This exam contains 7 pages (including this cover page) and 9 questions. Total of points is 100 plus 10 bonus points. The grade of the final exam is the smaller value between your total points, including the bonus points, and 100 points. You need to get 60 points, in Questions 1 through 8, to pass the course.

Instructions

1. You may not use any electronic devices with computation or network functionalities during this exam.

2. You may use one page of notes, but no books or other assistance during this exam.

3. Write your Name and PID on the lines above.

4. Write your solutions below each questions. If you need to rewrite your solution, or if the solution overflows, ask the TA or instructor for a replacement page or an overflow page.

5. Read each question carefully, and answer each question completely.

6. Show all of your work; no credit will be given for unsupported answers.

Grade Table (for teacher use only)

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1. (10 points) Evaluate the integral
\[ \int_0^9 \int_{\sqrt{x}}^3 e^{y^3} \, dy \, dx. \]

2. (10 points) Evaluate the integral
\[ \iint_R e^{y-3x} \sin(y-x) \, dx \, dy \]
where \( R \) is the region bounded by \( y-x=0, \ y-x=2, \ y-3x=0, \ y-3x=1. \)
(Hint: you may want to change the variables.)

3. (11 points) Given the vector field \( \mathbf{F}(x,y,z) = (\sin y - z \cos x, x \cos y + \sin z, y \cos z - \sin x) \).
   (a) (6 points) Find a scalar function \( f(x,y,z) \) such that \( \nabla f = \mathbf{F} \).
   (b) (5 points) Evaluate the line integral \( \int_c \mathbf{F} \cdot d\mathbf{s} \) along the path \( c(t) = (\frac{\pi}{2} t, t^3 \sin(\frac{\pi}{2} t), t^4 \cos(2\pi t)) \), with \( t \) elapsing from 0 to 1.

4. (12 points) Let \( A \) be the astroid, enclosed by the curve \( x^{2/3} + y^{2/3} = 4 \) on the plane.
   (a) (10 points) Parametrize the curve \( \partial A \).
   (Hint: let \( x = 8 \cos^3 \theta \) for \( \theta \in [0, 2\pi] \).)
   (b) (15 points) Find the area of \( A \).
   (Hint: you may want to use \( \cos 2\theta = 1 - 2\sin^2 \theta \) and/or \( \sin 2\theta = 2\sin \theta \cos \theta \).)

5. (13 points) Find the area of the graph of the function \( f(x,y) = \frac{2}{3}(x^{3/2} + y^{3/2}) \) that lies over the domain \([0,1] \times [0,1] \).
   (Hint: the parametrization of the graph is \( (x,y,f(x,y)) \).)

6. (14 points) Calculate the integral \( \iint_S \mathbf{F} \cdot d\mathbf{S} \), where \( S \) is the sphere \( x^2 + y^2 + z^2 = 4, \) and \( \mathbf{F} = (x + 3y^5, y + 10xz, z^2 - xy) \). Here \( S \) is oriented by the outward unit normal.
   (Hint: you may want to use the Gauss' theorem)

7. (15 points) Let \( Q \) be the part of the cone \( x^2 + y^2 = \frac{1}{3}z^2 \) with \( 0 \leq z \leq 1-x \), oriented by the downward unit normal. We have a parametrization \( \Phi: D \to Q \),
\[ \Phi(\rho,\theta) = \left( \frac{1}{2}\rho \cos \theta, \frac{1}{2}\rho \sin \theta, \frac{\sqrt{3}}{2} \rho \right). \]
Let \( \mathbf{F}(x,y,z) = (3z - 2, 0, -2x) \) be a vector field.
   (a) (5 points) Find an elementary region \( D \in \mathbb{R}^2 \) such that \( \Phi \) is one to one except possibly on boundaries, and \( \Phi \) is onto.
**Solution:** The given parametrization $\Phi$ indeed satisfies the equation $x^2 + y^2 = \frac{1}{3}z^2$. In order to find the range of parameters $\rho$ and $\theta$, we need to make use of the inequality $0 \leq z \leq 1 - x$. Plug $(x, y, z) = \Phi(\rho, \theta)$ into the inequality we get

$$0 \leq \frac{\sqrt{3}}{2} \rho \leq 1 - \frac{1}{2} \rho \cos \theta \quad \Rightarrow \quad 0 \leq \rho \leq \frac{2}{\sqrt{3} + \cos \theta}.$$

By drawing the figure of $Q$, or observation (the denominator of the right hand side is always positive, we know $\theta \in [0, 2\pi]$. Hence

$$D = \left\{ (\rho, \theta) \mid \theta \in [0, 2\pi], 0 \leq \rho \leq \frac{2}{\sqrt{3} + \cos \theta} \right\}.$$

(b) (5 points) Find a parametrization of the ellipse $\partial Q$.

**Solution:** Usually, if a surface is represented by inequalities then its boundary is given by the corresponding equalities. If $z = 0$, we get the origin which is not a curve. The equation of the boundary ellipse $\partial Q$ is $x^2 + y^2 = \frac{1}{3}z^2$ and $z = 1 - x$.

We do not have to solve the system of equations. Actually we can simply plug the parametrization $(x, y, z) = \Phi(\rho, \theta)$ as provided into $z = 1 - x$ and we get

$$\frac{\sqrt{3}}{2} \rho = 1 - \frac{1}{2} \rho \cos \theta \quad \Rightarrow \quad \rho = \rho(\theta) = \frac{2}{\sqrt{3} + \cos \theta}.$$

Then by plugging this identity into $\Phi(\rho, \theta)$ again, we get the parametrization $c : [0, 2\pi] \to \partial Q$,

$$c(\theta) = \Phi(\rho(\theta), \theta) = \left( \frac{\cos \theta}{\sqrt{3} + \cos \theta}, \frac{\sin \theta}{\sqrt{3} + \cos \theta}, \frac{\sqrt{3}}{\sqrt{3} + \cos \theta} \right).$$

This is just one of many different parametrizations.

(c) (5 points) Use the Stokes’ theorem to evaluate the integral $\iint_Q (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$.

**Solution:** Use the Stokes’ theorem,

$$\iint_Q (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = - \int_{\partial Q} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = - \int_c \mathbf{F} \cdot ds.$$

Note that $c$ has the opposite orientation as the derived orientation from that of $Q$. To see this, note that the projection of $c$ goes counterclockwise, while $Q$ is oriented downward. (Caution: The normal vector field $T_\rho \times T_\theta$ points upward, so $\Phi$ is an orientation reversing parametrization of $Q$!)
We have
\[ F(c(\theta)) = \left( \frac{\sqrt{3} - 2 \cos \theta}{\sqrt{3} + \cos \theta}, 0, \frac{-2 \cos \theta}{\sqrt{3} + \cos \theta} \right) \]
and
\[ c'(\theta) = \left( -\sqrt{3} \sin \theta, \frac{1 + \sqrt{3} \cos \theta}{(\sqrt{3} + \cos \theta)^2}, \frac{-\sqrt{3} \sin \theta}{(\sqrt{3} + \cos \theta)^2} \right). \]

(You do not have to compute \( y'(t) \).) Now compute the line integral
\[
\int_c F \cdot ds = \int_0^{2\pi} F(c(\theta)) \cdot c'(\theta) d\theta \\
= \int_0^{2\pi} \frac{\sqrt{3} - 2 \cos \theta}{\sqrt{3} + \cos \theta} \cdot (-\sqrt{3} \sin \theta) d\theta + \frac{-2 \cos \theta}{\sqrt{3} + \cos \theta} \cdot \frac{-\sqrt{3} \sin \theta}{(\sqrt{3} + \cos \theta)^2} d\theta \\
= \int_0^{2\pi} \frac{-3 \sin \theta}{(\sqrt{3} + \cos \theta)^2} d\theta \\
= \int_{\theta=0}^{2\pi} \frac{-3}{(\sqrt{3} + u)^2} du \quad (\text{Let } u = \cos \theta) \\
= \left[ \frac{1}{2 (\sqrt{3} + \cos^2 \theta)^2} \right]_{\theta=0}^{2\pi} = 0.
\]

A “late substitution” trick may help you compute the integral faster. Plug \( x, y, z \) in first, then simplify the integral. Note that \( z = 1 - x \) and \( dz = -dx \).

\[
\int_c F \cdot ds = \int_c (3z - 2, 0, -2x) \cdot (dx, dy, dz) \\
= \int_c (3z - 2) dx - 2x dz \\
= \int_c (-3z + 2 - 2x) dz \\
= \int_c -z dz \\
= -\frac{1}{2} z(\theta)^2 \bigg|_{\theta=0}^{2\pi} = 0
\]

Hence
\[
\int \int_Q (\nabla \times F) \cdot dS = - \int_c F \cdot ds = 0.
\]

8. (15 points) Find the integral of vector field \( F \) over the sphere \( x^2 + y^2 + z^2 = 1 \), where
\[ F(\mathbf{r}) = \frac{\mathbf{r} - \mathbf{p}}{|| \mathbf{r} - \mathbf{p} ||^3}, \]
and \( r = (x, y, z), \ p = (0, 0, c) \), with \( 0 \leq c \neq 1 \). You are not allowed to use the Gauss’ law directly. (Hint: distinguish between the cases \( 0 \leq c < 1 \) and \( c > 1 \).)

**Solution:** Do not be scared. This is a standard surface integral of vector fields.

Use the “late substitution” trick (not required) to simplify the integral. Since \( S = \{x^2 + y^2 + z^2 = 1\} \) is a sphere, the outward normal vector field \( n = r \).\( r \). Compute

\[
(r - p) \cdot r = (x, y, z - c) \cdot (x, y, z) = x^2 + y^2 + z^2 - cz = 1 - cz
\]

\[
\|r - p\|^2 = \|(x, y, z - c)\|^2 = x^2 + y^2 + z^2 - 2cz + c^2 = 1 + c^2 - 2cz
\]

Let \( I(c) \) denote the integral

\[
I(c) = \iiint_S F \cdot dS = \iiint_S \|r - p\|^2 dS = \iiint_S \|r - p\|^2 dS = \iiint_S \frac{1 - cz}{(1 + c^2 - 2cz)^{\frac{3}{2}}} dS.
\]

We are not lucky enough to get a constant function over the sphere, so a parametrization of \( S \) is necessary. My personal preference is the parametrization by \( \theta \) and \( z \), which come from the cylindrical coordinates, since in this case we have \( dS = d\theta dz \).

Write \( \Phi: [0, 2\pi] \times [-1, 1] \rightarrow S, \)

\[
(x, y, z) = \Phi(\theta, z) = \left(\sqrt{1 - z^2} \cos \theta, \sqrt{1 - z^2} \cos \theta, z\right).
\]

By computation

\[
\|T_\theta \times T_z\| = 1.
\]

Hence the surface integral

\[
I(c) = \int_{-1}^{1} \int_{0}^{2\pi} \frac{1 - cz}{(1 + c^2 - 2cz)^{\frac{3}{2}}} \ d\theta \ dz
\]

\[
= 2\pi \int_{-1}^{1} \frac{1 - cz}{(1 + c^2 - 2cz)^{\frac{3}{2}}} \ dz
\]

\[
= 2\pi \left\{ \int_{z=-1}^{1} \frac{1 - c^2 + u - \frac{1}{2}}{2u^{\frac{3}{2}}} \ du \right\} (\text{Let } u = 1 + c^2 - 2cz)
\]

\[
= 2\pi \left\{ \left. \left(\frac{1 - c^2}{4c} u^{-\frac{3}{2}} - \frac{1}{4c} u^{-\frac{1}{2}} \right) \right|_{z=-1}^{1} \right\}
\]

\[
= 2\pi \left(\frac{1 - c^2}{2c} \right) \left[ (1 + c^2 - 2cz)^{-\frac{1}{2}} - \frac{1}{2c} (1 + c^2 - 2cz)^{-\frac{3}{2}} \right]_{z=-1}^{1}
\]

\[
= 2\pi \left(\frac{1 - c^2}{2c} \right) \left[ 1 - c^{-1} - \frac{1}{2c} |1 - c| - \frac{1}{2c} |1 + c| \right) - 2\pi \left(\frac{1 - c^2}{2c} \right) \left[ 1 - c^{-1} - \frac{1}{2c} |1 + c| \right)
\]

\[
= \left\{ \begin{align*}
2\pi \left(\frac{1 + c}{2c} - \frac{1 - c}{2c} \right) - 2\pi \left(\frac{1 - c}{2c} - \frac{1 + c}{2c} \right), & \quad 0 < c < 1, \\
2\pi \left(\frac{1 - c}{2c} - \frac{1 + c}{2c} \right) - 2\pi \left(\frac{1 - c}{2c} - \frac{1 + c}{2c} \right), & \quad c > 1,
\end{align*} \right.
\]

\[
= \left\{ \begin{align*}
4\pi, & \quad 0 < c < 1, \\
0, & \quad c > 1.
\end{align*} \right.
\]
If $c = 0$, then

$$I(0) = \int \int_S 1 \, dS = \text{Area}(S) = 4\pi.$$ 

9. (10 points (bonus)) Let $R > r > 0$. A torus with the major radius $R$ and the minor radius $r$ is the surface $S$ defined by the equation

$$\left( R - \sqrt{x^2 + y^2} \right)^2 + z^2 = r^2.$$ 

A solid torus $T$ is the region enclosed by the surface $S$.

(a) (2 points (bonus)) Write a parametrization of $S$.

(b) (3 points (bonus)) Find the area of $S$. 
(c) (5 points (bonus)) Find the volume of $T$. 