Day: 5.1-5.2  - Introduce Xiudi Tang - office hours right after

Chapter 5 is review in theory. So, I'm going to try to add a few things.

5.2.2: Double integrals over rectangular regions

To start, let's review some 2OB.

What is \( \int_a^b f(x) \, dx \)?

By definition, it is the signed area of the graph of \( f(x) \), as measured by the limit of Riemann sums, i.e., by adding up thinner and thinner rectangles.

By Theorem, it is \( F(b) - F(a) \), i.e., the total change in the antiderivative from \( a \) to \( b \).

Why does this make sense? Let's say, \( p(t) \) is position of car at time \( t \), in km.

Then \( p'(t) = \text{speed} \), in \( \text{Km/hr} \), say. (\( t \) in hr)

So, consider rectangles! in above: horiz is \( t \), so \( \text{km} \); vert is speed, so \( \text{Km/hr} \),

area = width \times height, so "area" of rect is \( \text{hr} \times \text{Km/hr} = \text{Km} = \text{distance traveled} \)

So, our approximation of area under a curve gives total change in antiderivative position.

(For formal proof, take real analysis.)

An important point: Area is an important visualization tool, but the integral could represent anything, not to do with area!

So, also: An integral adds up infinitesimal pieces of the integrand, \( f(x) \, dx \).

for \( p'(t) \, dt \), these pieces were tiny changes in position.

Q1: Consider \( \int f(t) \, dt \), where \( f(t) \) is rate of global deforestation since 1900, in \( \text{Km}^2/\text{yr} \).

What is units for integral? What does the integral represent?

Integral adds up \( f(t) \, dt \Rightarrow \frac{\text{Km}^2}{\text{yr}} \cdot \text{yr} = \text{Km}^2 \) of forest deforested.

So integral is total deforestation 1900 to 2017.

Let's leave 2OB, go back to this class.

Let \( f : \mathbb{R}^2 \to \mathbb{R} \), i.e., \( f(x,y) \).

What is \( \iint_R f(x,y) \, dA \)?

Q2: Tell your neighbor, by definition, what it is/should be.

By definition, it is signed volume under graph of \( f(x,y) \) over the region \( R \), as measured by the limit of ("box") Riemann sums, i.e., by adding up thinner and thinner boxes.

(open up CDF. Show ex. Thinner boxes, choose height same how. Cool, but we won't directly use.) \( dA \) is area of base of box.

Great! Define. And volume is a good mental model. But 1. how do we calculate it? 2. What else can it mean?
Rectangular boxes are convenient for calculations, at least.

But: (Double) Integrals just add up infinitesimal quantities, i.e., the integrand!

Q3: Let \( r(x,y) \) be the rate of rainfall in cm/hr at the location \((x,y)\), at one particular time.

What are the units of \( \iint_{\text{CA}} r(x,y) \, dA \), and what does it represent?

\[
\iint_{\text{CA}} r(x,y) \, dA = \text{volume} \quad \text{cm}^3/\text{hr} \cdot \text{km}^2 = 10000 \text{ m}^3/\text{hr} = \text{rate of water coming from Sky to all of CA.}
\]

Don't get stuck in trap of thinking of double integrals only as volume!

How to calculate? Rectangular boxes not convenient.

Cavalieri's principle: The volume of a shape can be found by finding the volumes of thin, parallel slices, and adding them up. (Note: not formal statement from book, but idea. When you read a formal def'n, you should always try to rewrite it formally.)

Show: Stack of pennies. Explain finding total volume.

How does this help us? To find volume, take slices of shape, then add up.

\[
\int_{x=-1}^{x=1} A(x) \, dx \quad \text{area of slice} \quad \int_{x=-1}^{x=1} \text{volume of slice.}
\]

Show CDF. Slice is single integral, i.e.,

\[
A(x) = \int_{y=-?}^{y=?} f(x,y) \, dy
\]

So: If rectangle region \( R = [a,b] \times [c,d] \)

\[
\iint_{R} f(x,y) \, dA = \int_{x=a}^{x=b} \left( \int_{y=c}^{y=d} f(x,y) \, dy \right) \, dx
\]

Q4: Calculate

\[
\iint_{[1,2] \times [0,1]} 6xy^2 \, dA = \int_{0}^{2} \left[ \int_{1}^{2} 6xy^2 \, dx \right] \, dy = \int_{0}^{2} \left[ 3x^2y^2 \right]_{y=1}^{y=2} \, dy = \int_{0}^{2} 2x \, dx = 3.
\]

Could have sliced other way: vol of slice in other way

\[
\iint_{R} f(x,y) \, dA = \int_{x=0}^{x=b} \left( \int_{y=c}^{y=d} A(y) \, dy \right) \, dx = \int_{0}^{b} \left[ \int_{x=a}^{x=c} f(x,y) \, dx \right] \, dy.
\]

Some answer: \( f \times \text{rect!} \)

Fubini's Theorem: (Not a def'n!)

\[
\iint_{R} f(x,y) \, dA = \int_{a}^{b} \int_{c}^{d} f(x,y) \, dx \, dy = \int_{c}^{d} \int_{a}^{b} f(x,y) \, dy \, dx = \text{a calculation that gives volume}. \quad \text{Sometimes see}
\]

This is "Fubini's theorem," though no one calls it that.

Continuity is restrictive.

Also true if \( f \) is continuous! i.e., 1. repeated integrals give volume.

2. Can change order of integration.

(See book for details, but point is: be careful you fulfill conditions!)

(measured in kg/m.

If \( f \) is the inner integral exists for all mild values of outer variable.

(see book for details, but point is: be careful you fulfill conditions!)
Day 2 5.3 Non-rectangular regions.

So, how we can do integration over rectangular regions.
But often want to integrate over non-rectangular regions.
For ex, last time, we did \( \int f(x,y) \, dA \), rainfall over all of \( \text{CA} \). Unlike \( \text{CO} \), \( \text{CA} \) is not a rect!

Idea is simple: pretend it’s a rectangle.

Suppose we want to integrate \( f(x,y) \) over a disc, \( D \)

Instead, integrate a modified function: \( f^*(x,y) = \begin{cases} f(x,y) & \text{on } D \\ 0 & \text{elsewhere} \end{cases} \)

Then integrate that on a rectangle.

It’s discontinuous, but that’s fine. Careful results from last section said it’s fine.

But, to be honest, this is just a technical trick for proving it works. Actually doing it is obvious.

I mostly notice this bc I’ve come across similar tricks a few times.

How do we calculate? Slices again! (slightly different than book, same punchline.)

Volume = \( \int_a^b A(x) \, dx \) show CDF again. Same as before, just not all the same size.

Q1: Suppose integrating \( f(x,y) = \sqrt{1-x^2-y^2} \) (unit sphere) over unit circle \( x^2+y^2 \leq 1 \), like in that picture. Using Cavalieri’s principle, know \( \int_D f(x,y) \, dA = \int_a^b A(x) \, dx \).

What are \( a, b \)?

Integrating over that, \( x \) goes from \(-1\) to \( 1 \), not function. Why? Cuz adding up volumes of slices in that direction. The volumes don’t care what shape it’s.

Q2: Same. What is the \( A(x) \)?

Need to integrate function to find area. Show CDF’s graph again.

Size depends on \( x \), i.e., on which slice. Left end is bottom on other one.

So, integrating from bottom to top of \( x^2+y^2 = \), \( y = \pm \sqrt{1-x^2} \), so

\( A(x) = \int \sqrt{1-x^2-y^2} \, dy \) — notice, the \( y \) integrates away, so only a function of \( x \).

So volume of (half) sphere is \( \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{2} \, dy \, dx \).

Could have sliced other way. Q3: Which one is other way? (one is just switched).

\( \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \, dx \, dy \).

\( \int_{-1}^1 \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \, dx \, dy \) is obviously wrong for one important reason.

Integral gives a function of \( x \), not a number!
The book goes on about x-simple, y-simple regions. It's silly.
There's very little special about their "elementary regions."

How to do it? (For general, "reasonable" shapes.)

- Draw your region(s).
- Figure out how you'll slice it.
- Figure out integral(s) you'll have to do, for outside and inside.

Ex.

Vertical slices easiest. If horizontal, we'd have multiple integrals.

\[
\int_a^b \int_{\phi(x)}^{\delta_b(x)} f(x,y) \, dy \, dx
\]

So, \( \int_a^b \frac{\phi(x)}{\text{area of slice}} \, dx \).

If did horizontally, the "area of slice" would have to be multiple integrals.

Ex:

Annulus, symmetric, so doesn't matter which way (though, may be easier to integrate, depending on \( f(x) \)).

Q4: \( \int_a^b A(x) \, dx \). What is \( a \), \( b \)? -2 to 2

But \( A \) is complicated! I'd break into 4 integrals:

Part 1: \( \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y) \, dy \, dx \)

Part 2: \( \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{f(x,y)}{y-x^2} \, dy \, dx \)

No need to worry about elementary. Just break into "easy" pieces.
Day 3: 5.4 Changing order of integration.

You already know how to do this. If you have a region, you can slice either way.

Q1: If \( R = \text{region b/w } y = 1 \) and \( y = x^2 \), we could integrate \( f \) over \( R \) by \( \int_{-1}^{1} \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y) \, dx \, dy \).

Which of these is equivalent?

\[ \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y) \, dx \, dy. \]

So, overall, have \( \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y) \, dx \, dy. \)

So, now, the only trick is to take a double integral, and figure out the shape!

\[ \int_0^1 \int_{e^{-1}x+1}^{e^x} f(y) \, dy \, dx \]

\[ \text{So, outer is } dx, \text{ so stretches from } x = 0 \text{ to } x = 1. \]

So, how to switch orders? Slice other way!

\[ \int_R f \, dA = \int_{y=1}^{e} A(y) \, dy = \int_{y=1}^{e} \int_{x=\frac{y-1}{e-1}}^{1} f \, dx \, dy. \]

\[ \text{So, } \int_0^1 \int_{e^{-1}x+1}^{e^x} f(y) \, dy \, dx = \int_{y=1}^{e} \int_{x=\frac{y-1}{e-1}}^{1} f \, dx \, dy. \]

Why would you do this? 1. You feel like it.

2. Sometimes greatly simplifies integral.

\[ \text{ex: } \int_0^2 \int_y^2 e^x \, dx \, dy. \text{ This is impossible! No antideriv of } e^{x^2}! \]

Q2: Which is equivalent?

\[ x \text{ goes from } y \text{ to } 2 \]

\[ y \text{ goes from } 0 \text{ to } 2 \]

So, other way is \( \int_0^2 \int_0^{e^{x+y}} f(x,y) \, dy \, dx \)

What did this by us? \( \int_0^2 \left[ ye^{x^2} \right]_0^x \, dx = \int_0^2 ye^{x^2} \, dx = \frac{1}{2} (e^2 - e^0) \approx 26.8. \)
So, switching order is sometimes difference b/w possible or impossible.

Q3: Switch order of \[ \int \int_{-y}^{y} f(x,y) \, dx \, dy. \]

- X from \(-y\) to \(y\)
- Y from 0 to 1

Cut other way, get two kinds:

\[ \int_{-1}^{0} \int_{-x}^{x} f(y) \, dy \, dx + \int_{0}^{1} \int_{0}^{x} f(y) \, dy \, dx \]

or \[ \int_{-1}^{1} \int_{-x}^{x} f(y) \, dy \, dx \]
(determine which is easier)

Now for some "obvious" things.

From ZOB: How to find average velocity on a trip? Well, find total distance/total time. 

If v(t) is velocity, average velocity = \[ \frac{\int_{a}^{b} v(t) \, dt}{b-a} \]

\[ \int_{a}^{b} v(t) \, dt \] is area, \( b-a \) is width, so average is average height.

So, what about multiple integrals?

\[ \iint_{CA} f(x,y) \, dA = \text{total rainfall rate in CA, in m}^3/\text{s}. \]

\[ \frac{1}{\text{area}(CA)} \int_{CA} r(x,y) \, dA = \text{average rainfall rate over CA in m}^3/\text{s}, \text{i.e., original units}. \]

Average of a function on a region \( R \) is \[ \frac{1}{\text{area}(R)} \iint_{R} f(x,y) \, dA. \]

Rules: (from book, reworded) The average of a function is between its max value and min value.

i.e., if \( m \leq f < M \), then \[ m \cdot \text{area}(R) \leq \iint_{R} f \, dA \leq M \cdot \text{area}(R). \]

Mean value thm: A continuous function achieves its mean.

i.e., For cont. \( f \), there is a point \((x_0, y_0)\) s.t.

\[ \frac{1}{\text{area}(R)} \iint_{R} f \, dA = f(x_0, y_0). \]
Day 4: 5.5 Triple integrals

Double integrals: \( \iint f \, dA \) had 2 main (related) interpretations.

1. volume: \( f \, dA \) represents a box, added up. Also is def'n.

2. Adding up bits of \( f \, dA \) could be anything.

Triple integrals: \( \iiint f \, dV \) :

By def'n: four dim volume as defined by a Riemann sum of \( f \cdot \text{tiny 3-d boxes} \) is, 4-d boxes.

By Interpretation: Adding up bits of \( f \, dV \) could be anything.

Q1: Let \( p(x,y,z) \) be the density of a ball at pt \((x,y,z)\) (in cm\(^3\)). What is units of \( \iiint_B p \, dV \)? What does it mean?

\( f + dV \Rightarrow \frac{g}{\text{cm}^3} \cdot \text{cm}^2 = g, \) total mass.

This is a very common interpretation: The function is a density of something, (mass, energy, electric field strength) and you use a triple integral to add it up. Also, often used to get averages:

Q2: Let \( T(x,y,z) \) be the temperature of a ball \( B \) at pt \((x,y,z)\) (in \(^\circ\)C). What is Unit of \( \frac{1}{\text{vol}(B)} \iiint_B T \, dV \)? What does it mean.

\( T \, dV \Rightarrow \text{\( \circ\)C} \cdot \text{cm}^2 \)? Weird units - some kind of heat content. Sometimes integral itself is hard to interpret, outside of average. Fortunately, \( \text{vol}(B) \Rightarrow \text{cm}^3 \), so overall have \( \text{\( \circ\)C} \). Makes sense: average should have same unit as input func.

So: 4-d volume is usually not best way to think about it: add up integral bits of something

How to calculate? Not surprisingly, repeated integrals.

\[
\iiint \text{Box} = \int_a^b \int_c^d \int_p^q f \, dx \, dy \, dz
\]

To find over all, adding up, add up on slices, then add those slices together.

Slice = \( \int_c^d \int_a^b f \, dx \, dy \)

Like 2-d, but we have another dimension of slices to deal with.
Fubini's Thm still holds: If \( f \) is "integrable," then \( \iiint_R f \, dv \) is equal to the iterated integral and you can integrate in any order.

So, rectangles are not hard to set up.

**Q3:** For \( R = [0,1] \times [-1,0] \times [-1,1] \), which of these equals \( \iiint_R f \, dv \) (if \( f \) is integrable).

\[
\int_{-1}^{1} \int_{-1}^{0} \int_{0}^{1} f \, dy \, dz \, dx
\]

What about more complicated regions? Do slices.

This is hard mostly thanks to visualizing and drawing. Actually doing the calculation is either impossible, or incredibly tedious.

**Ex:** Integrate \( f \) over the region bounded by the cylinder \( x^2 + y^2 \leq 1 \), the plane \( z = 0 \) and the plane \( x+y+z = 3 \).

Which way to slice? Horizotnal slices seem bad cuz down low and up high are diff. (Sketch, redraw).

Better: that kind of slice, perhaps. \( y \) slices start at \( y = -1 \), end at \( y = 1 \)

\[
\int_{-1}^{1} f \, dy
\]

\[
\int_{-1-\sqrt{y^2-1}}^{1-\sqrt{y^2-1}} f \, dx
\]

So \( \iiint_R f \, dv = \int_{-1}^{1} \int_{-1-\sqrt{y^2-1}}^{1-\sqrt{y^2-1}} f \, dx \, dy \).

Could we slice other way? Sure, but it'd be more difficult.

**Q4:** For \( R = \text{unit sphere } \left( x^2 + y^2 + z^2 \leq 1 \right) \), which is \( \iiint_R f \, dv \)?

Horiz slices (same as my choice). \( z \) slices go from \(-1\) to \(1\).

\[
\int_{-1}^{1} \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-z^2-y^2}}^{\sqrt{1-z^2-y^2}} f \, dy \, dx \, dz
\]

So \( \iiint_R f \, dv = \int_{-1}^{1} \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-z^2-y^2}}^{\sqrt{1-z^2-y^2}} f \, dy \, dx \, dz \).
Day 5: 6.1
Motivation: We saw last time that integration over a ball is ugly:
\[
\iiint_{B} f \, dV = \int_{-1}^{1} \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} f \, dx \, dy \, dz.
\]
Even if \( f \) were super simple, like 1, so we were just calculating the volume, this integral looks pretty intimidating.
But it's a ball! It should be easy! It's just a simple shape!
Over the next week we'll discuss how to make these much easier.
But to do that, we need to talk about how to turn rectangles into other shapes!
Because rectangles are easy to integrate over.

6.1: Let's look at maps \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \). So two inputs, 2 outputs.
We could think of this as vector valued functions put in a point, get out the vector at that point.
Instead, we want to think of this as a transformation of one region into another.
Or, as another way, think of it as coordinates. I'll get there.

\[
\begin{array}{ccc}
D^2 & \xrightarrow{T} & D \\
\end{array}
\]

\( T \) deforms or transforms \( D^2 \) into \( D \). (show this pt to that, or deforming.
For this class, \( T \) should be \( \text{C}^1 \), i.e., it has continuous derivatives.
(we'll need derivs. later for integrals).

Most basic kind is linear map \( T \).
Recall: \( T \) is linear if \( T(\alpha \mathbf{u} + \beta \mathbf{v}) = \alpha T(\mathbf{u}) + \beta T(\mathbf{v}) \).

Thm: Every linear transformation \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) can be expressed as matrix multiplication. Thus, for instance, parallelograms always get sent to parallelograms.
Q1: Which of these is linear? Only \( T(x,y) = (3x+7y, 2x-7y) \).

ex: \( T(x,y) = (3x+7y, 2x-7y) \)
Coordinates: When looking at \( \mathbb{D} \), we could use the standard coor
ds.

There's nothing inherently wrong with this, but a tilted paralle
\( \text{gram is not as nice as a rectangle, eh?} \)

Instead of using standard coor
ds, use inputs to \( T \) as coordinates.

So \( (1,0) \) \( \text{sent point at} \) \( (3,2) \), just like new
\( \text{name.} \)

\( \text{etc. Now, D is kind of like a rectangle, to visualize it!} \)

\( \text{Hint: in 6.2, we'll use this so we can make} \)
\( \text{many weird integrals just integrate "over a rectangle" instead of a funny shape.} \)

This worked well for this transformation. But you can't just use any transformation
as coordinates.

**Q2:** The map \( T: \mathbb{R}^2 \to \mathbb{R}^2 \); \( x,y \geq 0 \) given by \( T(x,y) = (x^2, y^2) \), is a \( C^1 \) map.

Explain to your neighbor why \( T \) does not provide good coordinates for the 1st quadrant.

\( \text{Why not? Well, let's look at (4,1) in standard coor
ds. We could think of that as (2,2) in the new coor
ds defined by T.} \)
\( \text{Or maybe it should be (2,-2), or (-2, 2), or (-2,-2)?} \)
\( \text{Uh... more than one possible name! Bad.} \)
\( \text{Show w/ handkerchief what the transformation was}. \)

**Def:** A function/map/transformation is **one-to-one** (injective) if no two inputs
go to the same output. i.e., if \( T(\vec{p}) = T(\vec{q}) \), then \( \vec{p} = \vec{q} \).

Another important thing is that \( T(D^\bullet) \) completely covers \( D \) so that every
point in \( D \) can be expressed by smth in \( D^\bullet \). Otherwise we only have
coordinates for part of \( D \).

**Def:** \( T \) is **onto** (surjective) \( D \) if every point in \( D \) is a valid output
of \( T \), i.e., if \( \vec{q} \in D \), then there exists \( \vec{p} \in D^\bullet \) s.t. \( T(\vec{p}) = \vec{q} \).

**Def:** \( T \) is **bijective** if it is one-to-one and onto.

**Principle:** A map \( T: D^\bullet \to D \) makes good coordinates if and only if it is
continuously differentiable \( (C^1) \) and is bijective, though bijectivity is... interesting
for polar, etc. coords.

On Monday, we'll discuss some very important examples. (1.4)

If time (hah!) explain \( \int \int \int \text{div} = \int \int \text{height dx dy dz} \) as slices.

\( \text{volume.} \)
Day 6: 1.4 Polar/cylindrical/spherical coordinates.

Rectangular coordinates work great... for boxes. But many, many things are not boxes. In many cases, other coordinates more easily describe a situation. For instance, circular things:

In rectangular coords, a circle is described by $x^2 + y^2 = 1$, or $y = \pm \sqrt{1 - x^2}$.
Polar coords are better! Recall: polar coords: $(r, \theta)$.

$$x = r \cos \theta \quad y = r \sin \theta$$

Then a circle is just $r = 1$.

Indeed using ideas from Friday, we can think of a circle as a rectangle instead!

$T(r, \theta) = (r \cos \theta, r \sin \theta)$ is the transformation for polar coods.

**Q1:** Which of these domains is $T$ bijective as a map $\mathbb{R} \to \mathbb{B}_1$, the unit ball? $(\mathbb{B}_1 = \{(x,y); x^2 + y^2 \leq 1\})$.

None, actually! Problems: central point is for any angle, so either get it many times or none. Also, $2\pi$ is equivalent to $0$, so angles overlap too.

Usually have $(r, \theta) \mapsto (x, y)$

$[0, \text{Radius}] \times [0, 2\pi] \to \text{Unit ball w/ center point.}$

This is bijective, $C'$, so good coordinates. We're missing the center point, but that's okay. Most of the time we can safely ignore single points.

Gives new names to each point. What's nice is that the ball is now a coordinate rectangle in these coords.

Easier to work w/ integrate.

When are polar coordinates useful? Usually when things are circular, i.e., symmetric around a central point.

**Cylindrical coordinates** - similar idea, just add a $z$-coord.

So, instead of $T(r, \theta) = (r \cos \theta, r \sin \theta)$, we have $T(r, \theta, z) = (r \cos \theta, r \sin \theta, z)$.

$x = r \cos \theta, y = r \sin \theta, z = z$.

$(x, y, z) = (r \cos \theta, r \sin \theta, z)$ polar for $x, y$, $z$ stays same.

**Q2:** If I want to cylindrical coords $T: \mathbb{R} \to \mathbb{C}$ be good coods, where $\mathbb{C}$ is the unit cylinder $x^2 + y^2 \leq 1, 0z \leq 1$.

Which of these is the best choice for $T$? Explain to your neighbor.

Why it isn't a perfect choice.

We use $R = [0, 1] \times [0, 2\pi] \times [0, 1]$.

We miss central line ($r = 0$) but that's not a huge deal most of the time.
Gives new names, and now, cylinder is a coordinate box, so it'll be easier to integrate over.

When would you use cylindrical coords? It suggests symmetry around the z-axis.
So, used for things that are symmetric around a line, like electric field around a wire, for instance.

Spherical coords: Another way to generalize polar coords.

\[
(x, y, z) = (p, \theta, \phi)
\]

\(\theta\) is same as polar coords. Angle in \((x, y)\) plane, as measured from \(x\)-axis.

We need one more coord. Use \(\phi\), angle from north pole.

Look at that triangle.

\[
\begin{align*}
A &= \sqrt{p^2 + z^2} \\
\sin \phi &= \frac{z}{A} = \frac{z}{\sqrt{p^2 + z^2}} \\
\cos \phi &= \frac{p}{A} = \frac{p}{\sqrt{p^2 + z^2}}
\end{align*}
\]

So, since \(x = r \cos \theta \), \(y = r \sin \theta\) as before, then

\[
\begin{align*}
x &= p \cos \theta \sin \phi, \\
y &= p \sin \theta \sin \phi, \\
z &= p \cos \phi
\end{align*}
\]

\(r\), in terms of a transformation,

\[
T(p, \theta, \phi) = (p \cos \theta \sin \phi, p \sin \theta \sin \phi, p \cos \phi).
\]

Warning: physicists are crazy and switch \(\theta, \phi\) in spherical.

Q3: If we want \(T: D^* \to B\) to be good coords for \(B = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}\),
what should \(D^*\) be?

\((p, \theta, \phi) \Rightarrow [0, 1] \times [0, 2\pi) \times (0, \pi]\). This gets everything but the \(z\)-axis.
Again, not perfect, but good enough.

Sometimes useful to allow \(p < 0\) or \(\theta \geq 2\pi\), or smaller, but usually just use the north pole to south pole.

Again, note: The ball is now a coordinate box!

What is it good for? For spherically symmetric things, often, like gravity around a star. The angle doesn't really matter, just distance to star.

Homework will give lots of chances to think about these.
Day 7: 6.2 Change of variables.

Let's talk about change of variables in 1-dim. It's gonna be weird. 

\[ \int_0^4 e^{\sqrt{x}} \, dx \]

\[ = \int_0^2 e^{u^2} \cdot 2u \, du \]

So, actually kind of adding up some rectangles over some interval!

\[ u \text{ just are new coords for domain.} \]

\[ T: [0,2] \rightarrow [0,4] \quad T(u) = u^2 \]

\[ u \rightarrow x \]

\[ 0 \rightarrow 0 \quad 2 \rightarrow 4 \]

\[ \text{gives coords for [0,4]!} \]

Notice \( dx = \frac{dT}{du} \, du \). Why? well, symbol wise \( T \) is \( x \), so reasonable.

Better: \( dx \) represents a bit of width in \( x \), say think of as \( 1 \).

\[ T(x) = \begin{cases} \sqrt{x} & x \in [0,2] \\ x^2 & x \in [2,4] \end{cases} \]

\[ 0 \rightarrow 0 \quad 2 \rightarrow 2 \quad 4 \rightarrow 4 \]

\[ \text{du bit of width in u. But u in u is different} \]

\[ \text{du amounts in x.} \]

\[ \frac{dT}{du} \text{controls how much x is changing when u is changing, so control relative widths in x's vs u's.} \]

If we want to have equivalent integrals, we need same height \( x \) same width.

\[ dx \text{ is "true" width, so fine.} \]

\[ du \text{ is not true width} \Rightarrow \text{it's coordinate width. So,} \]

\[ \text{multiply by} \quad \frac{dT}{du} = 2u \quad \text{to get "true" width} \quad 2udu \quad (= dx) \]

\[ \int \int f \, dA \]

You may have noticed I always use \( dA \) except in iterated integrals. That's on purpose.

\( dA \) is a thing, not just a place holder. It means a tiny bit of area, essentially.

Now, the standard interpretation is \( dA = dx \, dy \).

(Math majors, technically, \( dA = dx \, dy \), wedge product, and \( dx \, dy \) is different, but we'll fudge.)

Let's look at polar coords. What is a "bit of area in polar?"

Not just \( dr \, d\theta \). That's a bit of coordinate area, but not true area.

\[ dA \text{ is infinitesimal, so while not quite rectangle, can treat it like it is.} \]

\[ \text{So:} \quad dA = r \, dr \, d\theta \quad \text{(again, really \( r \, dr \, d\theta \))} \]

Thus, if we want to integrate, can use \( dx \, dy \) with rect coords, or \( r \, dr \, d\theta \) with polar coords.

Q2: Which of these gives area of circle?
Circle is \( 0 < r \leq R \), \( 0 \leq \theta \leq 2\pi \), so
\[
\text{area} = \int_0^R \int_0^{2\pi} r \ dr \ d\theta = \int_0^R \frac{r^2}{2} \ dr \int_0^{2\pi} \ d\theta = \int_0^R \frac{r^2}{2} \ d\theta \ = \pi \cdot r \cdot r^2 \ 
\]

Let's look at this a different way. What we did was look at an infinitesimal coord rect, and figured out its true area directly. Harder to generalize. Let's do it another way.

\[
\text{How does } T \text{ (coord transform) transform inf. coord rect? Since infinitesimal, can think of everything as linear. Want to find area of shape on right, so need how dr, d\theta transform.}
\]

Think of sides as vectors. dr side is \((dr, 0)\), other is \((0, d\theta)\). \((dr, 0)\) becomes one side of parallelogram. What is that vector? Well, how much does it point in x direction?
\[
\frac{dx}{dr}, \frac{dy}{dr} \ 	ext{is how much } x \text{ changes when } r \text{ does, dr is how much } r \text{ changes.}
\]
For y direction, \(\frac{dy}{dr}, \frac{dy}{d\theta}\), so \((dr, 0) \xrightarrow{dr} (\frac{dx}{dr} \ dr, \frac{dy}{dr} \ d\theta)\)
Similarly,
\( (0, d\theta) \xrightarrow{d\theta} (\frac{dx}{d\theta} \ d\theta, \frac{dy}{d\theta} \ d\theta)\)

What is area of parallelogram? From \(2\times2\) C, determinant of matrix with those:
\[
\text{area} = \begin{vmatrix} \frac{dx}{dr} & \frac{dy}{dr} \\ \frac{dx}{d\theta} & \frac{dy}{d\theta} \end{vmatrix} dr \ d\theta = \begin{vmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{vmatrix} \ dr \ d\theta = \sqrt{r^2 \ dr \ d\theta}, \text{ as before.}
\]

(Math majors have a bit of fudge, really use wedges here too.)

What's nice is this argument works for other coords as well!

**Thm:** If \( u, v \) are new coords for \( O \), then \( dA = \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix} du \ dv = |JT| du \ dv \), where \( T(u,v) = (x,y) \),

Then, to integrate, just need to write function and limits of int. in terms of new coords,

**Q3:** Suppose \( T: Q_1 \rightarrow Q_2 \) is a map defining coords for the 1st quadrant \( Q_1 \), by \( T(u,v) = (u^2, v^2) \).

What is \( dA \) in terms of new coords?
\[
dA = \begin{vmatrix} \frac{dx}{du} & \frac{dy}{du} \\ \frac{dx}{dv} & \frac{dy}{dv} \end{vmatrix} du \ dv = \begin{vmatrix} 2u & 0 \\ 0 & 2v \end{vmatrix} du \ dv = 4uv \ du \ dv.
\]
Day 8: Change of variables part 2

Thm: If \( u, v \) are good cords for \( D \), then \( dA = \left| \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array} \right| \ dudv \) or \( \sqrt{\text{det} \left( \frac{\partial T}{\partial u,v} \right)} \ dudv \), where \( T(u,v) = (x,y) \).

Then to integrate, just need to write function and limits of int. in terms of new cords.

Two notes: 1. \( \sqrt{\text{det} \left( \frac{\partial T}{\partial u,v} \right)} \) is called the Jacobian.

2. Actually want absolute value of determinant, since area should be positive.

Q1: Suppose \( T: Q_1 \to Q_2 \) is a map defining cords for the 1st quadrant \( Q_1 \) by \( T(u,v) = (u^2, v^2) \). What is \( dA \) in terms of new cords?

\[ dA = \left| \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array} \right| \ dudv = \left| \begin{array}{cc} 2u & 0 \\ 0 & 1 \end{array} \right| dudv = 4uv \ dudv \]

Q2: Which is equivalent to \( \int_1^2 \int_0^3 e^{xy} \ dx \ dy \)?

\[ \text{Region is } \begin{array}{c|c|c} x & y \\ \hline 0 & 0 & 1 \\ 1 & 1 & 2 \end{array} \]

Well, \( dA = dxdy \). If I use \( T(u,v) = (u^2, v^2) \), have \( dA = 4uv \ dudv \).

\[ \text{If } 0 \leq u \leq 1, 0 \leq v \leq \sqrt{u} \Rightarrow 0 \leq u \leq 1, \ 0 \leq v \leq \sqrt{u} \]

\[ \Rightarrow \int_0^1 \int_0^{\sqrt{u}} 4uv \ dudv \]

3-dim: Very similar. \( \iiint_R f \ dV \). \( dV \) is a bit of volume.

\[ dV = dx \ dy \ dz \] is one interpretation.

Cylindrical cords.

\( r \theta dr dz \) as before, for polar, area of base is \( r \theta dr \), so volume is \( r \theta dr dz \).

Thm: For cylindrical cords \( T(r, \theta, z) = (r \ \cos \theta, r \ \sin \theta, z) \), \( dV = r \theta dr dz \).

Q3: Which is equivalent to \( \iiint_R e^{x+y+z} \ dV \), where \( R \) is cylinder \( x^2+y^2 \leq 1, -z \leq z \leq 1 \)?

\[ \iiint_0^1 \int_0^{2\pi} \int_0^1 e^{r \theta} r \theta dr \ d\theta \ dz \]

Other way: Have \( T \) as above for cords.

\[ (0,0,0) \rightarrow (0,0,0) \quad \text{and} \quad (a,0,0) \rightarrow (a,0,0) \]

To find real volume, need area volume of their parallelepipeds. By Section 1.3, that's just \( \sqrt{\text{det} \left( \frac{\partial T}{\partial u,v} \right)} \) of the transformed coord vectors.

What does \( (dr,0,0) \) transform to? As before, \( \left( \frac{\partial x}{\partial r} dr, \frac{\partial y}{\partial r} dr, \frac{\partial z}{\partial r} dr \right) \)

So, as before, \( dV = \sqrt{\text{det} \left( \frac{\partial T}{\partial u,v} \right)} \ dr \ d\theta \ dz \). Can check, get \( r \) again.

Thm: If \( u, v, w \) are good cords for \( D \), then \( dA = \left| \begin{array}{ccc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{array} \right| \ dudvdw = \sqrt{\text{det} \left( \frac{\partial T}{\partial u,v,w} \right)} \ dudvdw \)

where \( T(u,v,w) = (x,y,z) \).
Q4: The function for spherical co-ords is $T(r, \theta, \phi) = (r \cos \theta \sin \phi, r \sin \theta \sin \phi, r \cos \phi)$

Find $dV$ in spherical co-ords. Set up matrix to find Jacobian.

Doing this geometrically is hard, so better to just use Thm:

$$
\begin{vmatrix}
\cos \theta \sin \phi & -r \sin \theta \sin \phi & r \cos \phi \\
\sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \\
0 & \cos \phi & -r \sin \phi
\end{vmatrix} = r^2 \sin \theta
$$

Takes a while to simplify. Uses $\sin^2 \phi + \cos^2 \phi = 1$ a bunch of times.

So: Thm: For spherical co-ords $dV = r^2 \sin \theta \, dr \, d\phi \, d\theta$

- Cylindrical $dV = r \, dr \, d\theta \, dz$
- Polar $dA = r \, dr \, d\theta$

In "real life," 95% of the time you use change of variables, it is because your problem, your function, and/or your domain are cylindrically or spherically symmetric, in which case it is natural to change co-ords.

In those cases, you just use the new $dA$, $dV$, and make function/limits in terms of new co-ords.

Occasionally, you may use other co-ords adapted to your domain/function/problem to simplify things, in which case you'd need to calculate $dV$, $dA$ yourself. HW has examples.

Q5: Suppose you want to know the mass of Earth's atmosphere. You might assume that the atmosphere is roughly spherically symmetric. If $p$ is the density of the atmosphere at any location is $\text{Kg/m}^3$, write an integral giving the mass of the atmosphere. (Rad of Earth $\approx 6371$ km, atmosphere essentially gone at 100km)

$$
\int_0^{6371} \int_0^{6471} \int_0^{\pi} p(r) \, dr \, d\phi \, d\theta
$$

To change units, our problem, domain, and function are all close to spherical symmetry, so reasonable.