Discussion 5 Solutions

1. \( \mathbf{\gamma}' = (1, 2t, 0) \), \( \| \mathbf{\gamma}' \| = \sqrt{1^2 + (2t)^2} = \sqrt{1 + 4t^2} \)

\[ f(\mathbf{\gamma}) = 8t \]

\[ \int_{\mathbf{\gamma}} f \, ds = \int_0^1 8t \sqrt{1 + 4t^2} \, dt \]

\[ u = 1 + 4t^2 \quad t = 0 \Rightarrow u = 1 \]
\[ du = 8t \, dt \quad t = 1 \Rightarrow u = 5 \]

\[ \int_1^5 u \frac{1}{u^{3/2}} \, du = \frac{2}{3} \left[ u^{3/2} \right]_1^5 = \frac{2}{3} \left[ 5^{3/2} - 1 \right] \]

2. \( F(\mathbf{\gamma}(t)) = (t^2, -t, 1) \)

\[ F \cdot \mathbf{\gamma}' = t^2 - 2t^2 + 0 = -t^2 \]

\[ \int_{\mathbf{\gamma}} F \cdot d\mathbf{s} = \int_0^1 -t^2 \, dt = -\frac{t^3}{3} \bigg|_0^1 = -\frac{1}{3} \]

Graded 1: As we said in class, \( \int_{\mathbf{\gamma}} \nabla f \cdot d\mathbf{s} = f(\mathbf{\gamma}(b)) - f(\mathbf{\gamma}(a)) \).

For the first way, let \( F(t) = f(\mathbf{\gamma}(t)) \). Then \( F'(t) = \nabla f(\mathbf{\gamma}(t)) \cdot \mathbf{\gamma}'(t) \) by the chain rule. Recall that

\[ \int_{\mathbf{\gamma}} \nabla f \cdot d\mathbf{s} = \int_{a}^{b} \nabla f(\mathbf{\gamma}(t)) \cdot \mathbf{\gamma}'(t) \, dt, \]

and so

\[ \int_{\mathbf{\gamma}} \nabla f \cdot d\mathbf{s} = \int_{a}^{b} F'(t) \, dt = F(b) - F(a) \] by the fundamental theorem of calculus.

By definition, this is \( f(\mathbf{\gamma}(b)) - f(\mathbf{\gamma}(a)) \).

For the second way, recall that \( \nabla f \) dotted with a vector gives the slope of \( f \) in that direction. Thus \( \nabla f \cdot \mathbf{\gamma}' \) gives the slope of \( f \) as you travel along \( \mathbf{\gamma} \). Thus, when you add that up along \( \mathbf{\gamma} \) (using the integral), you get the total change in height of \( f \) along the path \( \mathbf{\gamma} \). That’s exactly given by \( f(\mathbf{\gamma}(b)) - f(\mathbf{\gamma}(a)) \).

Thus

\[ \int_{\mathbf{\gamma}} \nabla f \cdot d\mathbf{s} = f(\mathbf{\gamma}(b)) - f(\mathbf{\gamma}(a)). \]