of ρ_0) was found by Hegyi, Lee and Cohen [28] in 1975.

In this article we shall attempt to present in a unified development the useful parts of this and other recent work on bounds on the mass of general relativistic, perfect fluid, non-rotating neutron stars. In addition, we shall review bounds on neutron star moments of inertia and estimates of the effects of non-perfect fluid matter, rotation and other theories of gravity on the mass bounds. The plan is as follows: sections 1–5 consider bounds on the mass and moment of inertia of perfect fluid non-rotating neutron stars in Einstein's general relativity. These restrictions are the basis of the most complete investigations and only in section 6 will we discuss the work on relaxing these assumptions. Section 1 contains a discussion of the assumptions on neutron star matter and a derivation of some non-optimum bounds notable for the immediacy of their derivation. In sections 2–4 optimum bounds are derived under the assumptions of positive energy and microscopic stability alone and then, under the additional assumption $(\mathrm{d}p/\mathrm{d}\rho)^{1/2} \leqslant c$. Section 5 contains a review of bounds on the moment of inertia while section 7 contains some brief conclusions and suggestions for further work.

1. Basic assumptions on the matter and simple bounds on the allowed cores

1.1. The matter

No bounds can be obtained on the mass or moment of inertia of non-rotating neutron stars without some assumption as to the properties of the matter from which these stars are made. Classical general relativity alone contains no parameter having the dimension of mass or length. The minimal general assumptions we consider are the following:

(1) *The matter is a perfect fluid described by a one parameter equation of state which relates the pressure p to the energy density ρ.*

Neutron stars are made from matter which may be idealized as being in its ground state. This condition fixes the constituents and requires zero temperature. The question of whether, at a given density, the matter is a solid or a fluid can only be settled by finding which of these states minimizes the energy. Some calculations (see [5, 6] for reviews) have suggested that neutron star matter becomes solid at densities above nuclear densities. These calculations typically [29] give elastic constants governing shear stresses which are comparable to those governing the isotropic compressibility so that appreciable shear stresses could in principle be supported inside neutron stars. What and how large these stresses are depend on the mechanism of formation and the subsequent history of the neutron star. If neutron stars are formed in a hot, liquid state then the subsequent buildup of shear stresses and nonisotropic pressure through cooling and spindown might not be very great and therefore have little influence on the equilibrium structure and total mass. For these reasons it is reasonable to begin the analysis of bounds on neutron star structural parameters by assuming that the matter is a perfect fluid. By this we mean matter with vanishing shear stresses and isotropic pressure. In a frame in which the fluid is at rest off diagonal elements of the stress energy tensor of the matter will vanish and the diagonal elements will be (ρ, p, p, p). The pressure of this zero temperature matter will then be related to the energy density by a one parameter equation of state $p = p(\rho)$. We will make these assumptions here and discuss the effect of non-perfect fluid matter in section 6.

(2) *The density is positive,*

$$\rho \geqslant 0. \tag{1.1}$$

This is the statement that gravity is attractive. While not immune from question, this usually posited principle of most gravitational theories is a reasonable assumption in the absence of any experimental evidence or theoretical argument to the contrary.

(3) *The matter is miscroscopically stable.*

In order to have an equilibrium star the matter must be stable against the collapse of local regions. This requires that

$$\mathrm{d}p/\mathrm{d}\rho \geqslant 0. \tag{1.2}$$

A small contraction of a region in which this condition is violated leads to forces on the boundary which increase the contraction rather than to ones which oppose it, and the region is unstable to collapse. This is true for both signs of the pressure. Since pressure is certainly positive at low densities where the equation of state is known, eq. (1.2) also implies that the pressure is *always* positive,

$$p \geqslant 0. \tag{1.3}$$

(4) *The equation of state is known below a fiducial density ρ_0.*

We assume that for densities below ρ_0 and for pressure below the corresponding pressure p_0 the equation of state is a known function $p = p(\rho)$ which satisfies assumptions (2) and (3) above.

These four assumptions are minimal restrictions on the properties of neutron star matter. As we shall see, however, even these are enough to put fairly stringent bounds on neutron star masses and moments of inertia. Any further general principles which restrict the equation of state will only lead to improvement in the bounds. An example of such a restriction is the condition†

$$(\mathrm{d}p/\mathrm{d}\rho)^{1/2} \leqslant 1. \tag{1.4}$$

The quantity $(\mathrm{d}p/\mathrm{d}\rho)^{1/2}$ is the hydrodynamic phase velocity of sound waves in the neutron star matter. In the absence of dispersion and absorption it would be the velocity of signals in the medium. Condition (1.4) would then be the condition that the speed of these signals not exceed that of light. Unfortunately, neutron star matter is dispersive and, because it is at zero temperature, the first sound waves will be infinitely damped. From general and direct considerations of causality alone there seems as yet no compelling reason to require eq. (1.4). Whether it is a fact that eq. (1.4) can be demonstrated for matter in its ground state from more detailed considerations remains an open question (see [31] for discussion), although most calculations [5] of the properties of matter at high densities do yield equations of state which satisfy it. In view of this, we will use the condition (1.4) throughout the following to illustrate how the upper bound on the mass of non-rotating neutron stars is affected by the imposition of more restrictive assumptions than those contained in (1)–(4) above, first calculating bounds with assumptions (1)–(4) alone and then with the additional assumption $(\mathrm{d}p/\mathrm{d}\rho)^{1/2} \leqslant 1$.

1.2. Core and envelope

The four assumptions on the matter outlined above already allow a number of useful deductions on the structure of spherical neutron stars in general relativity. The spacetime geometry of a spherical neutron star is described by a metric which in Schwarzschild coordinates has the form

$$\mathrm{d}s^2 = -\mathrm{e}^{\nu(r)}\mathrm{d}t^2 + \left[1 - \frac{2m(r)}{r}\right]^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2). \tag{1.5}$$

†Here and throughout we use units in which $c = G = 1$. As far as possible we follow the conventions of [30].

The equations which determine the star's structure and the geometry are

$$-\frac{\mathrm{d}p}{\mathrm{d}r} = (\rho + p)\frac{m + 4\pi r^3 p}{r^2(1 - 2m/r)}, \tag{1.6a}$$

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho, \tag{1.6b}$$

$$\frac{\mathrm{d}\nu}{\mathrm{d}r} = 2\frac{m + 4\pi r^3 p}{r^2(1 - 2m/r)}. \tag{1.6c}$$

To construct a given stellar model, these equations are to be integrated starting from the center $r = 0$ with a central density ρ_c, out to the surface $r = R$ where the pressure vanishes. The three boundary conditions which fix a solution are

$$p(0) = p(\rho_c) \equiv p_c, \tag{1.7a}$$

$$m(0) = 0, \tag{1.7b}$$

$$e^{\nu(R)} = 1 - 2m(R)/R. \tag{1.7c}$$

An important property which follows from assumptions (1) to (4) on the matter is that the inequality

$$2m(r)/r < 1, \tag{1.8}$$

is always satisfied. This can be proved by contradiction. Suppose that r_* is the first radius (moving out from the center) where $2m(r_*) = r_*$. In the immediate vicinity of r_*, eq. (1.6a) becomes

$$\frac{1}{\rho + p}\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{1}{2(r_* - r)}\frac{1 + 8\pi r_*^2 p(r_*)}{1 - 8\pi r_*^2 \rho(r_*)}. \tag{1.9}$$

The right hand side of this equation is negative for $r < r_*$. The left hand side is the logarithmic derivative of the relativistic enthalpy η defined by

$$\eta = (\rho + p)/n, \tag{1.10}$$

where n is the baryon number density. To check this one need only compute $\mathrm{d}(\log \eta)/\mathrm{d}p$ using the first law of thermodynamics

$$\mathrm{d}\rho/\mathrm{d}n = (\rho + p)/n. \tag{1.11}$$

Integrating both sides of eq. (1.9) one concludes that $\eta(r_*) = 0$. For no realistic equation of state, however, does the enthalpy ever vanish. At low pressures the pressure vanishes faster than the energy density and the enthalpy approaches the mass per baryon. Integration of eq. (1.11) then shows that n can never become infinite unless ρ and p do also. Since η never vanishes, $2m/r$ can never assume the value unity and the inequality (1.8) always holds.

Another important fact which can be deduced from our assumptions on the matter and from the equation of structure is that the density is non-increasing outwards. In fact,

$$\frac{\mathrm{d}\rho}{\mathrm{d}r} = \frac{\mathrm{d}\rho}{\mathrm{d}p}\frac{\mathrm{d}p}{\mathrm{d}r} = -\left(\frac{\mathrm{d}p}{\mathrm{d}\rho}\right)^{-1}(\rho + p)\left[\frac{m + 4\pi r^3 p}{r^2(1 - 2m/r)}\right]. \tag{1.12}$$

The quantities ρ, p, $\mathrm{d}p/\mathrm{d}\rho$ are positive by assumptions (2) and (3) and the positivity of the term in brackets follows from eq. (1.8) and from

$$m(r) = \int_0^r \mathrm{d}r\, 4\pi r^2 \rho. \tag{1.13}$$

Thus $(\mathrm{d}\rho/\mathrm{d}r) \leqslant 0$.

Because the density is non-increasing outwards, the radius r_0 at which the density assumes the fiducial value ρ_0 divides the star into two parts: An *envelope* $(r \geqslant r_0, \rho < \rho_0)$ where the equation of state is known and a *core* $(r \leqslant r_0, \rho \geqslant \rho_0)$ where it is only known to satisfy the general restrictions (1)–(4) above. It is this division into core and envelope which underlies all of our further analysis. The properties of the cores are restricted only by the minimal general assumptions on the matter while the properties of the envelope can be predicted from the known equation of state below ρ_0.

The mass of the core, which we denote by M_0, is simply the integral in eq. (1.13) out to the radius r_0. If M_0 is specified, the equations of structure eqs. (1.6a) and (1.6b) can be integrated outwards from r_0 using the known equation of state to give the mass in the envelope and the total mass of the star. The two boundary conditions needed to integrate eqs. (1.6a) and (1.6b) are supplied by $p(r_0) = p_0$ and $m(r_0) = M_0$. The total mass of the star can thus be written

$$M = M_0 + M_{\mathrm{env}}(r_0, M_0). \tag{1.14}$$

The mass in the envelope, M_{env}, is thus a function which is computable from the assumed equation of state at densities below ρ_0 and is therefore to be considered a known function of r_0 and M_0. The total mass itself is likewise a known function of r_0 and M_0.

To bound the mass of spherical neutron stars one must first determine what range of values are allowed the mass and radius of the core by general assumptions (1)–(4) above and by any additional assumptions which may be imposed. This range of possible cores we will call the *allowed region* in the r_0–M_0 plane. To find an optimum upper bound on the total mass, the function M in eq. (1.14) is maximized over the minimal allowed range of the variables r_0 and M_0. Lower bounds and bounds of more restrictive type can be found in similar ways.

1.3. Simple bounds on the allowed cores

The constraints that the density is non-increasing outwards and that $2m(r)/r < 1$ already obtained from the general assumptions are enough to place simple bounds on the allowed region but not optimal ones [26]. Applying eq. (1.8) to the whole core one has

$$M_0 < \tfrac{1}{2} r_0. \tag{1.15}$$

A non-increasing density means that the lowest value in the core is assumed on its boundary. Thus

$$M_0 = \int_0^{r_0} 4\pi r^2 \rho\, \mathrm{d}r \geqslant \int_0^{r_0} 4\pi r^2 \rho_0\, \mathrm{d}r, \tag{1.16}$$

or

$$M_0 \geqslant \tfrac{4}{3}\pi\, r_0^3 \rho_0. \tag{1.17}$$

The inequalities (1.15) and (1.17) define a finite allowed region in the r_0–M_0 plane. The core mass and radius themselves are bounded by

$$M_0 < \tfrac{1}{2}\left(\frac{3}{8\pi\rho_0}\right)^{1/2}, \qquad r_0 < \left(\frac{3}{8\pi\rho_0}\right)^{1/2} \tag{1.18}$$

for example, with a density of $\rho_0 = 5 \times 10^{14}$ g/cm^3 these imply a maximum core mass of $M_0 \leqslant 6M_\odot$ and a maximum radius $r_0 \leqslant 18$ km.

It would now be possible to find an upper bound on the total mass: Calculate the mass of core plus envelope $M(r_0, M_0)$ [eq. (1.14)] and maximize this function over the allowed region defined by eqs. (1.15) and (1.17). The resulting bound, however, would not be an optimum one. Eqs. (1.15) and (1.17) are only *bounds* on the allowed region; they do not define the minimal allowed region such that every point inside it corresponds to a core which can be constructed from an equation of state obeying assumptions (1)–(4). Some points in the region defined by eqs. (1.15) and (1.17) correspond to cores which are inconsistent with assumptions (1)–(4). In the next section we shall outline how to obtain the true allowed region.

2. The allowed cores and the optimum bound on the mass

To obtain optimum bounds on the mass and moment of inertia of non-rotating neutron stars, the minimal region allowed to cores constructed from equations of state satisfying assumptions (1)–(4) must be found. To find the minimal region, eq. (1.15) giving a bound on the upper boundary of the allowed region must be improved. Eq. (1.17) which supplies the lower boundary of the allowed region cannot be improved because cores made from matter with constant density ρ_0 – matter which satisfies assumptions (1)–(4) – lie along this lower boundary. To improve eq. (1.15) we apply to the cores a method developed by Buchdahl [32] for obtaining bounds on the redshift of a whole star following the development in Sabbadini and Hartle [26]. The method starts from the observation that the pressure in a spherical star is everywhere finite except possibly at the center where it may become infinite as a limiting case. As a consequence the redshift $z(r) = \exp\,[-\nu(r)/2] - 1$ is finite everywhere except possibly at the center because eq. (1.6c) can be integrated inward from the surface without diverging using the boundary condition in eq. (1.7c). The quantity

$$\zeta(r) = e^{\nu(r)/2}\,, \tag{2.1}$$

is then everywhere positive and finite in an equilibrium star vanishing only at the center of an infinite central pressure limiting configuration.

A second order differential equation relating $\zeta(r)$ and $m(r)$ can be found by combining the equations of structure (1.6a) and (1.6c). It is

$$\left(1-\frac{2m}{r}\right)^{1/2}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[\left(1-\frac{2m}{r}\right)^{1/2}\frac{1}{r}\frac{\mathrm{d}\zeta}{\mathrm{d}r}\right]=\frac{\zeta}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{m}{r^3}\right). \tag{2.2}$$

If the density is non-increasing outwards, then the average density is also non-increasing outwards, so that $\mathrm{d}(m/r^3)/\mathrm{d}r \leqslant 0$. The right hand side of eq. (2.2) is then negative or at most equal to zero if the density distribution is uniform. Changing to the independent variable

$$\xi = \int_0^r \mathrm{d}r\, r\left(1-\frac{2m}{r}\right)^{-1/2}, \tag{2.3}$$

this inequality takes the simple form

$$\mathrm{d}^2\zeta/\mathrm{d}\xi^2 \leqslant 0. \tag{2.4}$$

For such a convex downward curve the slope at any point is always less than that of the chord joining $\zeta(\xi)$ and $\zeta(0)$ i.e.,

$$\mathrm{d}\zeta/\mathrm{d}\xi \leqslant [\zeta(\xi) - \zeta(0)]/\xi. \tag{2.5}$$

The equality holds for a uniform density core. Since $\zeta(0) \geqslant 0$

$$\frac{1}{\zeta}\frac{\mathrm{d}\zeta}{\mathrm{d}\xi} \leqslant \frac{1}{\xi}. \tag{2.6}$$

The equality holds only for a uniform density core with infinite central pressure. Rewritten in terms of the variables ν and r, eq. (2.6) becomes

$$\frac{1}{2}\left(1 - \frac{2m}{r}\right)^{1/2} \frac{1}{r}\frac{\mathrm{d}\nu}{\mathrm{d}r} \leqslant \left[\int_0^r \mathrm{d}r\, r\left(1 - \frac{2m}{r}\right)^{-1/2}\right]^{-1} \tag{2.7}$$

The right hand side of eq. (2.7) is as large as possible for a uniform density star. Indeed, since m/r^3 is non-increasing outward one has for all $r_1 \leqslant r$ the inequality $2m(r_1)/r_1 \geqslant [2m(r)/r]\,(r_1/r)^2$ and therefore

$$\int_0^r \mathrm{d}r_1\, r_1\left(1 - \frac{2m(r_1)}{r_1}\right)^{-1/2} \geqslant \int_0^r \mathrm{d}r_1\, r_1\left(1 - \frac{2m(r)}{r^3}\, r_1^2\right)^{-1/2} \geqslant \frac{r^3}{2m(r)}\left[1 - \left(1 - \frac{2m(r)}{r}\right)^{1/2}\right]. \tag{2.8}$$

The relation is an equality for a uniform density star. Eq. (2.8) may be used to bound the right hand side of eq. (2.7). The left hand side may be reexpressed in terms of m and p using the equation of structure eq. (1.6c). The result is the following bound on $2m(r)/r$

$$m(r)/r \leqslant \tfrac{2}{9}\{1 - 6\pi r^2 p(r) + [1 + 6\pi r^2 p(r)]^{1/2}\}. \tag{2.9}$$

The inequality becomes an equality for a uniform density star with infinite central pressure.

When eq. (2.9) is evaluated at the surface of the whole star where $p = 0$ one finds

$$2M/R \leqslant 8/9, \tag{2.10}$$

M now being the total mass of the star. This is the origin of the surface redshift bound $z_{\mathrm{surf}} \leqslant 2$ obtained by Buchdahl [32] and Bondi [33]. When evaluated at the boundary of the core where $p = p_{\mathrm{o}}$, eq. (2.9) becomes

$$M_{\mathrm{o}} \leqslant \tfrac{2}{9} r_{\mathrm{o}}[1 - 6\pi r_{\mathrm{o}}^2 p_{\mathrm{o}} + (1 + 6\pi r_{\mathrm{o}}^2 p_{\mathrm{o}})^{1/2}]. \tag{2.11}$$

Equation (2.11) is the sought for improvement in eq. (1.15). It is a bound which cannot be improved because it is satisfied by a uniform density star with infinite central pressure. Eq. (2.11) together with eq. (1.17)

$$M_{\mathrm{o}} \geqslant \tfrac{4}{3}\pi r_{\mathrm{o}}^3 \rho_{\mathrm{o}}, \tag{2.12}$$

define the upper and lower boundaries respectively of the allowed region in the r_{o}–M_{o} plane for cores constructed from matter which satisfies our assumptions (1)–(4).

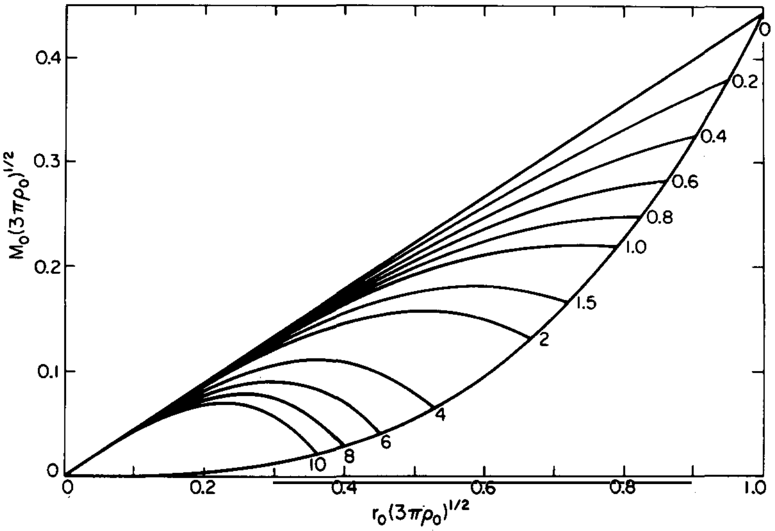

Fig. 1. The allowed region in the r_0–M_0 plane for cores constructed from equations of state satisfying $\rho \geqslant 0, p \geqslant 0$ and $(dp/d\rho) \geqslant 0$. Plotted here are the upper and lower boundaries of the allowed region using the dimensionless variables $(3\pi\rho_0)^{1/2}r_0$ and $(3\pi\rho_0)^{1/2}M_0$. The lower boundary is the curve $M_0 = \frac{4}{3}\pi r_0^3$. The upper boundary is given by eq. (2.11) and is shown for various values of the ratio p_0/ρ_0 as indicated on the graph. The allowed region becomes smaller the larger p_0/ρ_0 becomes.

The allowed region is illustrated in fig. 1. The overall scale may be regarded as set by ρ_0 so we plot $M_0(3\pi\rho_0)^{1/2}$. The shape of the region is then determined by the ratio p_0/ρ_0 and we have illustrated how the shape changes for several values of this quantity.

The physical origin of the allowed region is not difficult to see. To construct a core with a lower mass than one having a constant density equal to the boundary density ρ_0 [the equality in eq. (2.12)], the density would have to increase outward in some region and the matter there be microscopically unstable. The core with the largest mass is constructed from the matter with the stiffest equation of state – incompressible, constant density matter. Even here, however, there is a limit to the mass which can be contained in a radius r_0 without having the pressure become infinite at the center and equilibrium lost. That limit is given by the equality in eq. (2.11). It is this limit which is the special feature of Einstein's general relativity and which leads to the ability to impose a bound on the total mass.

The improvement in going from eq. (1.15) to (2.11) is significant. For $p_0 \ll \rho_0$, a condition which is satisfied for typical calculated equations of state in the regime of a few times nuclear densities and below, the maximum core mass and radius are

$$M_0 < \frac{4}{9}\left(\frac{1}{3\pi\rho_0}\right)^{1/2}, \qquad r_0 < \left(\frac{1}{3\pi\rho_0}\right)^{1/2} \tag{2.13}$$

an improvement of a factor 0.84 in the mass and 0.94 in the radius.

With the region allowed to cores constructed from matter obeying assumptions (1)–(4) now demarked, one only has to calculate the total mass as a function of r_0 and M_0 and maximize it over the allowed region in order to obtain an upper bound to the mass of non-rotating neutron stars. In section 4 the results of this procedure are discussed for several envelope equations of state. First, however, we consider a second derivation of the allowed region by a technique which may be more easily applicable when there are more restrictions on the equation of state than those contained in assumptions (1)–(4).

3. Variational approach to the allowed region

3.1. The method of Rhoades and Ruffini

This section outlines another approach to the problem of determining the region in the r_0–M_0 plane allowed cores constructed from matter obeying our basic assumptions. This is the variational method pioneered by Rhoades and Ruffini [24, 25]. In this method the problem of finding the boundary of the allowed region is considered as the problem of extremizing the mass of a core of given radius with respect to variations in the equation of state treating its restrictions and the equations of structure as constraints. For the simple general assumptions (1)–(4) the method is more cumbersome than that given in the previous section. It has the advantage, however, that further restrictions on the equation of state, such as eq. (1.4), can be incorporated in a straightforward manner, whereas Buchdahl's arguments do not appear to be so easily extendable.

3.2. Recovery of the allowed region

To illustrate the variational method we begin by recovering the allowed region derived under assumptions (1)–(4) in the previous section. To find the upper and lower boundary of the allowed region at a given r_0 we extremize the mass of the core M_0 with respect to variations in the equation of state treating the general restrictions (1.1) and (1.2) and the equations of structure (1.6a, b) as constraints. The extremum values of M_0 give the limits of the allowed region for that value of r_0. The equation of structure constraints we shall write as

$$\mathrm{d}m/\mathrm{d}r = 4\pi r^2 \rho \tag{3.1a}$$

and

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -(\rho + p)\frac{m + 4\pi r^3 p}{r^2(1 - 2m/r)} \equiv G(r, \rho, p, m). \tag{3.1b}$$

The microscopic causality constraint is conveniently taken into account by introducing an auxiliarly variable c defined by

$$\frac{\mathrm{d}\rho}{\mathrm{d}r} = \frac{1}{c^2}\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{G}{c^2}\,. \tag{3.1c}$$

The constraint is then simply

$$0 \leqslant c \leqslant \infty. \tag{3.2}$$

The three constraints of eq. (3.1) will be enforced by the method of Lagrange multipliers. The remaining constraints $\rho \geqslant 0$ and eq. (3.2) will be enforced explicitly. The variational condition that the mass of the core be an extremum for fixed r_0 is thus

$$\delta\left\{M_0 - \int_0^{r_0} \mathrm{d}r\left[\lambda_1(r)\left(\frac{\mathrm{d}m}{\mathrm{d}r} - 4\pi r^2\rho\right) + \lambda_2(r)\left(\frac{\mathrm{d}\rho}{\mathrm{d}r} - \frac{G}{c^2}\right) + \lambda_3(r)\left(\frac{\mathrm{d}p}{\mathrm{d}r} - G\right)\right]\right\} = 0, \tag{3.3}$$

where p, ρ, c and m are to be varied independently. The Lagrange multipliers are functions of r because the constraints in eq. (3.1) must be enforced at each value of r.

The variations in p, ρ and m can be decomposed into variations of their values at the endpoints of the interval $[0, r_o]$ and the variations in the interior. The values of p and m at the endpoints are constrained by the conditions $m(0) = 0$ (space locally flat at the origin) and $p(r_o) = p_o$ (no finite force on a fluid element of vanishing volume). The density at the endpoint is constrained through condition (1.2) by $\rho(r_o) \geqslant \rho_o$. Except for these constraints the endpoint values are free to vary.

The variation of $m(r_o)$ yields the condition

$$\lambda_1(r_o) = 1, \tag{3.4a}$$

while the variations of $\rho(0)$ and $p(0)$ yield respectively

$$\lambda_2(0) = 0, \tag{3.4b}$$

$$\lambda_3(0) = 0. \tag{3.4c}$$

The inequality constraint on $\rho(r_o)$ implies that the stationary configurations either satisfy

$$\rho(r_o) = \rho_o, \tag{3.5a}$$

or satisfy $\rho(r_o) > \rho_o$ in which case variation of its value yields

$$\lambda_2(r_o) = 0. \tag{3.5b}$$

Equations (3.4) and either of equations (3.5) are the boundary conditions on the Lagrange multipliers.

Three differential equations on the Lagrange multipliers result from variations of eq. (3.3) with respect to p, ρ and m keeping the endpoint values fixed. They are respectively

$$\frac{d\lambda_1}{dr} + \frac{\partial G}{\partial m}\left(\frac{\lambda_2}{c^2} + \lambda_3\right) = 0, \tag{3.6a}$$

$$\frac{d\lambda_2}{dr} + 4\pi r^2 \lambda_1 + \frac{\partial G}{\partial \rho}\left(\frac{\lambda_2}{c^2} + \lambda_3\right) = 0, \tag{3.6b}$$

$$\frac{d\lambda_3}{dr} + \frac{\partial G}{\partial p}\left(\frac{\lambda_2}{c^2} + \lambda_3\right) = 0. \tag{3.6c}$$

There remains only the variation of c. Since c is constrained by the inequality $c \geqslant 0$, in the core one must have *either*

$$c = 0, \tag{3.7a}$$

or eq. (3.3) must be satisfied with variations with respect to c. The latter condition is

$$\lambda_2 G/c^3 = 0. \tag{3.7b}$$

The possible configurations which extremize the mass M_o at a given r_o are found by integrating eqs. (3.1), (3.6) and (3.7) subject to the endpoint boundary conditions. There are seven equations for the seven unknowns p, ρ, m, c, λ_1, λ_2 and λ_3.

To classify the stationary solutions we begin by analyzing the implications of eq. (3.7). Eq. (3.7b) implies that either $\lambda_2 = 0$ or $c = \infty$. It is not difficult to show, following the argument of Rhoades

[24], that λ_2 cannot vanish in any finite range of r. Take eq. (3.6) and put $\lambda_2 = 0$. Eq. (3.6b) becomes an algebraic relation between λ_1 and λ_3 so that both of the remaining equations can be expressed in terms of one of them. The consistency of these two equations requires

$$v^2 + 2v(1-u) + 4u - 7u^2 = 0, \tag{3.8}$$

where we have written $v = 4\pi r^2 p$ and $u = m/r$. Since $v \geqslant 0$ and $u \leqslant \frac{1}{2}$, this equality can never be satisfied. Thus λ_2 can vanish only at isolated radii. The condition (3.7a) can also be satisfied only at isolated radii since it implies a density discontinuity. For finite ranges of r there remains only the possibility $c = \infty$, that is, a region of constant density. The possible configurations which locally extremize the mass of the core are thus stars of constant density regions separated by density discontinuities.

The position of a density discontinuity in a extremum configuration is not arbitrary since the core mass must still be stationary with respect to variations in the density on either side of the discontinuity. To see what this implies let ρ be the density distribution of a stationary configuration with a density discontinuity as $r = r_1$. Consider a variation $\delta\rho$ which vanishes for $r > r_1$. The stationary condition, eq. (3.3) gives for such a variation

$$\int_0^{r_1} \mathrm{d}r \left\{ \left[\lambda_1 4\pi r^2 + \left(\frac{\lambda_2}{c^2} + \lambda_3 \right) \frac{\partial G}{\partial \rho} \right] \delta\rho - \lambda_2 \frac{\mathrm{d}}{\mathrm{d}r} (\delta\rho) \right\} = 0. \tag{3.9}$$

Integrating this by parts and using eq. (3.6b) one finds

$$[\lambda_2\ \delta\rho]_{r_1} - [\lambda_2\ \delta\rho]_0 = 0. \tag{3.10}$$

The endpoint condition eq. (3.4b) implies that the second term vanishes. Since $\delta\rho(r_1)$ is arbitrary,

$$\lambda_2(r_1) = 0. \tag{3.11}$$

We conclude that the candidate configurations which extremize M_0 for a given r_0 are those for which the density is constant except at density discontinuities and that these discontinuities can occur only at the zeros of λ_2 considered as a solution of the differential equations (3.1) and (3.6) with boundary conditions (3.4) and (3.5). We now integrate these equations.

When the density is constant, eqs. (3.6) take the form

$$\frac{\mathrm{d}\lambda_1}{\mathrm{d}r} = \lambda_3 \frac{(\rho + p)(1 + 8\pi r^2 p)}{(r - 2m)^2}, \tag{3.12a}$$

$$\frac{\mathrm{d}\lambda_2}{\mathrm{d}r} = -4\pi r^2 \lambda_1 + \lambda_3 \frac{(m + 4\pi r^3 p)}{r(r - 2m)}, \tag{3.12b}$$

$$\frac{\mathrm{d}\lambda_3}{\mathrm{d}r} = \lambda_3 \frac{[m + 4\pi r^3(\rho + 2p)]}{r(r - 2m)}. \tag{3.12c}$$

Start at the origin with central density ρ_c and central pressure p_c. Out as far as the first density discontinuity $m(r) = \frac{4}{3}\pi r^3 \rho_c$. If p_c were finite then, near the origin, the right hand side of eq. (3.12c) would vary as $\lambda_3(r) 4\pi (\frac{4}{3}\rho_c + 2p_c) r$, and λ_3 itself would be proportional to $\exp[2\pi r^2 (\frac{4}{3}\rho_c + 2p_c)]$. No configuration with finite p_c can therefore satisfy the boundary condition $\lambda_3(0) = 0$ [eq. (3.3c)]. The configurations which extremize the core mass must therefore have infinite central pressure.

To calculate the candidate extrema with infinite central pressure we begin at the center with $\rho = \rho_c$, $p = \infty$, $m = 0$, $\lambda_2 = 0$, $\lambda_3 = 0$ and integrate eqs. (3.1) and (3.12) outward maintaining the density constant. The leading terms of the solution in a power series about the origin are

$$p(r) = (2\pi r^2)^{-1} + \dots \tag{3.13a}$$

$$\lambda_1(r) = CD + (5C/2\pi)r + \dots \tag{3.13b}$$

$$\lambda_2(r) = -\tfrac{4}{3}\pi CDr^3 + \dots \tag{3.13c}$$

$$\lambda_3(r) = Cr^4 + \dots \tag{3.13d}$$

where the constants C and D may be chosen arbitrarily. (There are two arbitrary constants because having fixed $p(0)$ at one end of its range, condition (3.4c) is now automatically satisfied.) Since eqs. (3.12) are homogeneous, the constant C sets the overall scale of the Lagrange multipliers and is fixed eventually by $\lambda_1(r_0) = 1$. Its value however does not affect the zeros of λ_2. Therefore with these boundary conditions and an arbitrary choice for C integrate eq. (3.1) and (3.12) outward. The constant D can be chosen so that the first zero of $\lambda_2(r_0)$ occurs at any radius of the core. At that radius insert an arbitrary downwards density discontinuity and continue the integration outward until the boundary pressure p_0 or a new zero of λ_2 is reached. This procedure does not have to be repeated for all different central densities since all quantities can be dimensionally scaled by that number. Typical curves of $\lambda_2(x)$, $x = (8\pi\rho_c/3)^{1/2} r$ are shown in fig. 2. The important point is that,

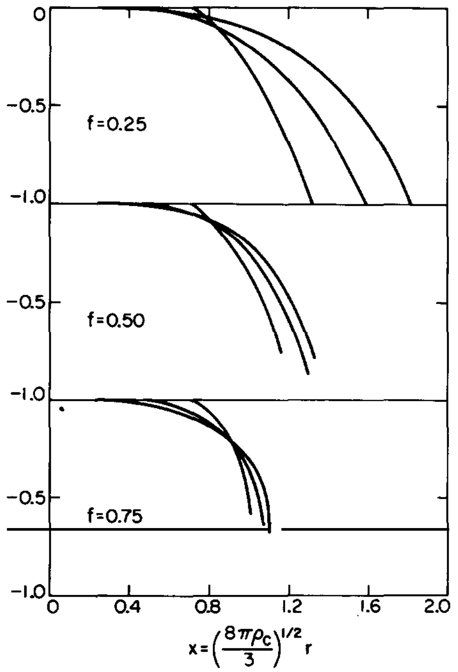

Fig. 2. The Lagrange multiplier λ_2 as a function of $x = (8\pi\rho_c/3)^{1/2}r$. This figure shows typical curves of $\lambda_2(x)$ which result from integrating eqs. (3.12) with the boundary conditions of eq. (3.13) with $C = 1$. The density distribution is uniform out to a radius x_d at which it discontinuously decreases by a fraction f to a new uniform value. At x_d, λ_2 is made zero by appropriate choice of the constant D in eq. (3.13). The integration is continued until the pressure drops to zero. Three different values of f are illustrated and for each three different values of x_d are chosen, $x_d = 0.25x_{max}$, $0.50x_{max}$ and $0.75x_{max}$ where x_{max} is $(8/9)^{1/2}$, the maximum value of x allowed to cores with no discontinuity. As these typical curves show there are no further zeros of λ_2. The cores which extremize M_0 at a fixed r_0 thus have uniform density with at most one discontinuity.

while D can be chosen to give a zero of $\lambda_2(r_0)$ at any point in the range $(0, r_0)$, there are then *no further zeros.* Taking into account the boundary condition (3.5) one then sees that the only possible stationary configurations are two layer stars with a constant density ρ_0 in the outer layer, a density discontinuity at any radius $0 \leqslant r \leqslant r_0$, and a constant density in the interior layer chosen to make $p(0) = \infty$. Only explicit computation can decide which of these gives the absolute extremum of M_0. The result is completely plausible. The minimum core mass is obtained by having the discontinuity arbitrarily close to the center so that the minimizing configuration is simply a constant density core with density ρ_0. The maximum value of M_0 is obtained by having the discontinuity at the core boundary and is a core with constant density and infinite central pressure. These are exactly the two configurations which saturate the bounds (2.11) and (2.12) already obtained by Buchdahl's method.

By repeating this calculation for different values of r_0 we recover the allowed region obtained in the previous section. The derivation, using the variational techniques of Rhoades and Ruffini, however was considerably less transparent and involved a moderate amount of computation. The great advantage of this method is that it is essentially straightforward and therefore applicable to situations where the restrictions on the equation of state are such that a direct derivation of the bounds has yet to be found. We shall now illustrate this.

3.3. The restriction $(\mathrm{d}p/\mathrm{d}\rho)^{1/2} \leqslant 1$

Any restriction on the equation of state in addition to assumptions (1)–(4) of section 1 will make the allowed region for cores in the r_0–M_0 plane smaller. We illustrate this by considering the effect of the restriction $(\mathrm{d}p/\mathrm{d}\rho)^{1/2} \leqslant 1$ discussed in section 1. The result is shown in fig.3. This was the problem to which Rhoades and Ruffini originally applied their variational technique. Actually Rhoades and Ruffini [24] attempted to apply their variational method to the whole star rather

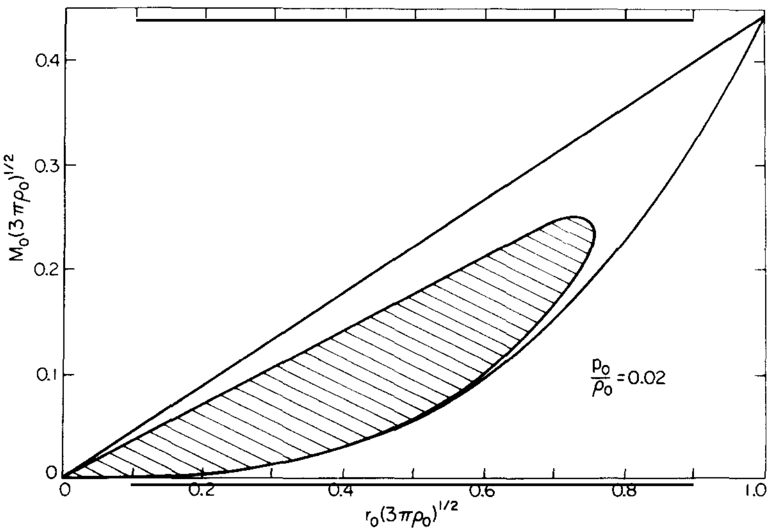

Fig. 3. The allowed region with the restriction $(\mathrm{d}p/\mathrm{d}\rho)^{1/2} \leqslant 1$. Shown here is the allowed region for cores constructed from equations of state satisfying $\rho \geqslant 0$, $p \geqslant 0$, $0 \leqslant (\mathrm{d}p/\mathrm{d}\rho)^{1/2} \leqslant 1$. The particular example shown here corresponds to a ratio $p_0/\rho_0 = 0.0162$ and is taken from [23]. The outer lines are the boundary of the allowed region without the assumption $(\mathrm{d}p/\mathrm{d}\rho)^{1/2} \leqslant 1$. The allowed region with this restriction is the shaded area.

than just to the core and thereby obtain the extremum of the total mass for stars whose equations of state satisfy assumptions (1)–(4) and eq. (1.4). This is in principle possible to do but the resulting variational equations which specify the possible stationary configurations have never been completely analyzed, leaving the derivation of Rhoades and Ruffini incomplete. When applying the technique to the core we follow Chitre and Hartle [25]. The derivation of the allowed region is essentially the same as that given above except that the constraint on c [eq. (3.2)] is now replaced by

$$0 \leqslant c \leqslant 1. \tag{3.14}$$

The configurations which extremize the core mass at a given r_0 must either lie on the boundary of the possible range of c or be stationary with respect to variations of c. The latter condition leads to eq. (3.7b) which can be satisfied only at isolated radii. The same is true for the condition $c = 0$ (density discontinuity). In any finite region c must therefore lie on the upper boundary of the range in eq. (3.14). The stationary configurations for this problem therefore consist of layers in which the equation of state satisfies $\mathrm{d}p/\mathrm{d}\rho = 1$ separated by density discontinuities. The identical argument to that preceding eq. (3.11) shows that these density discontinuities can only occur at a zero of λ_2.

The procedure for calculating the candidate configurations for the extremum of M_0 for a given r_0 is as follows: Begin at the center with a given central density ρ_c, an equation of state $p = \rho - k$ and the boundary conditions $p_c = \rho_c - k$, $m(0) = 0$, $\lambda_2(0) = 0$ and $\lambda_3(0) = 0$ [eqs. (1.7) and (3.4)]. The value of λ_1 at the center is an arbitrary constant which sets the scale of the Lagrange multipliers later to be fixed by eq. (3.4a). Proceed with the integration until a zero of λ_2 is reached. Since eqs. (3.6) are homogeneous, the position of this zero is not affected by the arbitrary value of $\lambda_1(0)$. Choose a new (lower) value of k in the equation of state $p = \rho - k$ and continue the integration until a new zero of λ_2 is found and a further discontinuity inserted. This process is repeated until the core boundary at r_0 is reached. There, either of the conditions (3.5) must be satisfied. These conditions can be thought of as fixing ρ_c. The possible stationary configurations are thus characterized by (1) which of conditions (3.5) is satisfied at the core boundary, (2) the number of zeros of λ_2 and (3) the possible discontinuities in ρ at each zero of λ_2.

In the case where the upper limit of c was ∞ there was only one zero of λ_2 and the above program could be carried out completely. Here, unfortunately, an arbitrary number of zeros of λ_2 are possible. To see this and also to provide a convenient way of solving the extremum equations, it is convenient to dimensionally scale all the variables and write

$$p(r) = s^2 p^*(r^*) \tag{3.15a}$$

$$\rho(r) = s^2 \rho^*(r^*) \tag{3.15b}$$

$$m(r) = s^{-1} m^*(r^*) \tag{3.15c}$$

$$\lambda_i(r) = \lambda_i^*(r^*) \tag{3.15d}$$

$$r = r^*/s. \tag{3.15e}$$

The form of the equations (3.1) and (3.6) is unchanged by this transformation but s can be chosen so that the core boundary occurs at an arbitrarily large value of r^*. Imagine adjusting the density discontinuity at a given zero of λ_2^* so that subsequently the equation of state is $p^* = \rho^* - \epsilon$ is an arbitrarily small number. If $\epsilon = 0$, it is not difficult to show that for large r the asymptotic solution to the equation of structure has the form [18, 33]

$$m^* \to r^*/4, \tag{3.16a}$$

$$p^* \to 1/(16\pi r^{*2}), \tag{3.16b}$$

and that the corresponding behavior for $\lambda_2(r)$ is

$$\lambda_2^* \to A(r^*/r_s^*)^3 \cos[(\tfrac{3}{2})^{1/2} \log(r^*/r_s^*)], \tag{3.16}$$

where A and r_s^* are constants determined by the small r integration. This behavior will hold as long as $p^* \gg \epsilon$, that is for $r^* \ll (16\pi\epsilon)^{-2}$. When p^* becomes comparable to ϵ, $p^*/(\rho^* + \epsilon)$ will drop so that it can become less than the scale invariant ratio p_0/ρ_0 at the core boundary as is required by $p(r_0) = p_0$, $\rho(r_0) \geq \rho_0$. The important point, however, is that since ϵ can be made arbitrarily small, the behavior of eq. (3.16) shows that an arbitrarily large number of zeros of λ_2 can be contained within the core. There is no hope of testing the infinite number of stationary configurations to see which provide the boundaries of the allowed region. Chitre and Hartle [25] have numerically computed a large number of them when the dimensionless ratio p_0/ρ_0 had the value 0.0162 corresponding to a density $\rho_0 = 5.09 \times 10^{14}$ g/cm^3 in the BBPS equation of state which we shall discuss later. Specifically cores where either of the conditions (3.5) is enforced at the core boundary were examined with zero, one, and two density discontinuities of varying amounts in the interior. The results are shown in fig. 4. In every case the stationary configurations were contained within or lay on the boundary of the domain occupied by those core configurations constructed from an equation of state with $(dp/d\rho)^{1/2} = 1$ and with a density discontinuity *only* at the core boundary. These configurations

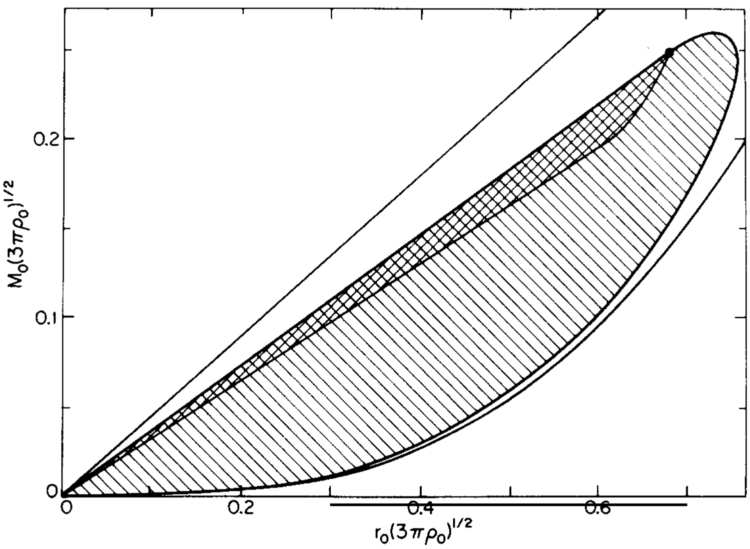

Fig. 4. The construction of the allowed region shown in fig. 3. The allowed region was determined by solving eqs. (3.6) with the constraint (3.14) for the configurations which extremize M_0 at a given r_0. The extremizing configurations with $\rho(r_0) = \rho_0$ are located as follows: Those with no interior discontinuity lie along the lower boundary of the allowed region from the origin to the heavy dot; those with one or two discontinuities in the core interior fill completely the allowed region (hatched area). The extremizing configurations with $\lambda_2(r_0) = 0$ at the core boundary are located as follows: Those with no interior discontinuity lie on this scale along the upper core boundary from the origin to the heavy dot; those with two and three interior discontinuities lie in the cross hatched region. This figure shows that all the calculated extremizing configurations lie within the heavy line which is therefore plausibly the boundary of the allowed region.

therefore appear to define the allowed region for cores obeying assumptions (1)–(4) and also $(\mathrm{d}p/\mathrm{d}\rho)^{1/2} \leqslant 1$.

It should be emphasized that as yet we do not have a rigorous derivation that the allowed region is coincident with the domain occupied by cores with $(\mathrm{d}p/\mathrm{d}\rho)^{1/2} = 1$ and a density discontinuity only at the core boundary because all possible stationary configurations have not been calculated. Still the large number which have been calculated at $p_0/\rho_0 = 0.0162$ make this result very plausible. However, the limited nature of this result should be kept in mind, especially for larger values of p_0/ρ_0. This situation makes it clear that it would be very desirable to have a direct calculation of the allowed region along the lines of that in section 3. As yet none has been presented.

4. Calculation of the upper bound on the mass

4.1. A particular ρ_0

Having found in section 2 the region in the r_0–M_0 plane allowed to cores constructed from matter obeying the minimal assumptions (1)–(4) of section 1, and having found in section 3 the restriction of this region when the additional condition $(\mathrm{d}p/\mathrm{d}\rho)^{1/2} \leqslant 1$ is imposed, we can now proceed to calculate optimum upper bounds on the mass of non-rotating neutron stars under both these sets of conditions. To do this we need to choose a ρ_0, choose an equation of state below this density, calculate the mass in the envelope as a function of r_0 and M_0, and maximize the total mass – core plus envelope – over the allowed core region. As an illustration of this calculation we may take that of ref. [26]. The equation of state in the envelope used in this calculation is that of Baym, Bethe, Pethick and Sutherland [34] which we shall refer to as the BBPS equation of state. It is qualitatively similar to most recent calculations of the equation of state at nuclear densities and below [6, 7]. The fiducial density and pressure used were $\rho_0 = 5.1 \times 10^{14}$ g/cm^3, and $p_0 = 7.4 \times 10^{33}$ dynes/cm^2 respectively, giving a dimensionless ratio $p_0/\rho_0 = 0.016$ when units are used where $c = G = 1$. The fiducial density and pressure were chosen because they represent the largest value to which BBPS believe their nuclear matter calculation can be applied [35]. It represents a density only slightly larger than that of homogeneous saturated nuclear matter (3×10^{14} g/cm^3).

The allowed region which correspond to the values of p_0 and ρ_0 given above and to assumptions (1)–(4), is shown in fig. 5 both with and without the extra assumption $(\mathrm{d}p/\mathrm{d}\rho)^{1/2} \leqslant 1$. Superimposed are contours of the total mass calculated as explained above. For reasons which we shall discuss below, the limit at the origin is non-uniform depending on the slope of the line in the r_0–M_0 plane by which it is approached. This detail is on too small a scale to appear in fig. 5 and, in any event, the value of the total mass at the origin does not exceed $1.4M_\odot$. The optimum upper bound on non-rotating neutron star masses at this value of the fiducial density is $5.0M_\odot$ if assumptions (1)–(4) are taken, and is $3.0M_\odot$ if $(\mathrm{d}p/\mathrm{d}\rho)^{1/2} \leqslant 1$ is imposed in addition. To an accuracy of about 1% these are the values which are obtained by a number of authors using the same assumptions but different equations of state in the envelope, if their results are extrapolated to this particular value of ρ_0. In particular, Hegyi, Lee and Cohen [28] find a maximum mass of $5M_\odot$ under assumptions equivalent to (1)–(4). Nauenberg and Chapline [9] and Rhoades and Ruffini [24] find a maximum mass of $3M_\odot$ when the assumption $(\mathrm{d}p/\mathrm{d}\rho)^{1/2} \leqslant 1$ is added.

Two features are worth noting about this result. First, in both cases the bound occurs at the maximum possible value of the core mass in the allowed region. Second, the contribution of the

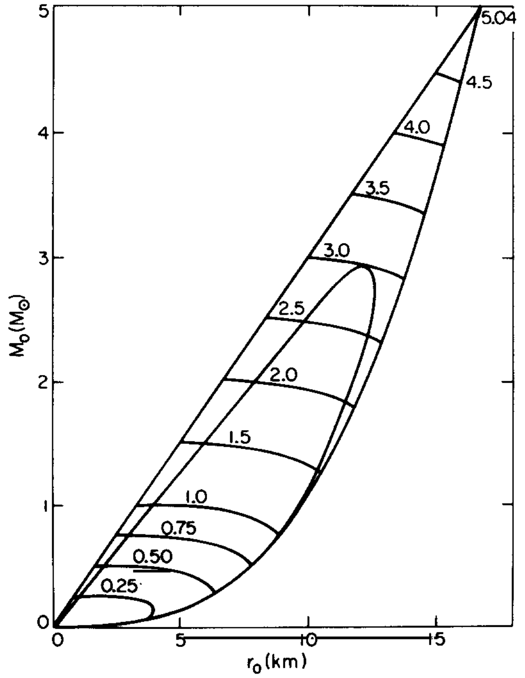

Fig. 5. The function $M(r_0, M_0)$ for the BBPS equation of state and a fiducial density $\rho_0 = 5 \times 10^{14}$ g/cm^3. Shown here are the contour lines of constant total mass of the star as a function of the mass and radius of its core over the allowed regions appropriate to the fiducial density $\rho_0 = 5 \times 10^{14}$ g/cm^3. Two allowed regions are shown: The larger one corresponding to the assumptions $\rho \geqslant 0$, $p \geqslant 0$, $(dp/d\rho)^{1/2} \geqslant 0$ and a smaller one corresponding to the additional assumption $(dp/d\rho)^{1/2} \leqslant 1$. Each contour is labeled by the value of $M/M_\odot$. The contours appropriate to the limiting behavior at the origin discussed in section 4.3 occur on too small a scale to appear here. The maximum of the function $M(r_0, M_0)$ gives the optimal upper bound to non-rotating neutron star masses. For both allowed regions this occurs at the largest core mass. The value of the bound is $3M_\odot$ if $(dp/d\rho)^{1/2} \leqslant 1$ is assumed and $5M_\odot$ if it is not. This figure is adapted from [42].

envelope to the bound is very small, less than 1% of the total mass in both cases. This latter feature is the reason for the general agreement of the authors who have considered the problem when their results are extrapolated to $\rho_0 = 5 \times 10^{14}$ g/cm^3. At this fiducial density the envelope gives an unimportant contribution to the bound so it does not matter which of several not too different equations of state are used to compute it. While these are certainly not general features, and *do not* persist at higher values of ρ_0, we can expect them when p/ρ is small throughout the envelope as it is here. In the following sections we will argue why this is the case.

4.2. The behavior of $M(r_0, M_0)$ for specific enthalpy near unity

The equations of structure [eq. (1.1)] which determine the envelope can be written in the form

$$dm/dr = 4\pi r^2 \rho \tag{4.1a}$$

$$\frac{1}{\eta}\frac{d\eta}{dr} = -\frac{M + 4\pi r^3 p}{r(r-2M)} \tag{4.1b}$$

where we have used the first law of thermodynamics eq. (1.11) to express the right hand side of eq. (1.1b) in terms of the relativistic enthalpy

$$\eta = (\rho + p)/n \tag{4.2}$$

where n is the baryon number density. At low pressures the pressure vanishes faster than the density so that η approaches a finite limit which is the mass per baryon μ of zero pressure matter. The enthalpy may be regarded as a function of the pressure determined from the relation $p = p(\rho)$ by the number μ and by the first law of thermodynamics [eq. (1.11)].

Let us now show that if $\eta(p_o)/\mu$ is sufficiently close to 1 and M_o/r_o is sufficiently big, the envelope will make a negligible contribution to the overall mass. The physical reason for this is that if envelope pressures are small and the core's surface gravity high, the scale height in the envelope will be small and the total envelope essentially a negligible atmosphere on the massive core. We proceed self-consistently by assuming that the thickness of the envelope $h = R - r_o$ is small compared to r_o

$$h/r_o \ll 1. \tag{4.6}$$

Then, since the density decreases outward,

$$M_{\text{env}} = \int_{r_o}^{R} 4\pi r^2 \rho \, \mathrm{d}r < \frac{4\pi}{3}(R^3 - r_o^3)\rho_o \approx \left(\frac{3h}{r_o}\right)\left(\frac{4\pi}{3} r_o^3 \rho_o\right). \tag{4.7}$$

Since $M_o \geqslant \frac{4}{3}\pi r_o^3 \rho_o$ [eq. (2.12)], we have from eq. (4.7)

$$M_{\text{env}} \lesssim (3h/r_o)M_o \ll M_o. \tag{4.8}$$

The mass of the envelope is thus negligible in comparison with the core.

It remains to show that this result implies eq. (4.6). Everywhere in the envelope we can replace m by the constant M_o in the equation of structure eq. (4.1b) in view of eq. (4.8). Further,

$$4\pi r^3 p \approx 4\pi r_o^3 \rho_o (p/\rho_o) \leqslant 3(p_o/\rho_o)M_o, \tag{4.9}$$

since the pressure decreases outward. Now from the definition of η, the first law of thermodynamics, and the fact that the density is an increasing function of p,

$$\log\left(\frac{\eta(p_o)}{\mu}\right) = \int_0^{p_o} \frac{\mathrm{d}p}{(\rho + p)} > \int_0^{p_o} \frac{\mathrm{d}p}{(\rho_o + p)} > \frac{p_o}{\rho_o}. \tag{4.10}$$

Thus if $\eta(p_o)/\mu$ is close to 1, p_o/ρ_o will be small and the pressure term in eq. (4.1b) can be neglected in comparison with M_o.

With these approximations eq. (4.1b) can be integrated to give in the envelope

$$\frac{\eta(p_o)}{\eta(p)} = \left[\frac{1 - 2u_o(r_o/r)}{1 - 2u}\right]^{1/2}. \tag{4.11}$$

Here, $u = m/r$, $u_o = M_o/r_o$ and p is the pressure at radius r.

At the surface $r = R = r_o + h$ the pressure vanishes and $\eta(0) = \mu$.

Thus,

$$\frac{\eta(p_o)}{\mu} = \left[\frac{1 - 2u_o/(1 + h/r_o)}{1 - 2u_o}\right]^{1/2}. \tag{4.12}$$

The right hand is unity at $h = 0$ and approaches $(1 - 2u_o)^{-1/2}$ at $h = \infty$. By eq. (4.10), $\eta(p_o)$ is an increasing function of p_o. Eq. (4.12) will have a self-consistent solution with h/r_o small provided $\eta(p_o)/\mu$ is close to unity *and* provided u_o is not too close to zero. It is

$$\frac{h}{r_o} \approx \left(\frac{\eta(p_o)}{\mu} - 1\right)\left(\frac{1 - 2u_o}{u_o}\right), \tag{4.13}$$

assuming

$$\frac{\eta(p_o)}{\mu} - 1 \ll u_o < \tfrac{4}{9}. \tag{4.14}$$

The mass in the envelope is then negligible

$$M_{\text{env}} \approx 3\left(\frac{\eta(p_o)}{\mu} - 1\right)\left(\frac{1 - 2u_o}{u_o}\right) M_o. \tag{4.15}$$

The situation is illustrated in fig. 6. If $\eta(p_o)/\mu$ is sufficiently close to unity, then for all u_o of order unity (shaded region) the envelope will be a negligible contribution to the total mass. The maximum total mass in this region will occur at and be very nearly equal to the maximum core mass. This, however, does not prove that the upper bound is given by the maximum mass core since the analysis does not touch the unshaded region. Considerable insight on the behavior here can be obtained by examining the behavior of $M(r_o, M_o)$ in the neighborhood of the origin where exact results can be obtained.

4.3. *The origin of the r_o–M_o plane*

One can insert a core of arbitrarily high density but negligible mass and radius at the center of any star without significantly affecting the star's structure or equilibrium. As a consequence, the

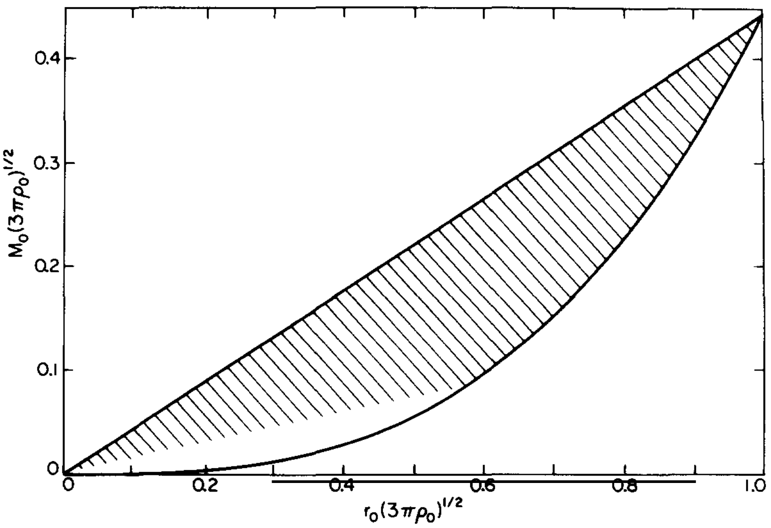

Fig. 6. If the enthalpy at the fiducial density ρ_o is close to unity the envelope will yield a negligible contribution to the core mass over much of the allowed region. The shaded area in this figure shows schematically where this will be the case.

function $M(r_o, M_o)$ should range over the masses of stars whose central density ρ_c is *less* than ρ_o. These masses will occur at the origin of the r_o–M_o plane corresponding to objects with vanishingly small but very dense cores. We shall show below how this is so.

The limit of $M(r_o, M_o)$ at the origin of the r_o–M_o plane is non-uniform and depends upon the slope $u_o = M_o/r_o$ with which it is approached. To demonstrate this and to calculate the limit, we note first that because

$$m_{\text{env}}(r) = \int_{r_o}^{r} \mathrm{d}r\, 4\pi r^2 \rho, \tag{4.16}$$

and because the density decreases outwards, the radius r_s at which m_{env} will be comparable to M_o is greater than the radius found by equating m_{env} to M_o in eq. (4.16) and replacing ρ by ρ_o

$$r_s^3 \gtrsim r_o^3 + 3M_o/4\pi\rho_o. \tag{4.17}$$

For fixed u_o and small r_o this becomes

$$r_s \gtrsim (3u_o/4\pi\rho_o)^{1/3}\, r_o^{1/3}. \tag{4.18}$$

The range $0 \leqslant r \ll r_s$ is the range for which the approximate solution, eq. (4.11), of the equations of structure is valid.

Consider the limit $r_o \to 0$, u_o fixed. Evaluate expressions (4.11) and (4.16) at a radius $r_c \sim (r_o)^p$ where $\frac{1}{3} < p < 1$. For $r_o \to 0$ this point lies within the range $0 \leqslant r_c \ll r_s$ to an increasingly better approximation. In the limit $r_o \to 0$ one has *exactly*

$$r_c = 0 \tag{4.19a}$$

$$m_{\text{env}}(r_c) = 0 \tag{4.19b}$$

$$\eta(p_c)/\eta(p_o) = (1 - 2u_o)^{1/2}. \tag{4.19c}$$

Thus, as the limit r_o is approached at a fixed value of u_o, the mass of the core vanishes and the envelope begins at $r_c = 0$ with $m_{\text{env}}(0) = 0$ and a value of the pressure p_c given by eq. (4.19c). In other words, in the limit the envelope becomes the entire star and has the same structure as the star on the non-rotating sequence with a central pressure p_c determined by the slope of the line in the r_o–M_o plane on which the origin is approached. Writing $M_{\text{origin}}(u_o)$ for the limiting value of $M(r_o, M_o)$, one sees that as u_o is increased from zero, $M_{\text{origin}}(u_o)$ ranges over the masses of all stars on the non-rotating sequence with central pressures less than p_o. If $\eta(p_o)/\mu$ is less than 3 then $p_c = 0$ will be reached before u_o reaches its maximum value of $\frac{4}{9}$. The limit will remain $m_{\text{origin}}(u_o) = 0$ for values of u_o larger than that corresponding to $p_c = 0$, thus reproducing at the origin the behavior obtained in section 4.2 above.

4.4. *The bound on the mass as a function of ρ_o*

Given the uncertainties in nuclear and hadronic physics there is no clear cut choice for the fiducial density ρ_o. Most workers would choose a value somewhere between 10^{14} g/cm^3 and 10^{15} g/cm^3 depending on how conservatively they view the extrapolations of nuclear matter calculations above nuclear densities. In such a situation the only reasonable course is to evaluate and state the bound on the mass of non-rotating neutron stars as a *function* of ρ_o. This not only will allow various possible views on nuclear matter calculations to be accommodated but also will indicate what improvements can be expected in the bound as our knowledge of superdense matter is pushed to higher densities.

The work of the previous two subsections leads to a qualitative understanding of how the bound on the mass varies with density. The crucial parameter is $\eta(p_0)/\mu$. For values slightly greater than unity, over a large region of the r_0–M_0 plane (the shaded region in fig. 6), the envelope will be a negligible contribution to the total mass so that the largest total mass will be given by the mass of the largest core. For $\eta(p_0)/\mu$ near 1 this largest core mass is given by eq. (2.13), decreases as $\rho_0^{-1/2}$ as ρ_0 is increased, and is independent of the details of the envelope equation of state.

Some indication of the behavior of the bound in the region (unshaded area in fig. 6) where this analysis does not apply can be found from the exact analyses of the behavior at the origin in section 4.3. The maximum mass at the origin is that of the most massive star constructed from the envelope equation of state with a central density *less than* ρ_0. As ρ_0 is increased from nuclear densities this number will typically increase through the sequence of neutron star masses until a maximum value is reached.

These qualitative considerations lead one to expect that the optimum upper bound on the mass of non-rotating neutron stars will be the result of a competition between the mass of the largest core which is decreasing with ρ_0 and the maximum mass of all stars central densities less than ρ_0 which is increasing or remaining constant with ρ_0. For values of $\eta(p_0)/\mu$ near unity, the optimum bound on the mass of non-rotating neutron stars under assumptions (1)–(4) should be given by eq. (2.13) *as long as this mass is greater than the mass of the most massive star constructed from the envelope equation of state with central densities less than* ρ_0. *Thus, to a good approximation,*

$$M_{\text{bound}} = \frac{4}{9}\left(\frac{1}{3\pi\rho_0}\right)^{1/2}, \tag{4.20a}$$

or

$$\frac{M_{\text{bound}}}{M_\odot} = 11.4\left(\frac{10^{14}\ \text{g/cm}^3}{\rho_0}\right)^{1/2}, \tag{4.20b}$$

under the necessary (but not sufficient) conditions

$$\eta(p_0)/\mu - 1 \ll 1, \tag{4.21a}$$

and

$$M_{\text{bound}} \geqslant \max\{M(\rho_c);\, \rho_c \leqslant \rho_0\}. \tag{4.21b}$$

Eventually, with increasing ρ_0, eq. (4.21b) will not be satisfied. The mass of the largest core becomes smaller and smaller but the largest mass represented at the origin is increasing or remaining fixed. Thus, eventually, we should reach a ρ_1 where

$$M_{\text{bound}} = \max\{M(\rho_c);\, \rho_c < \rho_1\}. \tag{4.22}$$

Since the right hand side of eq. (4.22) is a bound on the masses of all stars with central densities $\rho_0 > \rho_1$, the value $M(\rho_1)$ is *the maximum neutron star mass* and not simply a bound on that quantity. Thus, *if* the fiducial density can be pushed high enough so that the maximum of $M(r_0, M_0)$ is assumed at the origin of the allowed region, one will have computed the maximum neutron star mass because this maximum is at once a bound on the mass of all stars of higher central density and,

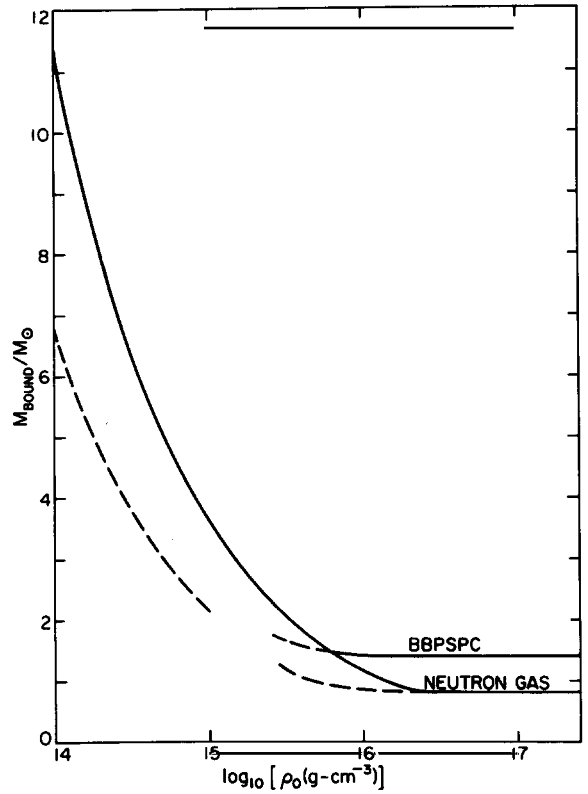

Fig. 7. Optimum upper bounds on the mass of non-rotating neutron stars as a function of the fiducial density ρ_0 above which the equation of state is restricted by minimal general assumptions and below which it is assumed known. The two solid curves were computed [27] with the assumptions $\rho > 0, p > 0, dp/d\rho > 0$. One was computed using the BBPSPC equation of state below ρ_0 and a second using the $n = \frac{3}{2}$ polytropic equation of state corresponding to a non-relativistic, free neutron gas below ρ_0. The dashed curves are the upper bounds computed with the additional assumption $(dp/d\rho)^{1/2} < 1$ and the same equation of state. The portions of these curves immediately above $\sim 10^{15}$ g/cm^3 are not shown since they have not actually been computed. They presumably join onto the horizontal portions which represent the maximum masses of stars below ρ_0 somewhat as shown. In any event they must be below the corresponding solid curve. Where the envelope is unimportant the calculations with the two different equations of state agree. There the bounds are given to an excellent approximation by $M_{\text{bound}}/M_{\odot} = 11.4[10^{14}\ \text{g-cm}^{-3})/\rho_0]^{1/2}$ when $(dp/d\rho)^{1/2} < 1$ is *not* assumed and by $M_{\text{bound}}/M_{\odot} = 6.8[10^{14}\ (\text{g-cm}^{-3})/\rho_0]^{1/2}$ when it is. These curves are not to be interpreted as accurate theoretical prediction of the optimum upper bounds on the mass of non-rotating neutron stars for high values of ρ_0, because they are computed from the above cited equations of state at densities where these particular relations may not accurately represent the properties of the matter. Rather, they should be taken as curves which indicate qualitatively the effect on the bounds a typical equation of state can have and in particular the dependence on ρ_0 which results.

at the same time, a physically realizible star computed from the known portion of the equation of state.

Qualitative considerations are no substitute for exact calculations. The above arguments do not prove the bound will behave in the way described because they do not state how the envelope behaves when $\eta(p_0)/\mu$ is close to unity and u_0 is small. In this case the envelope contributes significantly to the total mass.

Precise calculations have been carried out for two extrapolations of the equation of state above nuclear densities in [27]. The results are shown in fig. 7. The curve labeled BBPSPC was obtained using the Baym–Bethe–Pethick–Sutherland equation of state [34] joined onto Pandharipande's C

equation of state [36] above 5×10^{14} g/cm^3. The curve labeled neutron gas was obtained by using $n = \frac{3}{2}$ relativistic polytropic equation of state corresponding to a free non-relativistic neutron gas. With pressure and density expressed in units of km^{-2} this equation of state has the form

$$\rho = \tfrac{3}{2}p + 0.3032p^{3/5}. \tag{4.23}$$

The first choice is typical of many recent attempts at extrapolating the equation of state [5–7], while the second is a convenient analytic test relation neglecting all interactions.

The results of both calculations conform to the qualitative considerations above. Below about 4×10^{15} g/cm^3, the contribution of the envelope is negligible and the bound is given by eq. (4.20). The bound is assumed at the origin for a fiducial density $\rho_1 = 6.2 \times 10^{15}$ g/cm^3 for the BBPSPC equation of state, and $\rho_1 = 2 \times 10^{16}$ g/cm^3 for the $n = \frac{3}{2}$ polytropic equation of state. The bound then is equal to the actual maximum mass of $1.4M_\odot$ and $0.8M_\odot$ respectively.

If the assumption $(\mathrm{d}p/\mathrm{d}\rho)^{1/2} \leqslant 1$ is added to assumptions (1)–(4), the qualitative considerations above do not change except that the size of the allowed region is smaller. One actually does not know how this region varies with ρ_0 since the variational calculations described in section 3 have been carried out at only one fiducial density, $\rho_0 = 5 \times 10^{14}$ g/cm^3. It is plausible, however, that the results of this calculation extend to other densities for which eqs. (4.21) are satisfied and, in particular, the upper bound should occur at and be equal to the maximum mass core. The configuration which gives the maximum core mass has the equation of state $p = \rho + p_0 - \rho_0$ in the core interior. If $p_0 \ll \rho_0$ as is implied by eq. (4.21a), then dimensional considerations imply that the bound must vary as $\rho_0^{-1/2}$. Thus, with *the additional assumption* $(\mathrm{d}p/\mathrm{d}\rho)^{1/2} \leqslant 1$, we may plausibly put

$$M_{\text{bound}} = 6.8\left(\frac{10^{14}\ \text{g/cm}^3}{\rho_0}\right)^{1/2} M_\odot \tag{4.24}$$

under the necessary but not sufficient conditions of eq. (4.21). Within the 1% errors to which the calculations are accurate, these are the values obtained by Rhoades and Ruffini [24] and Nauenberg and Chapline [9]. This curve is shown in fig. 7, together with a guess based on the behavior at higher densities.

The curves shown in fig. 7 are not calculations of the bound based on known properties of matter at the endpoint of thermonuclear evolution. Rather they are qualitative indications on how the bound will behave with increasing ρ_0 based on current and test extrapolations of the equation of state to higher densities.

Most workers would place the fiducial density ρ_0 somewhere between 1×10^{14} g/cm^3 and 1×10^{15} g/cm^3 and calculate equations of state which are qualitatively the same as the BBPSC example used here. In this region the optimum upper bound on the mass of non-rotating neutron stars is largely independent of the details of the envelope equation of state. It is given by eq. (4.20) under assumptions (1)–(4) on the matter and by eq. (4.24) if $(\mathrm{d}p/\mathrm{d}\rho)^{1/2} \leqslant 1$ is additionally assumed. As ρ_0 ranges from 1×10^{14} g/cm^3 to 1×10^{15} g/cm^3 the bound ranges from $11.4M_\odot$ to $3.6M_\odot$ in the first case and from $6.8M_\odot$ to $2.1M_\odot$ in the second.

5. Bounds on the moment of inertia

5.1. The definition and importance of the moment of inertia

The moment of inertia is another gross structural parameter of neutron stars which may be important for observations. The pulsar neutron stars are rotating so slowly that deviations from spherical symmetry are small and it is therefore the moment of inertia of spherical stars which are of greatest interest. The moment of inertia of a non-rotating star may be defined generally as the ratio of the angular momentum J acquired by a star when it is given a slow and rigid rotation, to the angular velocity Ω of that rotation measured by an observer in an inertial frame a large distance from the star. More concretely the moment of inertia, I, of a spherical star is

$$I = (\partial J/\partial\Omega)_{\Omega=0}\,. \tag{5.1}$$

The importance of this number is not that it gives the connection between the angular momentum and angular velocity to *first* order in the angular velocity but rather that it gives the mass-energy of a rotating star with a given number of baryons to *second* order in the angular velocity. Exactly as in Newtonian theory,

$$M = M_{\mathrm{nr}} + \tfrac{1}{2}I\Omega^2 + \mathrm{O}(\Omega^4). \tag{5.2}$$

Here, M_{nr} is the non-rotating mass of a given collection of baryons and $\frac{1}{2}I\Omega^2$ is additional energy due to rotation to second order in Ω. This identity, first appreciated in relativity by Zel'dovich, is given a simple proof in [37]. It has been considerably elaborated by Carter [38]. Following [37], it can be seen directly from the variational principle for relativistic stellar structure. This principle [39] states that among all stationary axisymmetric configurations of matter and geometry with a given total angular momentum J and a given total baryon number A, those configurations which extremize the total mass M satisfy the equations of relativistic hydrostatic equilibrium. Introducing Lagrange multipliers Ω and μ to enforce the constraints on the total angular momentum and baryon number respectively, the principle can be written

$$\delta M - \Omega\delta J - \mu\delta A = 0, \tag{5.3}$$

where δM, δJ and δA are variations in the mass, angular momentum and baryon number respectively. Consider a particular variation in which the star with a fixed number of baryons is maintained in equilibrium but its angular velocity is increased by an amount $\delta\Omega$. Since eq. (5.1) implies

$$J = I\Omega + \mathrm{O}(\Omega^3), \tag{5.4a}$$

or

$$\delta J = I\,\delta\Omega + \mathrm{O}(\Omega^2)\,\delta\Omega, \tag{5.4b}$$

integration of eq. (5.3) implies the result in eq. (5.2). (For more details, see [37].) It is the moment of inertia of the spherical star which therefore controls the rotational energy of the slowly rotating star.

The defining relation, eq. (5.1) or eq. (5.4a), is only first order in the angular velocity and therefore only the first order equations of structure are needed to calculate I [40, 41]. To first order in the angular velocity the metric which describes the geometry outside a slowly rotating star may be written

$$\mathrm{d}s^2 = -\mathrm{e}^{\nu(r)}\mathrm{d}t^2 + [1 - 2m(r)/r]^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2) - 2\omega(r)r^2\sin^2\theta\,\mathrm{d}\varphi\mathrm{d}t + \mathrm{O}(\Omega^2). \tag{5.5}$$

Einstein's equation

$$R^{\mathrm{t}}_{\varphi} = 8\pi T^{\mathrm{t}}_{\varphi}, \tag{5.6}$$

taken to first order in Ω with a perfect fluid stress energy tensor gives an equation of structure for $\omega(r)$. Written in terms of the variable

$$f(r) = 1 - \omega(r)/\Omega, \tag{5.7}$$

this equation is

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^4 j \frac{\mathrm{d}f}{\mathrm{d}r}\right) + 4r^3 \frac{\mathrm{d}j}{\mathrm{d}r} f = 0, \tag{5.8}$$

where j is defined by

$$j(r) = \mathrm{e}^{-\nu/2}(1 - 2m/r)^{1/2}. \tag{5.9}$$

The boundary conditions which determine a solution of eq. (5.8) are that space be locally flat at the origin and asymptotically flat at infinity. The first condition implies $\omega(0)$ is finite and, hence, $f(0)$ is also. The second implies $\omega(r)$ vanishes at infinity. Since $j = 1$ outside the star, $\omega(r)$ must be proportional to r^{-3} and the constant of proportionality is twice the angular momentum of the star (see [30])

$$\omega(r) = 2J/r^3. \tag{5.10}$$

Thus, outside the star,

$$f(r) = 1 - 2I/r^3. \tag{5.11}$$

The moment of inertia I can thus be calculated by integrating eq. (5.8) outwards with an arbitrary value of $f(0)$, adjusting this value so that the constant term in eq. (5.11) is unity, and reading I off from the term which varies as r^{-3}.

The above procedure would give us the moment of inertia of any spherical star provided its spherical structure were known or, equivalently, provided the equation of state were known. In the absence of a detailed knowledge of the equation of state above a fiducial density ρ_0 we can bound the moment of inertia above and below by a variational technique similar to that employed for the mass following the work of Sabbadini and Hartle [42]. The basic idea simply is to vary the equation of state of the core of the star consistent with assumptions (1)–(4) of section 1 and see which configuration gives the largest and smallest value of I as computed above. The problem is how to take account of the equations of structure eqs. (3.1) and (5.8). The solution to this problem is greatly facilitated by having a variational principle for the moment of inertia which we now state.

5.2. A variational principle for the moment of inertia

For a given non-rotating star, among all trial functions $f(r)$ which satisfy the boundary conditions

$$r^4 j(\mathrm{d}f/\mathrm{d}r) \to 0, \qquad r \to 0 \tag{5.12a}$$

$$f \to 1 + O(1/r^3), \qquad r \to \infty \tag{5.12b}$$

that function $f(r)$ which extremizes the functional

$$I[f] = \frac{1}{6}\int_0^\infty \mathrm{d}r\left[r^4 j\left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)^2 - 4r^3 \frac{\mathrm{d}j}{\mathrm{d}r} f^2\right] \tag{5.13}$$

satisfies the equation of structure, eq. (5.8), and the value of the functional at the extremum in the moment of inertia of the spherical star [42]. The proofs of both parts of this statement are elementary. Variation of eq. (5.13) leads immediately to eq. (5.8), the surface term vanishing because of the boundary conditions (5.12). Integration of eq. (5.13) by parts with the extremum functional shows that the value of $I[f]$ is exactly the moment of inertia as defined through eq. (5.11).

The functional $I[f]$ is quadratic and positive definite since j is positive from eq. (5.9) and since $\mathrm{d}j/\mathrm{d}r$ can be calculated from the equations of structure (3.1) to be

$$\frac{\mathrm{d}j}{\mathrm{d}r} = -\frac{4\pi r(\rho + p)}{1 - 2m/r}\, j, \tag{5.14}$$

which is always negative. The unique extremum of eq. (5.13) is therefore a minimum and any trial function $f(r)$ satisfying eq. (5.12) will give an *upper bound* for I when inserted in the functional $I[f]$. For example, if one takes $f(r) = 1$ everywhere, one finds

$$I \leqslant \frac{8\pi}{3} \int_0^R \mathrm{d}r\, \frac{r^4(\rho + p)\mathrm{e}^{-\nu/2}}{(1 - 2m/r)^{1/2}}\,. \tag{5.15}$$

In the Newtonian limit this becomes an equality. This bound can also be derived directly from the equations of structure [37].

Bounds of this type, while straightforward to derive, are not optimum in the sense of being the actual moment of inertia for any spherical star. In the next section we shall outline how to use the variational principle to obtain optimum bounds.

5.3. *Optimum bounds*

5.3.1. *The extremizing density profile*

The basic problem from which all the other interesting bounds on the moment of inertia follow is to find the mass distribution $m(r)$ of a core of given mass M_0 and radius r_0 which extremizes the moment of inertia of the whole star. This is equivalent to extremizing I with respect to the equation of state because the density is related to $m(r)$ by

$$\rho(r) = \frac{1}{4\pi r^2}\frac{\mathrm{d}m}{\mathrm{d}r}\,, \tag{5.16}$$

and the pressure $p(r)$ can be recovered by integrating the equation of hydrostatic equilibrium [eq. (1.6a)] inward from the core boundary with the boundary condition $p(r_0) = p_0$. Restrictions must be imposed on the mass distribution in order that the resulting relation between p and ρ satisfies the chosen general assumptions on the equation of state. We will consider the problem only under assumptions (1)–(4) of section 1. These conditions, applied to a core of mass M_0 and radius r_0, lead to the constraints

$$m(0) = 0, \tag{5.17a}$$

$$m(r_0) = M_0, \tag{5.17b}$$

$$(\mathrm{d}m/\mathrm{d}r)_{r_0} \geqslant 4\pi r_0^2 \rho_0, \tag{5.17c}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^2}\frac{\mathrm{d}m}{\mathrm{d}r}\right) \leqslant 0, \qquad 0 < r < r_0, \tag{5.17d}$$

$$p(r) < \infty, \qquad 0 < r < r_0. \tag{5.17e}$$

The first two conditions are the boundary conditions on $m(r)$. Condition (c) states that $\rho(r_0) \geqslant \rho_0$. Conditions (d) and (e) when combined with the equation of hydrostatic equilibrium [eq. (1.6a)] imply both $p \geqslant 0$ and $\mathrm{d}p/\mathrm{d}\rho \geqslant 0$. By the last condition we do not exclude cores where $p(0) = \infty$, which are convenient to include as a limiting case.

Consider varying the moment of inertia I of a star having a core of mass M_0 and radius r_0 with respect to the mass distributions inside the core which satisfy the constraints of eq. (5.17). The variation in $m(r)$ will, through the equations of structure, produce variations in ρ, p, ν and thereby f and j. If we use the variational expression of eq. (5.13) for I we can write

$$\delta I = \int_0^\infty \mathrm{d}r \left[\frac{\delta I}{\delta f(r)}\,\delta f(r) + \frac{\delta I}{\delta j(r)}\,\delta j(r)\right]. \tag{5.18}$$

There are two important advantages to writing δI in this form. First, $\delta I/\delta f = 0$ by virtue of the variational principle for the moment of inertia so that variations in the angular velocity of the locally non-rotating frames, or of the associated equation of structure, do not have to be considered further. Second, since $\delta m(r_0) = 0$ and $\delta p(r_0) = 0$, the structure of the envelope is unaffected by variations of the mass distribution in the interior of the core. Since $\nu(r)$ is determined by the envelope structure through eq. (1.6c) and by a boundary condition that it vanish at infinity, it too will be unaffected by variations in the core mass distribution. Thus from eq. (5.9)

$$\delta j(r) = 0, \qquad r > r_0. \tag{5.19}$$

Combining these two facts one has

$$\delta I = \int_0^{r_0} \mathrm{d}r\,(\delta I/\delta j)\,\delta j(r), \tag{5.20}$$

or explicitly

$$\delta I = \frac{1}{6}\int_0^{r_0} \mathrm{d}r\left[r^4\,\delta j\left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)^2 - 4r^3 f^2\,\frac{\mathrm{d}}{\mathrm{d}r}(\delta j)\right]. \tag{5.21}$$

Thus, the variations in the moment of inertia of the *whole star* are related directly to variations in the quantities associated with the *core.*

The variations in j are not free. They are constrained by eq. (5.14) which relates them to variations in ρ and ν, by the equations of structure eqs. (1.6) which in turn relate these variations to the variations in $m(r)$, and finally by the constraints on the variations in $m(r)$ contained in eqs. (5.17). This makes for a very elaborate variational problem which in fact can be solved [42]. The solution is complicated by the need not only to consider free variations δm but also those which lie in the "boundary" of the space allowed $m(r)$ by the constraints (5.17), that is, variations in which one or more of the following conditions are satisfied

$$(\mathrm{d}m/\mathrm{d}r)_{r_0} = 4\pi r_0^2 \rho_0 \tag{5.22a}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^2}\frac{\mathrm{d}m}{\mathrm{d}r}\right)=0 \tag{5.22b}$$

$$p(0)=\infty. \tag{5.22c}$$

The multiplicity of these boundary constraints leads to a large variety of stationary conditions to be tested. In each case the question of the existence of a stationary solution can be reduced to a computable condition. We will not enter into the details of these conditions here because they are not especially instructive. The interested reader can find them in [42].

The end result of the analysis, however, is both very simple and physically reasonable: *The two mass distributions of a core with mass M_0 and radius r_0 which extremize the total moment of inertia of a star constructed from perfect fluid matter with $\rho \geqslant 0, p \geqslant 0$ and $\mathrm{d}p/\mathrm{d}\rho \geqslant 0$ are (1) the uniform density core, and (2) a core consisting of two uniform density layers separated by a density discontinuity, the outer layer having a density ρ_0 and the discontinuity being located as close to the origin as is consistent with finite pressure for $r > 0$.* The configuration (1) has the largest possible density at the core boundary consistent with the density not increasing outward. As much as possible of the matter is as far out as possible. The configuration on (2) has the smallest possible density at the core boundary and this density extends as far inwards as possible. As much as possible of the matter is as close to the center as possible. Unlike Newtonian theory all the matter cannot be put arbitrarily close to the center because this would violate the inequality $2m(r)/r < 1$. Intuitively one would expect the first configuration to give the upper bound on the moment of inertia and the second configuration to give the lower bound. The variational procedure by itself does not say which extremum gives the upper bound and which the lower. In actual calculations, however, ones intuitive expectations have always been born out.

We shall now illustrate the results quoted above by deriving several different types of bounds on the moment of inertia of spherical relativistic stars.

5.3.2. *Bounds on the moment of inertia of stars with a given mass and radius*

In Newtonian theory the upper bound on the moment of inertia of a star of radius R and mass M whose density is non-increasing outward is given by the moment of inertia of the uniform density sphere with that particular mass and radius. The lower bound is given by the configuration with all the mass in the center and is zero. Thus

$$0 \leqslant I_{\text{Newtonian}} \leqslant \tfrac{2}{5}MR^2. \tag{5.23}$$

In relativity we need only to set $\rho_0 = 0$ in the above solution to the variational problem and thus make the core the whole star to obtain upper and lower bounds to the moment of inertia of a star of a given mass and radius. The configuration which gives the upper bound is the uniform density star of mass M and radius R exactly as in the Newtonian case. The lower bound is given by the uniform density of mass M with infinite central pressure. Since for this star the inequality $M/R \leqslant \frac{4}{9}$ [eq. (2.10)] becomes an equality, this is the smallest possible star of mass M consistent with the general relativistic equations of stellar structure and assumptions (1)–(4) of section 1 on the matter.

Dimensional considerations imply that the upper and lower bounds on the moment of inertia of a star of given mass and radius can be written in the form

$$MR^2\theta_{\text{lower}}(M/R) \leqslant I \leqslant MR^2\theta_{\text{upper}}(M/R) \tag{5.24}$$

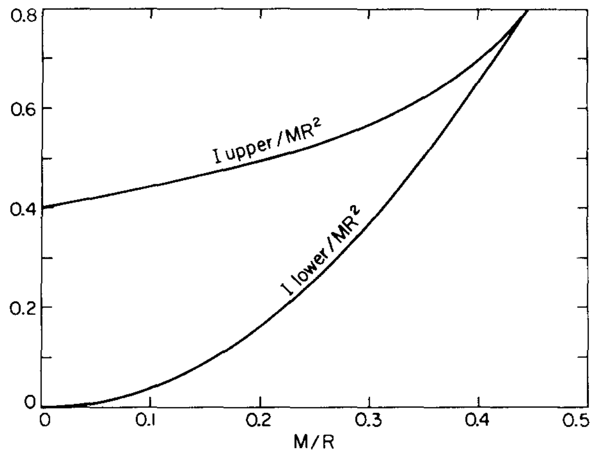

Fig. 8. The upper and lower bounds on the moment of inertia of a star of radius R and mass M expressed in units of MR^2. The assumptions underlying this result are $\rho \geqslant 0$, $p \geqslant 0$ and $\mathrm{d}p/\mathrm{d}\rho \geqslant 0$ for the perfect fluid equation of state. No detailed knowledge of the equation of state has been assumed in any density regime. When M/R is small the upper and lower bounds approach their Newtonian values $I_{\text{upper}} = \frac{2}{5}MR^2$ and $I_{\text{lower}} = 0$. As M/R approaches $\frac{4}{9}$ the upper and lower bound coalesce at the value $I = 0.799MR^2$. The relativistic upper bound is always larger than its Newtonian value. The figure is reproduced from [42].

where θ_{lower} and θ_{upper} are dimensionless functions of the ratio M/R. These functions are plotted in fig. 8. For nearly Newtonian stars, M/R is small and the values $\theta_{\text{upper}} = \frac{2}{5}$, $\theta_{\text{lower}} = 0$ are approached. As M/R approaches its maximum value of $\frac{4}{9}$, the configurations which give the upper and lower bounds coincide and $\theta_{\text{lower}}(\frac{4}{9}) = \theta_{\text{upper}}(\frac{4}{9}) = 0.799$.

Two points are especially worth noting about fig. 8. First, for all values of $M/R > 0$ there is a non-vanishing *lower* bound to the moment of inertia of a spherical star in contrast to the Newtonian situation. This is because in relativity there are no equilibrium configurations with all the mass at arbitrarily small rådii. Second, for all values of M/R, the upper bound for relativistic stars is greater than the corresponding Newtonian value of $\frac{2}{5}$, and at its greatest is almost double this. We know of no general explanation for this effect but some discussion of it is in [42].

5.3.3. *Absolute bounds on the moment of inertia of little use*

Optimum absolute upper and lower bounds to the moment of inertia of non-rotating stars can be obtained by combining the results for the extrema of the moment of inertia of stars with cores of given mass and radius, with the results on allowed values of these parameters derived in section 2. One could obtain absolute upper and lower bounds on the moment of inertia of spherical neutron stars by calculating the moment of inertia of the two extremizing cores for every core in the allowed region, and then taking the maximum and minimum of all these values. These results are not very interesting although some are calculated in [42]. The absolute minimum is zero, corresponding to a configuration with an arbitrarily small dense core but with a structure that is otherwise of an essentially zero mass, zero radius star. The upper bound for physically realistic equations of state will be given by white dwarf configurations with radii about 10^3 times larger than neutron star radii but with comparable masses. The resulting values of the upper bound therefore will be about 10^6 times greater than the actual values calculated from reasonable extrapolations of the equation of state. Restricting attention to configurations with central densities greater than typical neutron star central densities will not alter this value of the upper bound because there exist extremizing configurations with dense but arbitrarily small cores whose structure is otherwise the same as a white dwarf. Absolute upper and lower bounds on the moment of inertia thus yield little useful information.

5.3.4. Bounds on the mass-moment of inertia relation

Upper and lower bounds on neutron stars of a given mass are of greater interest than absolute bounds on the moment of inertia. Since the masses of white dwarf stars are less than the Chandrasekhar limit for fully catalyzed matter ($\sim 1M_\odot$), bounds on the moment of inertia of stars *more* massive than this will be in general much more restrictive since stars with dense cores but essentially white dwarf structure are then excluded. In addition, bounds of this type are interesting because the mass and moment of inertia are the two structural parameters of a neutron star most accessible to observation and any predicted correlation between these two quantities might be testable observationally.

Assumptions (1)–(4) of section 1 associate a unique total mass of a star to every possible core described by a mass M_0 and radius r_0. This is illustrated in fig. 5 where contours of constant total mass are shown over the allowed core region for the BBPS equation of state with a fiducial density of $\rho_0 = 5 \times 10^{14}$ g/cm^3. To find upper and lower bounds on the moment of inertia for stars of a given mass, one has only to compute its value at the two extremizing mass distributions for every core along the contour which corresponds to the particular mass value of interest and take the largest and smallest of the resulting numbers. The results for the BBPS equation of state and a fiducial density of $\rho_0 = 5 \times 10^{14}$ g/cm^3 are shown in fig. 9. Some values are given in table 1. As expected, below $1M_\odot$ the bounds are very generous because of the existence of the white dwarf stars. Above this value, however, they are interestingly restrictive. A star of $1.5M_\odot$, for example, cannot have a moment of inertia greater than $90M_\odot$ km^2 or less than $30M_\odot$ km^2 or, correspondingly, a radius of

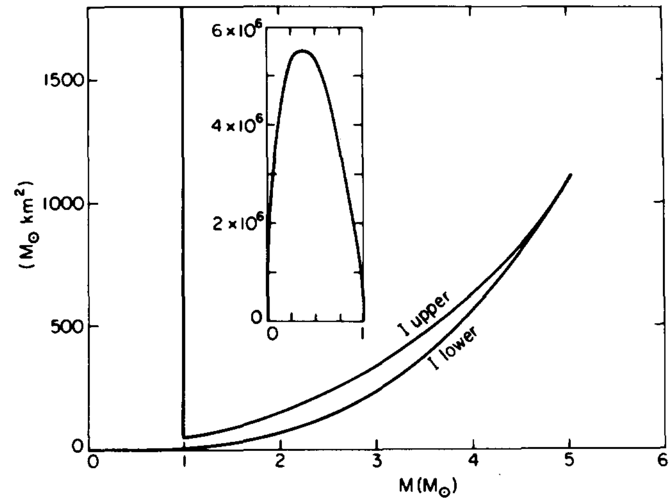

Fig. 9. Upper and lower bounds on the moment of inertia of a star of given mass when the BBPS equation of state is assumed to hold in the envelope. Above the Chandrasekhar limit $M = 1M_\odot$ the bounds are restrictive. For $M < 1M_\odot$ the upper bound is given by white dwarf configurations and is therefore much larger and correspondingly less useful. The upper bound versus mass plot in this region is shown in the inset on an appropriately reduced scale. Numerical values for these bounds are given in table 1. Reproduced from [42].

Table 1
Upper and lower bounds on the moment of inertia of neutron stars of a given mass*

$M/M_\odot$	$I_{\text{lower}}(M_\odot\ \text{km}^2)$	$I_{\text{upper}}(M_\odot\ \text{km}^2)$
1.25	17.2	65.7
1.50	29.8	90.2
1.75	47.3	120.
2.00	70.6	156.
2.25	100.	192.
2.50	138.	237.
2.75	183.	290.
3.00	238.	341.
3.25	303.	400.
3.50	378.	470.
3.75	465.	552.
4.00	566.	631.
4.25	677.	722.
4.50	804.	827.
4.75	945.	952.
5.04	1130.	1130.

* These bounds are taken from [42]. They assume $\rho \geqslant 0, p \geqslant 0, \mathrm{d}p/\mathrm{d}\rho \geqslant 0$ and the BBPS equation of state below $\rho_0 = 5.09 \times 10^{14}$ g/cm^3.

gyration greater than 7.7 km or less than 4.5 km. The tightness of this relation, especially the existence of a lower bound, would make the observed relation between M and I an interesting test of the general relativistic theory of stellar structure which is independent of the details of the equation of state at very high densities.

6. Other assumptions

In the preceding sections we have investigated bounds on the maximum mass of neutron stars under the assumptions that (1) the star is non-rotating, (2) the matter is a perfect fluid satisfying $p > 0$, $\rho > 0$ and $\mathrm{d}p/\mathrm{d}\rho \geqslant 0$, and (3) the correct theory of gravity is Einstein's general relativity. Since pulsar neutron stars are slowly rotating, since the matter inside them may not be precisely a perfect fluid [5, 6], and since there are competitors to Einstein's theory of gravity, it is of considerable interest to see how sensitive the bounds on the mass are to these assumptions. On the whole this problem has not been as systematically or as completely investigated as the bounds on the mass for perfect fluid stars in the general theory of relativity. Nevertheless, there is a considerable body of work which does indicate what one can expect when assumptions (1), (2) and (3) are relaxed. In the following we will summarize some of this work restricting ourselves for the most part to the conclusions and referring the reader to the original papers for details and derivations.

6.1. Rotation

One expects that rotation will in general increase the maximum neutron star mass for two reasons: First, the centrifugal forces act with the pressure to oppose the gravitational forces acting to collapse the star. Second, the rotational energy should give an additional positive contribution to the star's total mass.

The first point to appreciate is that there is no upper limit to the amount of matter which can be supported against gravity if an arbitrary rotation is allowed. Bardeen and Wagoner [43] showed that, in the extreme limit of a uniformly rotating disk, there is an upper limit to the amount of matter which can be supported for a given angular momentum. If the angular momentum increases without limit, however, the amount of matter which can be supported also increases without limit. Only in contexts in which a limit can be placed on the amount of rotation does it make sense to investigate a maximum rotating neutron star mass.

There are two considerations which lead naturally to a limitation on the amount of rotation. One is the observed pulsar angular velocities which are slow. The second is stability. While the latter situation is incompletely analyzed, it is generally believed that sequences of rotating stars become unstable by a variety of mechanisms if the rotation is too large. (For a small sample of the discussion of this subject see [45].)

The only true *bound* which has been obtained for the mass of rotating neutron stars follows from the upper bounds on the moment of inertia of spherical stars obtained in section 5. This is a bound on the increase in mass due to a slow and rigid rotation of a star with a given number of baryons. In the slow rotation limit, the mass of a uniformly rotating star containing A baryons is given by [37]

$$M(A) = M_{\text{nr}}(A) + \tfrac{1}{2}I(A)\Omega^2 + O(\Omega^4), \tag{6.1}$$

where Ω is the angular velocity of rotation and M_{nr} is the mass of the non-rotating configuration of A baryons. The quantity $I(A)$ is always positive as can be seen from eqs. (5.13) and (5.14), so that rotation always acts to increase the mass of A baryons over its non-rotating value. Defining the increase in mass due to rotation by $\delta M = \frac{1}{2}I\Omega^2$, one has from eqs. (5.24) and (6.1)

$$\frac{\delta M}{M} \leqslant \frac{1}{2}\,\theta_{\text{upper}}(M/R)\left(\frac{M}{R}\right)\frac{\Omega^2}{(M/R^3)}\,. \tag{6.2}$$

Now $(M/R^3)^{1/2}$ is approximately the angular velocity of mass shedding in Newtonian theory and, as such, is certainly an upper bound on the angular velocities at which the slow rotation theory applies. The maximum value of the first three factors in eq. (6.2) is 0.18 so that

$$\frac{\delta M}{M} < 0.2\left(\frac{\Omega^2}{M/R^3}\right) \ll 0.2. \tag{6.3}$$

A similar result appears in Rhoades and Ruffini [24]. Thus a slow and rigid rotation can increase the mass of a neutron star by only a small amount. Indeed, using the bounds obtained in section 5 for $I(M)$ and the angular velocity for the Crab pulsar, one finds $\delta M/M < 0.01$ so that, in realistic cases, the fractional increase in mass will be truly negligible.

It should be emphasized that the bound in eq. (6.3) on the rotational energy of a slowly rotating star with a given number of baryons is not the same thing as a bound on the maximum mass of a sequence of slowly rotating stars. This is because, in any rotating equilibrium sequence, there typically will be stars with baryon number A *greater* than the maximum value, A_{max}, which can be obtained along a non-rotating sequence constructed from the same equation of state. The situation is illustrated schematically in fig. 10. It is in the range $A > A_{\text{max}}$ that the actual maximum mass of a rotating sequence will be obtained and in this regime eq. (6.1) does not apply. Indeed, using only

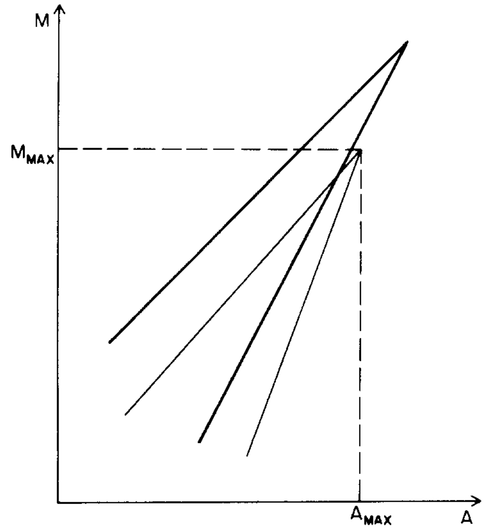

Fig. 10. A schematic representation of the behavior of the mass versus baryon number relation for a constant angular momentum sequence of rotating stars (heavy line) and a sequence non-rotating stars (light line) constructed from the same equation of state. There are cusps at the extrema of these curves because, considered as a function of central density, the mass and baryon number have simultaneous extrema along constant angular momentum sequences. In general, for a given baryon number, rotation increases the mass. There exist rotating equilibrium configurations with baryon number greater than the maximum allowed to non-rotating stars.

moderately stiff equations of state, Hartle and Thorne [45] obtained a fractional increase in the maximum mass of a constant Ω sequence of slowly rotating stars of $0.3\Omega^2/(M/R^3)$.

While as of this writing no bounds other than the one mentioned above have been proved for rotating neutron stars, there have been a number of calculations of the effect of rotation on the masses and moments of inertia of relativistic stars with specific equation of state. The most systematic of these studies is the work of Butterworth and Ipser [46] on the effects of a uniform rotation on constant density stars and relativistic polytropes. Their results for models with constant density – the stiffest possible equation of state – are of particular interest here. For this case they found that the largest fractional increase in mass along a sequence of uniformly rotating models of increasing Ω but fixed rest mass was about 0.15 if the sequence was terminated at the mass shedding stability, and was about 0.08 if it was terminated at an estimate of onset of the Dedekind instability. These results differ very little from the slow rotation results in eq. (6.3).

Butterworth and Ipser estimate that the largest fractional increase in the mass of a uniformly rotating constant density star over a non-rotating one with the same central pressure is about 0.3 if the rotation is limited by mass shedding and about 0.15 if limited by their estimate of the onset of the Dedekind instability. They argue this value is a good estimate of the increase in the maximum mass allowed to rotating stars over that for non-rotating ones. These results agree roughly with those of Shapiro and Lightman [47] who use post-Newtonian gravity and a free neutron gas equation of state, with Saenz [48] who uses slow rotation theory and a $(\mathrm{d}p/\mathrm{d}\rho)^{1/2} = 1$ equation of state above nuclear densities, and with Hartle and Thorne [45] who use the slow rotation theory and the Harrison–Wakano–Wheeler equation of state and the Tsuruta–Cameron V–γ equation of state.

These estimates suggest that uniform rotation does not increase very greatly the maximum mass allowed to stable neutron stars. It should be noted, however, that Wilson [49] produced differentially rotating configurations with fractional increases in mass at constant central pressure of 0.7

when limited by another estimate of secular instability although Hegyi [50] has argued that any significant differential rotation would be rapidly damped. In addition, Saenz [48] has made the point that the study of rotating models with a given equation of state may not be a good guide to the effect of rotation on any bound to the maximum mass since he can obtain greater masses from stability limited slowly rotating stars with $(\mathrm{d}p/\mathrm{d}\rho) = 0.5$ than with $\mathrm{d}p/\mathrm{d}\rho = 1$. The stiffest allowed equation of state does not necessarily give the maximum rotating mass.

The effects of rotation on the moment of inertia has been calculated for various equations of state by Butterworth and Ipser [46] and by Hartle [51] in the limit of slow rotation. Butterworth and Ipser find for both that the effect of rotation on the moment of inertia is generally more significant than it is on the mass. For the constant density equation of state a uniform rotation increases the moment of inertia over its non-rotating value by a factor of up to about 6. for sequences limited by the mass shedding instability and by up to about 0.4 for sequences terminated by an estimate of the onset of the Dedekind instability.

6.2. Non-perfect fluid matter

Recent calculations of ground state matter leave open the possibility of whether it becomes solid at densities above a few to ten times nuclear densities [5, 6]. It therefore is of interest to ask what the effect would be of relaxing the perfect fluid assumption on the bounds on the mass.

Bowers and Liang [52], Heintzmann, Hillebrandt and Steinmetz [53, 54] have considered the effects of anisotropic pressures on the masses of spherical stars choosing the difference between the tangential and radial pressures in a simple but largely ad hoc way. Bowers and Liang found incompressible models in which the ratio of $2M/R$ was not limited by 8/9 as it is in the perfect fluid case [eq. (2.10)], but could approach arbitrarily close to the Schwarzschild limiting value of 1. This means that the largest mass of a sphere of density ρ_0 is given by [cf. eq. (1.18)] $(3/32\pi\rho_0)^{1/2}$ rather than by $\frac{4}{9}(1/3\pi\rho_0)^{1/2}$ as in the fluid case [eq. (2.13)] – an increase by a factor of 1.19.

Heintzmann and Hillebrandt [53] were able to obtain configurations of arbitrarily high mass by fixing the ratio of transverse to radial pressure to be constant throughout the star and taking it arbitrarily large. In some regions of these stars the density increased outward. Hillebrandt and Steinmetz [54] found that these configurations were mechanically stable. Thus, there can exist no upper bound to the mass of configurations with anisotropic pressures without some further restriction on the types of configuration allowed or on the equation of state. Indeed, it is clear that if the density is permitted to increase outward then it is always possible to achieve stars of arbitrarily high mass simply by surrounding an existing star by a massive shell of low density material placed far enough away that it can support itself by its own shear stresses.

Mikkelsen [50] argued that, at the epoch of their formation, neutron stars would be hot and fluid. The density then would decrease outward and, as a consequence, the neutron stars existing today would have cooled in a state with outward decreasing density even if solid. For spherical stars, the equation of structure [eq. (1.6b)]

$$\mathrm{d}m/\mathrm{d}r = 4\pi r^2 \rho \tag{6.4}$$

is unaltered by any difference between the transverse and radial pressures. This relation, plus the assumption of non-increasing density, implies, as in section 1.3, that the allowed region for cores is given by

$$2M_0/r_0 \leqslant 1, \qquad M_0 \geqslant \tfrac{4}{3}\pi\rho_0 r_0^3, \tag{6.5}$$

and that in particular the mass of the core is bounded by

$$M_o \leqslant \left(\frac{3}{32\pi\rho_o}\right)^{1/2}. \tag{6.6}$$

The work of Bowers and Liang shows that there are configurations for which the equality is satisfied. The considerations which lead from the allowed region to a bound on the maximum mass now apply, as in section 4, provided ρ_o is chosen to be a density below which the matter is well approximated by a perfect fluid. This will be the case if ρ_o is nuclear density or slightly above. In this region all calculations on the equation of state give a specific enthalpy near unity so that as discussed in section 4, the bound is essentially given by the mass of the largest core. Thus if Mikkelsen's argument for *outwards non-increasing density is accepted*, we have *for spherical stars for ρ_o near nuclear densities or slightly above*

$$M_{\text{bound}} = \left(\frac{3}{32\pi\rho_o}\right)^{1/2} \tag{6.8}$$

or

$$\frac{M_{\text{bound}}}{M_\odot} = 13.6 \left(\frac{10^{14}\ \text{g/cm}^3}{\rho_o}\right)^{1/2} \tag{6.9}$$

even if the pressure in the core is anisotropic. This is about a 20% increase over the bounds given in eq. (4.20).

It should be noted, however, that if neutron star matter is solid, there is no reason that the star should be spherical. Bounds on the masses of non-spherical stars have not been obtained.

Magnetic fields will also cause the stress energy of a neutron star to deviate from a perfect fluid form. Munn [56] has investigated the effects of small magnetic fields on neutron star structure. In model calculations for stars constructed from the Harrison–Wakano–Wheeler (HWW) equation of state [20], he found that the largest fractional increase in mass at constant central density due to a dipole magnetic field embedded in the star is

$$\delta M/M \approx 0.2 B_s^2/(M/R^3) \tag{6.10a}$$

or, in order of magnitude,

$$\delta M/M \approx 0.01 (B_s/10^{18}\ \text{gauss})^2, \tag{6.10b}$$

where B_s is the surface magnetic field strength at the equator. Since most estimates put $B_s \approx 10^{12}$ gauss, the magnetic field has a truly negligible effect on the mass.

6.3. Other theories of gravity

The existence of a maximum mass for spherical stars constructed from reasonable equations of state in Einstein's general relativity is intimately related to the details of the equations of hydrostatic equilibrium. It therefore should not be surprising that the existence of a maximum mass and its value depend sensitively on which theory of gravity is used. In Newtonian theory there is no upper bound on the masses of stars constructed from incompressible matter, while there is a maximum mass using general relativity. Wagoner and Malone [57] have emphasized the sensitivity of the mass to the theory of gravity by constructing models of stars in the parametrized-post-Newtonian

(PPN) approximation to metric theories of gravity using a fixed equation of state and variable PPN parameters. They found the mass to be a sensitive function of some of the PPN parameters although the particular choice of values they used to generate large masses has effectively been ruled out by the recent lunar laser ranging experiments [58]. In the following we will briefly review the maximum mass situation in three theories of gravity — the Brans–Dicke theory, Rosen's Bimetric theory, and Ni's theory.

Saenz [48] has demonstrated the existence of a maximum mass to spherical stars constructed from equations of state satisfying assumptions (1)–(4) of the introduction and $(\mathrm{d}p/\mathrm{d}\rho)^{1/2} \leqslant 1$ in the Brans–Dicke theory. With these assumptions he has generalized the techniques of Rhoades and Ruffini [24] (reviewed in section 3 of this article) to calculate an upper bound to the maximum mass. For a $\rho_0 = 4.6 \times 10^{14}$ g/cm^3 and the HWW equation of state, he finds, as in relativity, that the envelope gives a negligible contribution to the bounding configuration so that the bound is given essentially by the largest mass core constructed from the equation of state $p = \rho - \rho_0 + p_0$. Assuming that this is true for other reasonable equations of state for which $\eta(\rho_0)/\mu - 1 \ll 1$, we may state his bound as follows

$$\frac{M_{\text{bound}}}{M_\odot} = B\left(\frac{10^{14}\ \mathrm{g/cm^3}}{\rho_0}\right)^{1/2}$$

where typical values of B are given in table 2 for several different values of the Brans–Dicke parameter ω. The general relativity value $B = 6.9$ is approached in the limit of large ω. Even for small values of ω the bound is not very different (~10%) from the general relativity result. Using the Brans–Dicke theory raises the bound on the maximum mass of spherical neutron stars, but not by much. This small change is consistent with the small changes in the masses of stellar models using particular equations of state found by Salmona [59], Heintzmann and Hillebrandt [60] and others. (For references to other calculations of neutron star models in the Brans–Dicke theory see [60].) Of particular interest are Heintzmann and Hillebrandt's [60] calculations of incompressible spheres in the Brans–Dicke theory. Calculations with incompressible matter may give some indication of the effect of Brans–Dicke theory on the bound on the maximum mass when the assumption $(\mathrm{d}p/\mathrm{d}\rho)^{1/2} \leqslant 1$ is relaxed. These authors find that as the Brans–Dicke parameter ω is *decreased* from 10^5, where the theory is essentially indistinguishable from general relativity, to the value $\omega = 6$, the maximum mass of incompressible spheres *decreases* by 13%. Interestingly, the maximum mass of an

Table 2
Bounds on the maximum mass assuming $(\mathrm{d}p/\mathrm{d}\rho)^{1/2} < 1$ in the Brans–Dicke theory of gravity†

ω	B	$M_{\text{bound}}(\rho_0 = 5 \times 10^{14}$ g/cm$^3)$
2.1	7.6	3.4
6	7.2	3.2
12	7.0	3.1
25	6.9	3.1
100	6.9	3.1

† Data taken from the work described in [48]. The author appreciates the help of Dr. Saenz in supplying more detailed numbers than those which appear in this paper.

incompressible sphere occurs at a finite value of the central pressure rather than at infinite central pressure as in relativity.

A completely contrasting situation is found in Rosen's bimetric theory of gravity. There Rosen and Rosen [61] have shown by explicit numerical calculation that the mass of an incompressible sphere tends to infinity as the central pressure tends to infinity. Thus, in the bimetric theory, there is no bound on the mass of spherical neutron stars if the equation of state is restricted only by assumptions (1)–(4) of section 1. Sequences of models with softer equations of state have been studied by Rosen and Rosen [62] and by Caporaso and Brecher [63] including, in particular, the analytic forms $p = c_s^2(\rho - \rho_1)$ for various values of c_s and ρ_1. Maximum masses exist for sequences of stars constructed from these trial equations of state but the maximum masses are generally larger then the corresponding general relativity result. The stiffer the equation of state the more pronounced this effect becomes. For example when $c_s^2 = 1/3$, Rosen and Rosen [62] find a maximum mass 1.7 times the general relativity value, while if $c_s^2 = 1$, the ratio is 17.1. It would be interesting to know if it is generally true that a maximum mass exists when $(\mathrm{d}p/\mathrm{d}\rho)^{1/2} \leqslant 1$ and what a bound on it would be.

Mikkelsen [64] has analyzed the existence of a maximum mass in some versions of Ni's theory of gravity which give the same results as general relativity in the post-Newtonian limit. He shows that even with reasonable choices for the equation of state (e.g., a free neutron gas), stars of arbitrarly high mass are possible in these theories.

7. Conclusions

7.1. *Summary of bounds on the mass in general relativity*

In general relativity an optimum upper bound can be established on the mass of non-rotating perfect fluid neutron stars when the equation of state is known below a fiducial density ρ_0, while above this value it is restricted only by the conditions of positive energy ($\rho \geqslant 0$), microscopic stability ($p \geqslant 0$, $\mathrm{d}p/\mathrm{d}\rho \geqslant 0$) and whatever further restrictions can be deduced from general physical principles. To illustrate these restrictions two cases were considered in this review: no further restrictions, and the additional restriction $(\mathrm{d}p/\mathrm{d}\rho)^{1/2} \leqslant 1$. Two things are necessary to rigorously establish a bound in the development given here: first, a determination must be made of the optimal region in the mass-radius plane allowed to cores constructed from equations of state obeying the restrictions. Second, the total mass must be computed as a function of the core mass and radius, and extremized over the allowed region. The maximum value gives the upper bound.

We know of no shortcut to this procedure for establishing an upper bound. When, however, it is applied with realistic estimates of the equation of state and a ρ_0 not too far above nuclear densities, very simple results emerge. The bound is given to a good approximation by

$$M_{\text{bound}} = 11.4\left[\frac{10^{14}\ \mathrm{g/cm^3}}{\rho_0}\right]^{1/2} M_\odot, \tag{7.1}$$

if no restrictions are assumed on the perfect fluid above ρ_0 other than positive energy and microscopic stability. This is changed to

$$M_{\text{bound}} = 6.8\left[\frac{10^{14}\ \mathrm{g/cm^3}}{\rho_0}\right]^{1/2} M_\odot, \tag{7.2}$$

if $(\mathrm{d}p/\mathrm{d}\rho)^{1/2} \leqslant 1$ is assumed in addition. The reason for this simplicity is that the envelope does not contribute significantly to the bound in these cases and this, in turn, can be traced to the small value of p/ρ at ρ_0 (or, more precisely, to the small difference between the specific enthalpy and unity).

If one asks which bound has been established by an airtight argument, then one has to conclude only eq. (7.1) for those equations of state for which the envelope structure has been explicitly integrated and shown to give a negligible contribution to the bound. Indeed, if one were to remain firmly rooted in *experimental* fact, then the only airtight bound would be eq. (7.1) with a ρ_0 of 8 g/cm^3 giving an upper bound of $2 \times 10^7\, M_\odot$!

The general agreement on the equation of state below nuclear densities and the analysis given in section 4 of when the envelope is important and when it is not, makes it very plausible, however, that equations (7.1) and (7.2) will give the bounds to a good approximation for all reasonable envelope equations of state as long as p_0/ρ_0 is small.

For the practically minded reader it therefore would seem very reasonable to accept eq. (7.1) as the most generally established bound and after a study of superdense matter calculations to arrive at a choice for ρ_0 obtain an actual number.

7.2. How close are we?

Table 3 shows the maximum neutron star mass and the central density at which it is assumed for 13 different equations of state all published since 1970 except for the free neutron gas equation of

Table 3
Neutron star maximum masses from selected equations of state*

Equation of state	$M_{max}/M_\odot$	ρ_c(g/cm^3)
A	1.66	$4. \times 10^{15}$
B	1.41	$6. \times 10^{15}$
C	1.85	$3. \times 10^{15}$
D	1.65	$4. \times 10^{15}$
E	1.73	$3. \times 10^{15}$
F	1.46	$5. \times 10^{15}$
G	1.36	$6. \times 10^{15}$
H	0.71	$4. \times 10^{15}$
I	2.45	$2. \times 10^{15}$
L	2.70	$1. \times 10^{15}$
M	1.96	$2. \times 10^{15}$
N	2.58	$2. \times 10^{15}$
O	2.39	$2. \times 10^{15}$

* Data taken from the review of Arnett and Bowers [7]. The equations of state are due to A: Pandharipande (neutron); B: Pandharipande (hyperon); C: Bethe and Johnson; D: Bethe and Johnson; E: Moszkowski; F: Arponen; G: Canuto and Chitre; I: Cameron, Cohen, Langer and Rosen; L: Pandharipande and Smith (mean field); M: Pandharipande and Smith (tensor interaction); N: Walecka, O: Bowers, Gleeson and Pedigo. H is the free neutron gas equation of state. See Arnett and Bowers [7] for discussion and references to the original literature.

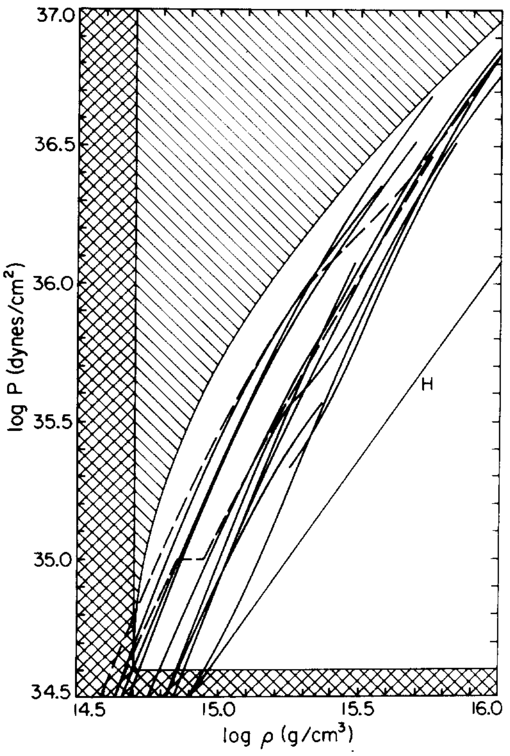

Fig. 11. Thirteen equations of state for neutron star matter. These equations of state have all been calculated since 1970 except for H which is the free neutron gas equation of state. Some equations of state are represented by solid lines and some by dashed lines in an attempt to distinguish them. The maximum masses which result from the models calculated with these equations of state are shown in table 4. This figure was reproduced from Arnett and Bowers [7]. The crosshatched area of the figure shows the regions *not* allowed to equations of state above a fiducial density $\rho_0 = 5 \times 10^{14}$ g/cm^3 and pressure $p_0 = 3 \times 10^{34}$ dynes/cm^2 by the assumptions $\rho \geqslant 0, p \geqslant 0$, and $dp/d\rho \geqslant 0$. The shaded area shows the additional region excluded by the assumption $(dp/d\rho)^{1/2} \leqslant 1$.

state (H) which is included for comparison. Figure 11 is a graph of the equations of state themselves. We show this data not to advocate some sort of statistical approach to establishing a reasonable value for the maximum mass but, rather, to indicate the range of recent results and uncertainties.

If ρ_0 is taken to be 5×10^{14} g/cm^3, just a little above nuclear densities where there is still general agreement on the equation of state, then the bound in eq. (7.1) is $5M_\odot$ which is from 2 to 4 times greater than the maximum masses quoted in table 3. If $(dp/d\rho)^{1/2} \leqslant 1$ could be argued, then this range would be decreased to only 1.1 to 2 times greater than the maximum mass in table 3. The bounds are thus close enough to the computed values to be extremely powerful in astrophysical arguments but are far enough away to allow for considerable improvement.

The sharp $\rho_0^{-1/2}$ dependence of the bounds in eqs. (7.1) and (7.2) emphasizes the utility of even modest increases in this number. If ρ_0 could be pushed to the values of ρ_c given in table 3, then the values of the bounds in eq. (7.1) become competitive with the actual maximum masses. The bounds would then say nothing additional about the maximum mass of neutron stars. This number would already have been calculated! What the bounds *would* guarantee is that no other endstates of stellar evolution would have masses greater than the maximum neutron star mass. The absolute maximum mass, as distinguished from a bound on it, thus would have been determined by computation of the equation of state to only a finite density. In the long run this may be the most useful application of the bounds on the mass.

It is clear from the difference between the present value of the bounds and the calculated values of the maximum mass that a great deal of information is being thrown away in imposing only the assumptions of positive energy and microscopic causality above the fiducial density ρ_0. This is illustrated in fig. 11. Assuming a ρ_0 of 5×10^{14} g/cm^3, the uncrosshatched region shows the range of equations of state allowed by the assumptions of positive energy and microscopic causality. Clearly, there is room for improvement here. Demonstration of $(\mathrm{d}p/\mathrm{d}\rho)^{1/2} \leqslant 1$ for example would eliminate the shaded region. The considerable restriction which would result in the allowed equations of state, and the improvement in eq. (7.2) over (7.1), emphasize the need for settling the question of whether $(\mathrm{d}p/\mathrm{d}\rho)^{1/2} \leqslant 1$ is a true property of superdense matter and also for removing the deficiences in the derivation of the bound. There are, however, other possibilities for restricting the region. Moderate but trustworthy *bounds* on the equation of state restricting this region could be expected to yield considerably tighter bounds on neutron star masses and, perhaps, bounds on relations like the mass radius relation as well. Any general restriction on the equation of state of matter at the endpoint of thermonuclear evolution at high densities can but give us a closer range within which lie the true properties of neutron stars. It is an important task of the relativistic theory of stellar structure to translate restrictions on the matter to restrictions on the stars themselves.

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