Instructions: Answers without work may be given no credit at the grader's discretion. The test is out of 30 points.

This cover page may be used at scratch paper. However, all final work must be on the page with the related question.
1. Calculate the QR decomposition of the matrix

\[ A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 1 & 2 \\ 1 & 0 \end{bmatrix} \] (5 pts)

Set \( V_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \) and \( V_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix} \). Then we set

\[ g_1 = \frac{V_1}{\|V_1\|} = \frac{1}{2} \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \]

Now find

\[ w_2 = V_2 - (g_1^T V_2)g_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 3 \\ 9 \\ 7 \\ -1 \end{bmatrix} \]

and

\[ g_2 = \frac{w_2}{\|w_2\|} = \frac{1}{2\sqrt{35}} \begin{bmatrix} 3 \\ 9 \\ 7 \\ -1 \end{bmatrix} \]

We then gather the terms

\[ r_{11} = \|V_1\| = 2, \quad r_{12} = g_1^T V_2 = \frac{1}{2}, \quad \text{and} \quad r_{22} = \|w_2\| = \frac{\sqrt{35}}{2} \]

and we have

\[ Q = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{3\sqrt{55}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{9\sqrt{55}}{2} \\ \frac{1}{2} & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \]

\[ R = \begin{bmatrix} 2 & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{55}}{2} \end{bmatrix} \]

We verify

\[ QR = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{3\sqrt{55}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{9\sqrt{55}}{2} \\ \frac{1}{2} & \frac{3\sqrt{2}}{2} \\ \frac{1}{2} & -\frac{3\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 2 & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{55}}{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 1 & 2 \\ 1 & 0 \end{bmatrix} = A. \]
2. Solve the least squares problem

\[ A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 1 & 2 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 4 \\ -1 \end{bmatrix}. \]

You may use the previous question’s result, but are not required to. (5 pts)

Setting \( b = \begin{bmatrix} 2 \\ 0 \\ 4 \\ -1 \end{bmatrix} \), we must solve

\[ R x = Q^T b, \] where \( Q \) and \( R \) are as before. This gives the system

\[ \begin{bmatrix} 2 & \frac{1}{2} \\ 0 & \sqrt{3}/2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3/2 \\ \sqrt{3}/2 \end{bmatrix}. \]

Back substitution yields

\[ \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \]

We verify \( A^T(Ax - b) = 0 \) (to check that

\[ Ax - b \in N_u(A^T) = Col(A)^\perp \]

Indeed,

\[ \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 2 & 2 & 0 \end{bmatrix} \left( \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \\ 0 \\ 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 0 \\ 4 \\ -1 \end{bmatrix} \]

\[ = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix} = 4 \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \]

so \( \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) and

\[ \text{proj}_{Col(b)}(b) = \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \end{bmatrix}. \]
3. In class, we discussed four fundamental subspaces related to each linear transformation \( A \). Explain what each of these subspaces are, how they are related to \( A \), and how they relate to each other. Your explanation should include a picture for, say, the case of \( A : \mathbb{R}^2 \to \mathbb{R}^2 \). (5 pts)

We identify a linear transformation \( A : \mathbb{R}^n \to \mathbb{R}^m \) with its representative matrix under the standard bases, \( A \in \mathbb{R}^{m \times n} \). Then the four fundamental subspaces are:

- \( \text{Col}(A) = \text{Range}(A) \subseteq \mathbb{R}^m \), which is the span of the columns of the matrix \( A \); alternatively, the set of all \( b \in \mathbb{R}^m \) such that there is an \( x \in \mathbb{R}^n \) where \( A(x) = b \).

- \( \text{Nul}(A) = \text{Kernel}(A) \subseteq \mathbb{R}^n \), the solution set to \( Ax = 0 \) (that is, all vectors \( x \in \mathbb{R}^n \) s.t. \( A(x) = 0 \in \mathbb{R}^m \)).

- \( \text{Col}(A^T) = \text{Row}(A) \subseteq \mathbb{R}^n \), identical to the column space or range of \( A^T : \mathbb{R}^m \to \mathbb{R}^n \), also the span of the rows of \( A \).

- \( \text{Nul}(A^T) = \text{Coker}(A) \subseteq \mathbb{R}^m \), the null space of \( A^T \); also the left null space of \( A \). The set of vectors \( y \) such that \( y^TA = 0 \).

We know \( \text{Col}(A)^\perp = \text{Nul}(A^T) \) and \( \text{Nul}(A)^\perp = \text{Col}(A^T) \), but \( \text{Col}(A) \) and \( \text{Nul}(A) \) are not related in general (same for \( \text{Col}(A^T), \text{Nul}(A^T) \)).
4. Given $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, explain what the “least squares problem” is for $A$ and $b$ (For instance, if the solution is $\hat{x}$, what goal does $\hat{x}$ achieve?). Explain the relationship between the projection and the solution of this problem. Your explanation should include a picture showing this relationship. (5 pts)

The goal is to find $\hat{x} \in \mathbb{R}^n$ such that

$$\|A\hat{x} - b\| \text{ is minimized.}$$

If $\hat{x}$ is a solution, then for any $x \in \mathbb{R}^n$

$$\|A\hat{x} - b\| \leq \|A\hat{x} - b\|.$$  

If $\hat{x}$ is a solution, then

$$A\hat{x} = \text{proj}_{\text{Col}(A)}(b) = "\text{the projection of } b \text{ onto } \text{Col}(A)".$$  

Pictorially,
5. Let $F : V \to W$ be a linear transformation between two (finite dimensional) vector spaces $V$ and $W$. Prove that if we know how $F$ acts on a basis for $V$, then we know how $F$ acts on every element of $V$. (5 pts)

Suppose $\dim(V) = k$, $\dim(W) = n$, and that

$\{v_1, \ldots, v_k\}$ is a basis for $V$ for which we know the values of $F$ (suppose $F(v_1) = w_1, \ldots, F(v_k) = w_k$).

Then for any $v \in V$, we may find $c_1, \ldots, c_k \in \mathbb{R}$ s.t.

$v = c_1 v_1 + \cdots + c_k v_k$;

by linearity,

$F(v) = c_1 F(v_1) + \cdots + c_k F(v_k)$.

Because $F(v_1), \ldots, F(v_k)$ are known, we have

$F(v) = c_1 w_1 + \cdots + c_k w_k$. 

6. Using only our definition of the determinant, and the extra property that “if two rows of A are equal, then det A = 0,” prove that row reduction (in particular, subtracting a multiple of one row from another row) does not change the determinant of a matrix. You may restrict your attention to 2x2 matrices. (5 pts)

We observe that

\[
\text{det}\left(\begin{bmatrix} a-ka & b-kb \\ c & d \end{bmatrix}\right) = \text{det}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) - k \text{det}\left(\begin{bmatrix} c & d \\ c & d \end{bmatrix}\right)
\]

as \(\begin{bmatrix} c & d \\ c & d \end{bmatrix}\) has two equal rows.

Similarly,

\[
\text{det}\left(\begin{bmatrix} a & b \\ c-ka & d-kb \end{bmatrix}\right) = -\text{det}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) - k \text{det}\left(\begin{bmatrix} c & d \\ c & d \end{bmatrix}\right)
\]

\[
= -\text{det}\left(\begin{bmatrix} c & d \\ a & b \end{bmatrix}\right) = \text{det}\left(\begin{bmatrix} c & d \\ c & d \end{bmatrix}\right),
\]

using the fact that switching two rows flips the +/- sign of the determinant.