Q1. Formula for $\int_S F \cdot d\mathbf{s}$ for $F = \nabla f$.

$$\int_S F \cdot d\mathbf{s} = f(\mathbf{g}(b)) - f(\mathbf{g}(a))$$

Why?

Intuitively: $\nabla f \cdot \mathbf{v}$ gives slope of $f$ in direction $\mathbf{v}$,
so $\nabla f \cdot d\mathbf{s}$ gives increase/decrease of $f$ along the path for a tiny step.
So $\int_S F \cdot d\mathbf{s}$ adds up to get total change in value of $f$. $= f(\mathbf{g}(b)) - f(\mathbf{g}(a))$

Calculation: Let $g(t) = f(\mathbf{g}(t))$. Then

$$\int_a^b g'(t) \, dt = g(b) - g(a).$$

But $g'(t) = (f(\mathbf{g}(t)))' = \nabla f(\mathbf{g}(t)) \cdot \mathbf{g}'(t)$, by chain rule. So equal.

Q2: Flow line satisfies which eqn? What is idea?

$\mathbf{x}'(t) = F(\mathbf{x}(t))$

i.e., velocity (LHS) = value of vector field where you are.

i.e., F provides the tangent vector along the path.

Q3. What is an orientation? What difference does it make?

(consistent) Choice of normal vector over my surface, say inward or outward.

For integrating SA or functions, it makes no difference, cuz $dA$ is a bit of area, which is always pos.

For flux, it changes the sign. Why? 1. changes focus of direction

2. $\iint_S F \cdot \hat{n} \, dA$, $dA$ stays same, but switching orientation switches sign of $\hat{n}$. 
Q4: What is $d\vec{s}$ intuitively? How to calculate?

$d\vec{s}$ represents a bit of displacement, i.e., change of position. It is given by $d\vec{s} = \gamma' \, dt$, where $\gamma'$ is the velocity of the path, and $dt$ is a bit of time, so velocity $\times$ time = displacement, as we wanted.

Q5: Explain why $dA = ||Tu \times Tv|| \, du \, dv$.

The bit of coordinate area becomes the bit of real area in the parallelogram defined by $Tu \, du$, $Tv \, dv$. The area of a parallelogram in $\mathbb{R}^2$ is $||Tu \times Tv|| = ||Tu \times Tv|| \, du \, dv$.

Q6: $\gamma = (\cos t, \sin t, t)$ on $t \in [0, 2\pi]$. $\rho(x,y,z) = 2z+3$ is density.

What is mass of cord?

$\gamma' = (-\sin t, \cos t, 1)$,

$$||\gamma'|| = \sqrt{(\cos^2 t + \sin^2 t + 1)} = \sqrt{2}$$

Mass

$$= \int_{0}^{2\pi} \rho(2z+3) \sqrt{2} \, dt = \sqrt{2} \int_{0}^{2\pi} (2z+3) \, dt$$

$$= \sqrt{2} \left[ z^2 + 3z \right]_0^{2\pi}$$

$$= \sqrt{2} \left[ 4\pi^2 + 6\pi \right]$$

Q7: Voltage $= \int \vec{E} \cdot d\vec{s}$.

$\vec{E} \cdot \gamma' = (\cos t, \sin t, t) \cdot (-\sin t, \cos t, 1) = t$

So

$$\int_{0}^{2\pi} t \, dt = \left[ \frac{t^2}{2} \right]_0^{2\pi} = \left[ 2\pi^2 \right]$$

Q8: $(\theta, \phi) = (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi)$

$(\gamma$ is a constant$)$

$T_\theta = (-\sin \theta \sin \phi, \cos \theta \sin \phi, 0)$

$T_\phi = (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \phi)$

So area

$$= \int_{0}^{2\pi} \int_{0}^{\pi} r^2 \sin \phi \, d\phi \, d\theta = 4\pi \cdot r^2$$
Q9: Surface area \( = 2\pi r^2 \).

\[
\text{Average} = \frac{1}{\text{area}} \int_0^{2\pi} \int_0^r z \, dA = \frac{1}{2\pi r^2} \int_0^{2\pi} \int_0^r \rho \cos \phi \, \rho^2 \sin \phi \, d\phi \, d\theta
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \int_0^r \cos \phi \, \sin \phi \, d\phi \, d\theta
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} \, d\theta = \frac{\pi}{2}
\]

Q10: \( T_u = (1,0,1) \quad T_v = (0,1,-3) \quad T_u \times T_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 1 \\ 0 & 1 & -3 \end{vmatrix} = (-1,3,1) \) \( \epsilon \) upward normal.

\[
\text{Flux} = \int_0^1 \int_0^2 (x,y,z) \cdot (-1,3,1) \, d\nu \, d\mu
\]

\[
= \int_0^1 \int_0^2 (u,v,u-3v+3)(-1,3,1) \, d\nu \, d\mu
\]

\[
= \int_0^1 \int_0^2 -u+3v+u-3v+3 \, d\nu \, d\mu
\]

\[
= \int_0^1 \int_0^2 3 \, d\nu \, d\mu = 3
\]