**Problem.** For two matrices $A$ and $B$, explain why $\text{rank}(AB) \leq \text{rank}(A)$. (Hint: Compare their column spaces.) Then explain why $\text{rank}(AB) \leq \text{rank}(B)$. (Hint: Use the first part to study $\text{rank}(AB)^T$.)

**Solution.** Let $b_1, \ldots, b_p$ denote the columns of $B$. Then by the definition of matrix multiplication, we have

$$AB = [Ab_1 \ \cdots \ Ab_p].$$

Note that for each $1 \leq i \leq p$, the column vector $Ab_i$ is a linear combination of the columns of $A$, with the scalars given by the entries of $b_i$. Thus each $Ab_i$ is in $\text{Col}(A)$. Since $\text{span}\{Ab_1, \ldots, Ab_p\} = \text{Col}(AB)$, this shows that $\text{Col}(AB)$ is a subspace of $\text{Col}(A)$. By Theorem 4.11, this implies that $\text{rank}(AB) \leq \text{rank}(A)$.

For the second inequality, observe that taking the transpose of the matrix does not change its rank, since the dimensions of the column space and row space of a matrix are equal. Hence, using the first part of the problem, (with $B^T$ playing the role of $A$), we have

$$\text{rank}(AB) = \text{rank}(AB)^T = \text{rank}(B^T A^T) \leq \text{rank}(B^T) = \text{rank}(B).$$

$\square$