Day 20: 4.4, 8.1

Ch 8 is like the whole point of this class. This is where we get the most exciting theorems. It's awe some!

Q1: You can use a line integral to calculate area of a contained shape.

True! Amazing. Seems bizarre that just badly is enough.

In fact, there are mechanical machines (planimeters) that will do the integral for you. Show video. (I just think they're cool.)

Anyway, what this means is that integration over boundary (line integral) can give you information about integration over bulk (area). That is the theme of Ch 8. We're going to talk about it a bunch of different ways.

From 20B: FTC: \[ \int_{a}^{b} f'(x) \, dx = f(b) - f(a) \]

Integration of derivative over bulk = "integral" of original over boundary.

or, \[ \int_{S} df = \int_{S} f \] (loosely)

Okay, to get to first example, let's first go back to 4.4 to talk about divergence.

First, "del" operator. This is more of a Physics thing to do, but it does come up.

\[ \nabla = \left( \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n} \right) \]

operators, so, take derivatives.

Def: Divergence of a vector field is \[ \text{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n} \]

\( \text{in 3D} \)

Q2: What is \( \text{div} \mathbf{F} \) if \( \mathbf{F} = (x, y) \)?

\[ \text{div} \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = 1 + 1 = 2. \]

Easy enough, but what does it mean? Why do we care?

Word in English just means spreading or separation, and mathematically, it's similar.

The divergence of \( \mathbf{V} \) represents how much something flowing by \( \mathbf{V} \) is expanding or contracting.
Ex: Consider $F(x,y) = (x, y)$, recall $\text{div} \ F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}$. If we flow the box along the vector field, it'll expand!

Pos $\text{div}$ means expanding. Neg means contracting.

How much expansion?

Notice, $F$ is velocity, so units are $\frac{m}{s}$. $\text{div} F = \frac{dF_1}{dx} + \frac{dF_2}{dy}$ implies units $\frac{1}{s}$. So $\iint_S \text{div} F \, dA \Rightarrow \frac{m^2}{s}$

So amount of expansion per sec is $\iint_S \text{div} F \, dA$.

If we thought of something like water, which doesn't expand or contract, $\text{div} F > 0$ must mean water is being created from nothing! And $\text{div} F < 0$ means being destroyed! (Incompressible/ conservation of mass is $\nabla \cdot F = 0$.)

Now for the exciting part.

$\iint_S \text{div} F \, dA$ is rate of creation of water in $S$.

But as we said, water can't be compressed, or expanded. So

If $\iint_S \text{div} F \, dA > 0$, it must be flowing out of the shape.

In fact, the flux through the surface must be exactly the same amount as $\text{div} F$ tells us is created or destroyed!

In other words, $\text{Thm} (\text{Div Thm})$:

$$\iint_S \text{div} F \, dA = \oint_S F \cdot \mathbf{n} \, ds$$

outward normal.

<table>
<thead>
<tr>
<th>$\text{total rate of}$</th>
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<tr>
<td>$\text{Creation/destro}</td>
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<td>$\text{y} (or \text{expansion/contraction})$</td>
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There's no where else to go but out!

We did not anywhere close to prove this. I'll probably talk briefly about proof next time.
Day 21: 8.1/4.4
had some fun during discussion.

Last time: Div Thm: If $F$ is $C^1$ vector field, $\hat{n}$ outward unit normal

$$\int_S \text{div} F \, dA = \int_{\partial S} F \cdot \hat{n} \, ds$$

(total expansion rate) (total flux across boundary)

These theorems of Ch 8 are generally hard-ish to prove. I am only going to talk about one proof, for this/Green's thm. We've given a heuristic argument,

So as $\int_S (\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}) \, dx \, dy = \int_{\partial S} (F_1 \frac{dx}{dt} - F_2 \frac{dy}{dt}) \, dt$

is of $x(t)$, for $ds$,

$\hat{n} = (\frac{dx}{dt}, -\frac{dy}{dt})$ norm

In simpler case, $F$ only points in $x$ direction. Thus $F = (F_1, 0)$, ie, $F_2 = 0$.
Then

$$\int_S \frac{dF_1}{dx} \, dx \, dy = \int_{\partial S} F_1 \frac{dy}{dt} \, dt$$

$\leftarrow$ Let's prove this.

pf: Let's look at a simple region $S$. It has (up to) 4 sides:

- curved left/right, and optional flat tops and bottoms.

Give them names $L, R, T, B$. $L, R$ are functions of $y$, $L[y], R[y]$.

So, $\int_S \frac{dF_1}{dx} \, dx \, dy = \int_T \int_B \frac{F_1}{dx} \, dx \, dy$

inner integral is just integral of deriv, so FTC gives

$$= \int_T \left[ F_1 (R(y), y) - F_1 (L(y), y) \right] \, dy$$

$\leftarrow$ Split up, $\int_T \int_B \frac{F_1}{R(y), y} \, dy = \int_T \int_B \frac{F_1}{L(y), y} \, dy$

This is really a line integral along the right side from Bot to Top.

ie, $\int_R \int_R \frac{F_1}{dx} \, dt$ $\leftarrow$ just a change of variables

$L \leftarrow$ changed sign, b/c I went $L$ from Bot to Top, not vice versa.

But, wait, we wanted $\int_S \frac{dF_1}{dt} \, dt = \int_L \int_B \frac{F_1}{dx} \, dt + \int_R \int_B \frac{F_1}{dx} \, dt + \int_R \int_R \frac{F_1}{dx} \, dt$ + $\int_T \int_B \frac{F_1}{dx} \, dt$.

What about $\int_B \int_T$?

Q1: Where do they come in?  

a. They cancel each other out.

b. They're both zero.

C. We made a mistake.
It turns out they're zero. \[ \int_B \oint F \frac{\partial y}{\partial t} \, dt \rightarrow 0, \text{ cuz flat!} \] so integral zero.

To summarize.

\[ \oint_S \frac{\partial F}{\partial x} \, dx \, dy = \int_B F_i(R(y), y) - F_i(L(y), y) \, dy \] by FTC.

\[ = \int_R F_i \frac{dy}{dt} \, dt + \int_L F_j \frac{dy}{dt} \, dt \] by interpreting as line integral, with \[ \nabla \times \mathbf{F} = (0, F_i) \]

\[ = \int_B F_i \frac{dy}{dt} \, dt \] since \[ \oint_B, \oint_L = 0. \]

To finish proof of Div. Thm, also prove \[ \oint_S \frac{\partial F}{\partial y} \, dA = \oint_S -\frac{\partial F}{\partial x} \, ds, \] and combine.

Book gives slightly different wording as Green’s Thm:

**Thm:** (Green’s) For C^1 functions P, Q (instead of F_i, F_j),

\[ \int_S (P, Q) \cdot ds = \iint_S (Q_x - P_y) \, dA, \] where ds is oriented so S is on its left.

Phew. That was the most technical thing we've done this term. Fortunately, I will not be doing anything else nearly so technical.

The next section, 8.2, is about a generalization of Green’s Thm to 3-d.

**Def:** For a vector field \( \mathbf{F} \), \( \text{curl} \ \mathbf{F} \) or \( \nabla \times \mathbf{F} \)

\[ \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & -x & 0 \end{vmatrix} = (0, 0, -2) \]

What does it mean? Next time: rotation: direction is axis of rotation, length is speed.

Looks like we’re behind. Okay. Covered more than “scheduled” in 2 days, but the exact schedule online will be fudged a bit.
Curl \( F = \nabla \times F = \left| \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & 0 \\ F_2 & F_3 & F_1 \end{array} \right| \) represents rotation/circulation.

\[ \text{ex: } F = (-y, x, 0), \quad \text{curl} \ F = (0, 0, 2) \quad \begin{array}{c} \uparrow \\ \downarrow \end{array} \] (note right hand rule)

\[ \text{ex: } F = (x, y, 0) \quad \begin{array}{c} \uparrow \\ \downarrow \end{array} \] (pure expansion) \quad \text{curl} \ F = (0, 0, 0) \quad \begin{array}{c} \uparrow \\ \downarrow \end{array} \] (up no rotation)

That's a good intuition, but you have to be careful. It's an infinitesimal bit of rotation.

\[ \text{ex: } F = \left( \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right), \quad \text{curl} \ F = (0, 0, 0) \quad \begin{array}{c} \uparrow \\ \downarrow \end{array} \] (wrong)

Curl tells you whether a ball would spin in place, not whether it would go around a great whirlpool.

**Curl** \( \mathbf{F} \): direction gives axis of rotation, length gives speed of rotation.

Let's look at this over a region, curl \( \mathbf{F} \) is like a tiny whirlpool. Let's zoom in near boundary. Since zoomed in, all about the same.

Note: Opposite direction, so cancel out: only thing left over is how much \( \mathbf{F} \) is flowing along \( dS \). All else canceled out!

So, heuristically, total infinitesimal rotation on surface = flow of \( \mathbf{F} \) along \( dS \).

If \( S \) is in \( xy \) plane, then the normal to \( S \) is \( (0, 0, 1) \), i.e., \( \mathbf{k} \).

So Thm: (Flat version of Stokes' Thm): For a surface \( S \) in \( xy \) plane, and \( \mathbf{F} \) a \( C^1 \) vector field,

\[ \iint_{S} (\text{curl} \mathbf{F} \cdot \mathbf{k}) \, dA = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} \] oriented so \( S \) is to your left!

It isn't immediately obvious, but if you look at formulas for everything, this is actually equivalent to Green's Thm.

Again, we thought of it as rotation, but it is actually a Thm you can prove, that has nothing to do with our interpretation.
Q2: Calculate the flow along the path \( r(t) = (17\cos(t), \sin(t)) \) of \( F(x,y) = (e^x, e^y) \).

\[
\text{Curl} F = \left| \begin{array}{ccc}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
 e^x & e^y & 0
\end{array} \right| = (0, 0, 0)
\]

So, by Stokes' Thm, \( \iint_S \text{curl} F \cdot d\mathbf{A} = 0 \).

It turns out that you can generalize this to non-flat surfaces in 3-d!

Like \( \text{curl} F \cdot \mathbf{n} \) was bit of circulation (per area) for surfaces in xy plane,

\( \text{curl} F \cdot \mathbf{n} \) gives the part of the circulation of \( F \) that's along the surface. (May not be all of it.)

So, can visualize same way. In middle, still cancel out. Along edges, still get total flow of \( F \) along boundary. So:

Theorem (Stokes' Thm): For any "nice" oriented surface \( S \) with "nice" boundary, \( \partial S \), and \( F \), then \( \iint_S \text{curl} F \cdot \mathbf{n} \, dA = \int_{\partial S} F \cdot d\mathbf{s} \) if you use the induced orientation on \( \partial S \).

"Nice" means it's only not-regular on a line/points. For instance, a cubes wouldn't be fine.

"Induced orientation" depends on the orientation of \( S \). Find it by: Pretend you are walking along the surface where "up" is given by the orientation of \( S \). Then the correct direction along \( \partial S \) is so that the surface \( S \) is to your left.

Q3: For a graph, it is often assumed that \( \mathbf{n} \) is the upward normal. Should you walk clockwise or counterclockwise around the boundary of a graph \( (x, y, f(x, y)) \) on the domain \( [0, 2] \times [0, 2] \)?

(As viewed in the xy plane).

Q4: For \( F(x, y) \) in the plane, when we think of \( \mathbf{F} \) as a 3-d VF \( (F_1, F_2, 0) \),

\( \text{curl} F \) always points in \( z \) direction.

True, can check with a calculation.

So \( \text{curl} F \cdot \mathbf{k} = \| \text{curl} F \| \) for \( \mathbf{k} \) unit normal to surface, i.e., \( (0, 0, 1) \).

So \( \text{curl} F \cdot \mathbf{k} \) gives the infinitesimal circulation of \( F \) (per area) at the point.
Stokes' Thm: For any "reasonable" oriented surface $S$, with "reasonable" boundary $\partial S$, if $F \in C^1$, then

$$\int_S \text{Curl} F \cdot \hat{n} \, dA = \int_{\partial S} F \cdot ds$$

(can also write $\int_S \text{Curl} F \cdot dS$)

So far, we've only done this on a surface in xy plane or a graph. But it works on any surface. Induced orientation is same. Walk on surface with "up" given by surface orientation. Then correct direction along boundary is so $F$ is to your left.

Q1: Let $S$ be the portion of the sphere with $y \geq 0$, which way is oriented? (Describing it by north pole, do you go pos x or neg x first?)

Draw picture. Correct: pos x direction.

We won't prove it, but the basic idea is:
1. Use change of variables to pretend you're working on a flat domain.
2. Use Green's Thm to show it works on that modified domain.

We could do some simple examples, but you already did some at discussion.

Basically, if one looks hard, try the other.

Usually, line integral easiest.

But there's something fascinating here: More than 1 surface can share a boundary!

Q2: T/F: For any $C^1$ vector field $F$, $\int_S \nabla \times F \cdot ds = \pm \int_{S_1} \nabla \times F \cdot ds = \pm \int_{S_2} \nabla \times F \cdot ds$.

True! By Stokes' Thm, both are equal to $\int_S F \cdot ds$!

With one caveat: orientation is important!

But first, notice similarities to $\int \nabla \times F \cdot ds = f(\Phi(b)) - f(\Phi(a))$, which was independent of which path, as long as the endpoints are the same.

That's super important!

Why? You can use a simpler surface to calculate the integral!

For instance, for this example, we could use the flat disc inside $dS$, and either integral is $\pm \int \nabla \times F \cdot ds$, which is probably much easier.

Disc (though line still might be best...)
Q3: Orientation matters. If we pick orientations for $S_1$, $S_2$, then which is correct? $+\text{ or } -$?

Remember, walk so is on left! get opposite orientations, cuz need opposite side to the left!

Another important question is "what is a boundary?" Fortunately, it turns out that any reasonable definition will work. Easiest: A boundary is where you can't continue walking along the surface. Can be parameterized by $(\theta, \phi)$ with the domain $[0, 2\pi] \times [0, \pi]$. Which is the bdy?

Trick question: It has no bdy. There is no direction you can walk to like on Earth, you can't walk off the edge.

Q4: A sphere normal formulas, on the boundary.

Q5: Stokes' Thm says $\oint \nabla \times \mathbf{F} \cdot d\mathbf{s} = \int_S \mathbf{F} \cdot d\mathbf{S}$. For $S$, a sphere, $\oint \nabla \times \mathbf{F} \cdot d\mathbf{s} = \int_S \mathbf{F} \cdot d\mathbf{S}$.

It's zero. Stokes' Thm still holds but integral over nothing is nothing!

So $\int_S \mathbf{F} \cdot d\mathbf{S} = 0$.

Now, I just want to mention a few words about one use of these. Maxwell's eqns. There's two forms of these eqns, a differential eqn version, and an integral version. Both are useful for different things. Stokes' Thm and relatives let you switch b/w them.

ex: $\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}$ magnetic field

$\mathbf{E}$ electric field

$\mathbf{H}$ $\oint_C \mathbf{E} \cdot d\mathbf{s} = -\frac{1}{c} \oint_S \mathbf{H} \cdot d\mathbf{S}$