Math 20E
Final Exam

Instructions: Answers without work may be given no credit at the grader’s discretion. The test is out of 69 points, with 2 extra credit points possible.

This cover page may be used at scratch paper. However, all final work must be on the page with the related question. Do not remove this sheet.
1. True/False. Answer must be in the box. (2 pts each)

(a) Recall that \( \nabla \times \nabla f = 0 \) for any \( f \). In fact, it turns out that \( \nabla \times F = 0 \) if and only if \( F \) is a gradient of some function, i.e., \( F = \nabla f \). (Assuming \( F \) is defined on the whole of \( \mathbb{R}^3 \).

   \[ \text{F} \]

(b) To define good coordinates, a map \( T(u,v) \) needs to be bijective (except for maybe at a few points), \( C^1 \), and linear.

   \[ \text{(We don’t need linear.)} \]

(c) You can always switch the order of iterated integrals over rectangular domains, i.e.,

   \[ \int_a^b \int_c^d f(x,y) dydx = \int_c^d \int_a^b f(x,y) dxdy. \]

   \[ \text{F.} \]

(f needs to be continuous.)

2. Explain why for any \( C^1 \) vector field \( F \), that \( \iiint_{\text{sphere}} \text{curl} F \cdot d\mathbf{S} = 0. \) (5 pts)

Way 1: By Stokes’ Thm, \( \int_{\text{sphere}} \text{curl} F \cdot d\mathbf{S} = \int_{\text{dsphere}} F \cdot d\mathbf{S} \). However, the sphere has no boundary, and so the integral on the right is 0.

Way 2: By Stokes’ Thm, \( \int_{\text{sphere}} \text{curl} F \cdot d\mathbf{S} = \int_{\text{dsphere}} F \cdot d\mathbf{S} \). The standard parametrization of the sphere has two boundary components, both at \( \theta = 0 \) or \( 2\pi \), but these two are oriented oppositely. Thus their contributions to the body integral cancel out, for a total of 0.

Way 3: \( \text{curl} F \cdot d\mathbf{S} \) represents rotation along the surface. We can visualize this with a little whirl. When we integrate \( \text{curl} F \cdot d\mathbf{S} \), the little bits of whirl cancel out when they are adjacent, because they go in opposite directions. In fact, they cancel out all around the sphere, and so the total integral is 0.
3. Integrate \( f(x, y) = \sqrt{x^2 + 4y^2} \) over the interior of the ellipse \( x^2 + 4y^2 \leq 4 \). (Hint: A variation of polar coordinates gives good coordinates for an ellipse.) (5 pts)

\[
T(r, \theta) = (2r \cos \theta, r \sin \theta) \quad \text{on} \quad r \in [0, 1], \quad \theta \in [0, 2\pi].
\]

\[
|D_T| = \begin{vmatrix} 2 \cos \theta & -2r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = 2r. \quad \text{Note} \quad \sqrt{x^2 + 4y^2} = 2r
\]

so

\[
\iint_{\text{ellipse}} \sqrt{x^2 + 4y^2} \, dA = \int_0^1 \int_0^{2\pi} 2r \cdot 2r \, d\theta \, dr
\]

\[
= \int_0^1 \int_0^{2\pi} 4r^2 \, d\theta \, dr
\]

\[
= 8\pi \int_0^1 r^2 \, dr
\]

\[
= \frac{8}{3}\pi
\]
4. Explain why Stokes' theorem is intuitively true. (You may use the flat or non-flat version of Stokes' theorem. (5 pts)

Stoke's Thm says that for a "nice" region in space and a $C^1$ vector field,

$$\int_{\text{region}} \text{curl} \mathbf{F} \cdot d\mathbf{s} = \int_{\partial \text{region}} \mathbf{F} \cdot d\mathbf{s}$$

The $\text{curl} \mathbf{F} \cdot d\mathbf{s}$ represents bits of rotation along the surface, and we can represent that graphically by a little whirl.

Main picture

In the middle of the region, where the whirls touch, they are going in opposite directions, and so cancel out. However, at the boundary, the portions of the whirls along the boundary are left over. Thus, the total $\text{curl} \mathbf{F} \cdot d\mathbf{s}$ gives the total flow of $\mathbf{F}$ along the boundary, exactly as Stokes' Thm says.
5. (a) State the two-dimensional version of the divergence theorem. Make sure you correctly give all the conditions for it to hold! (2 pts)

For a "nice" region $S$, with "nice" boundary $\partial S$, if $\mathbf{F}$ is $C^1$, then
$$\iint_S \text{div} \mathbf{F} \, dA = \oint_{\partial S} \mathbf{F} \cdot \mathbf{n} \, ds,$$
where $\mathbf{n}$ is the outward normal. (Simply giving) was not enough!)

(b) What does each side of the equality in the divergence theorem intuitively represent? Use that to explain why the theorem is true. (4 pts)

The left side represents the total rate of expansion of something inside $S$ flowing along $\mathbf{F}$. The right hand side is the flux of $\mathbf{F}$ out of $\partial S$.

Intuitively, the rate something in $S$ is expanding (when flowing by $\mathbf{F}$) should be how much of it is flowing out of the shape, since any expansion has to go somewhere. So, it makes sense the two sides are equal.
6. Physically, what does a conservative field conserve? (A word or 3 is sufficient.) (2 pts)

[total energy]

7. One of Maxwell's laws says that the divergence of the magnetic field is zero, i.e., that \( \text{div} \, H = 0 \). Explain why that means that the magnetic flux across any closed surface must be zero. (Assume that all fields are \( C^1 \).) (5 pts)

Gauss' divergence Thm says that, for this case,

\[
\iiint \text{div} \, H \, dV = \iint_{\text{Surface}} H \cdot \mathbf{n} \, dA
\]

(The "inside of the surface" is the 3-d region enclosed by the 2-d surface. For ex, a ball is the inside of the sphere.)

Since \( \text{div} \, H = 0 \), the left hand integral is always 0, and so the total magnetic flux is zero.

(Note: Any (orientable, embedded) closed surface has an interior as described.)
8. Another of Maxwell's laws, also called Faraday's law, says that the rate of change of the magnetic field is negative the curl of the electric field, i.e., that \( \frac{\partial H}{\partial t} = -\nabla \times E \).

Suppose that \( E(x, y, z) = (-z^2y, z^2x, \sin(x) \cos(y)) \), and that \( S \) is the part of the sphere of radius two, centered at the origin, with \( z \geq -\sqrt{3} \), and outwardly oriented. Thus, \( \partial S \) is a circle at \( z = -\sqrt{3} \).

Calculate the rate of change of the flux of the magnetic field through the surface \( S \). (5 pts)

\[
\frac{d}{dt} \iint_S H \cdot \hat{n} \, dA = \iint_S \frac{\partial H}{\partial t} \cdot \hat{n} \, dA
\]

\[
= -\iint_S \nabla \times E \cdot \hat{n} \, dA
\]

\[
= -\int_{\partial S} E \cdot d\vec{s}
\]

\( \partial S \) is parameterized by \( \vec{r}(t) = (\cos t, \sin t, -\sqrt{3}) \), with correct orientation.

\[
\vec{r}'(t) = (-\sin t, \cos t, 0)
\]

\[
E \cdot \vec{r}' = (-z^2y, z^2x, \sin x \cos y) \cdot (-\sin t, \cos t, 0)
\]

\[
= 3 \sin^2 t + 3 \cos^2 t = 3
\]

So

\[
\frac{d}{dt} \iint_S H \cdot \hat{n} \, dA = -\int_{\partial S} E \cdot d\vec{s}
\]

\[
= -\int_{0}^{2\pi} 3 \, dt
\]

\[
= -6\pi
\]
9. Calculate the area of the region contained in the curve \( \gamma(t) = (t^2, t^3 - t) \) on the domain \([-1, 1]\).

(Credit for the graph to Desmos.com.) (5 pts)

\[ \gamma' = (2t, 3t^2 - 1) \]

out normal = \((3t^2 - 1, -2t)\)

By div Thm, \( F = (x, y) \),

area = \( \frac{1}{2} \int_{\text{bnd}} (x, y) \cdot \vec{n} \, d\ell \)

\[ = \frac{1}{2} \int_{-1}^{1} (t^2, t^3 - t) \cdot (3t^2 - 1, -2t) \, dt \]

\[ = \frac{1}{2} \int_{-1}^{1} 3t^4 - t^2 - 2t^4 + 2t^2 \, dt \]

\[ = \frac{1}{2} \int_{-1}^{1} t^4 + t^2 \, dt \]

\[ = \frac{1}{2} \left[ \frac{t^5}{5} + \frac{t^3}{3} \right]_{-1}^{1} \]

\[ = \frac{1}{2} \left[ \frac{2}{5} + \frac{2}{3} \right] = \frac{1}{5} + \frac{1}{3} = \frac{3}{15} + \frac{5}{15} = \frac{8}{15} \]

(looking at the picture, that seems about right.)
10. When introducing multiple integrals, we said that a good way to understand that kind of integral is as a volume. For instance, the integral of \( f(x, y) = x^2 + y^2 \) over the unit disc was the volume under \( f \), above the disc. However, when we began to work with surface integrals over more general surfaces, we said that volume was no longer a valid interpretation, at least most of the time.

Explain why surface integrals \( \iint_S f \, dA \) cannot usually be interpreted as volume. As part of your explanation, give an explicit example where you get the wrong answer if you try to use that interpretation. (5 pts)

The problem is that, for a curved surface, \( f \, dA \) no longer represents a "bit of volume."

Essentially, we'll be missing, or double counting some volume, since \( f \, dA \) is not the correct bit of volume.

For example, if \( S = \text{sphere of radius } 1 \), and \( f = 1 \), if we tried to interpret \( \iint_S f \, dA \) as a volume, we'd try to find the volume of the region between the sphere of radius 1 and the one of radius 2. This is

\[
\frac{4}{3} \pi 2^3 - \frac{4}{3} \pi 1^3 = \frac{28}{3} \pi.
\]

However,

\[
\iint_S f \, dA = \iint_S dA = \text{surface area} = 4 \pi \cdot 1^2 = 4 \pi.
\]

So, it doesn't work here.
11. Let $S$ be the surface of the cube $[-1, 1] \times [-1, 1] \times [-1, 1]$. Let $F(x, y, z) = (x, y, z)$. Calculate the outward flux of $F$ across $S$. (5 pts)

Way 1: See sol'n from midterm 1.

Way 2: $\text{div}F = 3$.

$$\text{Flux} = \iint_S F \cdot \mathbf{n} \, dA = \iiint_{\text{cube}} \text{div}F \, dV$$

$$= 3 \cdot \text{vol}(\text{cube})$$

$$= 3 \cdot 8 = 24$$
12. Let $T(u, v) = (x, y)$ be a map that provides good coordinates. What does the Jacobian $||DT||$ represent about this map? (The answer is not “the quantity you need for change of variables.” What does it represent when you use it for change of variables?) (5 pts)

It represents the conversion (or “fudge”) factor for converting coordinate area (in $u, v$) to real area (in $x, y$).

13. Rewrite this integral with the other order of integration. You do not need to actually evaluate the integral. (5 pts)

\[
\int_1^2 \int_1^2 e^x \sin(y) dy dx = \int_1^4 \int_{\sqrt{y}}^2 e^x \sin(y) dx dy
\]
14. Integrate $f(x,y,z) = xy$ over the region $R$ between $z = x$ and the $xy$ plane, and in the rectangle $1 \leq x \leq 2$, $2 \leq y \leq 3$. (5 pts)

\[
\int_1^2 \int_2^3 \int_0^x xy \ dy \ dx = \int_1^2 \int_2^3 x^2 y \ dy \ dx
\]

\[
= \int_1^2 x^2 \left[ \frac{y^2}{2} \right]_2^3 \ dx
\]

\[
= \left[ \frac{9}{2} - 2 \right] \int_1^2 x^2 \ dx
\]

\[
= \frac{5}{2} \left[ \frac{x^3}{3} \right]_1^2
\]

\[
= \frac{5}{2} \cdot \frac{7}{3} = 3 \frac{5}{6}
\]
15. Extra credit: for correctness only. Answer must be in the box. (1 pt each)

(a) You have 100 kg of cucumbers. (Don’t ask me why...) At the beginning of the day, they are 99% water. However, you leave them in the sun all day, and at the end of the day, they are only 98% water. How many kg of cucumbers do you have at the end of the day?

\[ \text{50 kg.} \]

\[ \text{Start: 99 kg water, 1 kg dry.} \]
\[ \text{end: 49 kg water, 1 kg dry} \]

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(b) Alice can dig a hole in 2 hours, while Bob takes 3 hours. Assuming they can work at the same time as efficiently as they can work by themselves, how long will it take them to dig a hole together?

\[ \text{6/5 hour} \]

\[ \text{Alice works at \( \frac{1}{2} \) hole/hour.} \]
\[ \text{Bob works at \( \frac{1}{3} \) hole/hour.} \]
\[ \text{together, they dig \( \frac{5}{6} \) hole/hour,} \]
\[ \text{so \( \frac{6}{5} \) hour to dig one hole.} \]