Problem. In the Gram-Schmidt process, we start with a linearly independent set of vectors \( \{x_1, \ldots, x_n\} \) and at each step we get a new vector \( v_i \). Explain why we know that none of the \( v_i \) will be the zero vector. (A one sentence answer is not enough.)

Solution. In the Gram-Schmidt process applied to the linearly independent vectors \( \{x_1, \ldots, x_n\} \), we define

\[
v_1 = x_1
\]

\[
v_2 = x_2 - \frac{x_2 \cdot x_1}{x_1 \cdot x_1} v_1
\]

\[
v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2
\]

\[
\vdots
\]

\[
v_n = x_n - \frac{x_n \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_n \cdot v_2}{v_2 \cdot v_2} v_2 - \cdots - \frac{x_n \cdot v_{n-1}}{v_{n-1} \cdot v_{n-1}} v_{n-1}
\]

To see why none of the \( v_i \)'s are equal to the zero vector, first observe that since \( \{x_1, \ldots, x_n\} \) is a linearly independent set, none of the vectors \( x_1, \ldots, x_n \) are equal to \( 0 \). In particular, \( v_1 = x_1 \neq 0 \). Next, we see that \( v_2 \) is a linear combination of \( x_2 \) and \( v_1 \), where \( v_1 \) is equal to \( x_1 \). As we continue the process, we see that \( v_1 \) is in \( \text{span} \{x_1\} \), \( v_2 \) is in \( \text{span} \{x_1, x_2\} \), \( v_3 \) is in \( \text{span} \{x_1, x_2, x_3\} \), and so on. That is, \( v_i \) is in \( \text{span} \{x_1, \ldots, x_i\} \) for each \( 1 \leq i \leq n \).

We already know \( v_1 \neq 0 \). For \( 2 \leq i \leq n \), we have

\[
v_i = x_i + \left( -\frac{x_i \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_i \cdot v_2}{v_2 \cdot v_2} v_2 - \cdots - \frac{x_i \cdot v_{i-1}}{v_{i-1} \cdot v_{i-1}} v_{i-1} \right).
\]

Since the vectors \( v_1, \ldots, v_{i-1} \) are in \( \text{span} \{x_1, \ldots, x_{i-1}\} \), we also have that the vector

\[
-\frac{x_i \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_i \cdot v_2}{v_2 \cdot v_2} v_2 - \cdots - \frac{x_i \cdot v_{i-1}}{v_{i-1} \cdot v_{i-1}} v_{i-1},
\]

which we denote as \( \vec{w} \), is in \( \text{span} \{x_1, \ldots, x_{i-1}\} \). Hence there are real numbers \( a_1, \ldots, a_{i-1} \) such that \( \vec{w} = a_1 x_1 + \cdots + a_{i-1} x_{i-1} \), so

\[
v_i = x_i + a_1 x_1 + \cdots + a_{i-1} x_{i-1}.
\]

Since the coefficient of \( x_i \) is nonzero (and \( x_i \) does not appear in the subsequent terms), \( v_i \) is a nontrivial linear combination of the vectors \( x_1, \ldots, x_i \). By assumption, these vectors are linearly independent, so it follows that \( v_i \neq 0 \).