Homework 10 sol'ns

1. For correctness:

Let $R$ be the region in between $S_1$ and $S_2$.

Clearly, $\iiint_R \text{div } F \, dV = 0$. By the divergence Thm, this equals $\iint_{S_1} F \cdot \hat{n} \, dA$, where $\hat{n}$ is outward from $R$.

Thus $0 = \iint_{S_1} F \cdot \hat{n} \, dA + \iint_{S_2} F \cdot \hat{n} \, dA$

$\iint_{S_1} F \cdot \hat{n} \, dA = -\iint_{S_2} F \cdot \hat{n} \, dA$.

For the div. Thm to work, $\hat{n}$ on the left is outward pointing, while $\hat{n}$ on the right is inward pointing. Thus the outward fluxes for both are the same.

2. For any closed curve, there is a surface spanning it, like in my picture. (Proving this is hard---) By Stokes' Thm,

$\int \int_{\text{surface}} F \cdot d\vec{s} = \int \int_{\text{Surface}} \nabla \times F \cdot d\vec{S} = 0$ since $\text{curl } F = 0$.

3. The potential of a vector field $F$ is a function $f$ such that $F = \nabla f$. Such an $f$ exists if $F$ is irrotational, i.e., $\nabla \times F = 0$.

For completion:

8.3-16. a. on the unit circle, $x^2 + y^2 = 1$, so we just need to calculate $\int (-y, x) \cdot d\vec{s}$ which isn't too hard.

b. The Thm from class says $F$ is conservative if and only if the line integral around all closed paths is 0, so $F$ is not conservative.

c. It does not contradict $F$ is not C!
8.4.8a. \( F = (1, 1, 1) \). \( \text{Div} F = 0 \), so by \( \text{div} \text{Thm} \), flux is 0!

directly, you’d find that opposite sides cancel out.

16. \( \text{div} F = 3y^2 + 3x^2 + 3z^2 = 3r^2 \).

By \( \text{div} \text{Thm} \),
\[
\iint_S F \cdot dS = \iiint_B \text{div} F \, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 3r^2 \cdot r^2 \sin \phi \, dr \, d\phi \, d\theta
\]
\[
= \int_0^{2\pi} \int_0^{\pi/2} \frac{3}{2} \sin \phi \, d\phi \, d\theta
\]
\[
= \int_0^{2\pi} \frac{6}{5} \, d\theta = \frac{12}{5} \pi
\]

17: use \( \nabla (fF) = \nabla f \cdot F + f \nabla F \) and the \( \text{div} \text{Thm} \).

24. If it is tangent, then \( F \cdot \vec{n} = 0 \) on the surface \( S \), since \( F \) and \( \vec{n} \) are orthogonal. By the \( \text{div} \text{Thm} \),
\[
\iiint_W d\omega F \, dV = \iint_S F \cdot \vec{n} \, dA = 0.
\]