Math 102  
Midterm 1  

Instructions: Answers without work may be given no credit at the grader’s discretion. ALL work must be on this test. The test is out of 30 points.

1. Calculate a $P$, $L$, $D$, and $U$ that form an LDU decomposition of the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ -3 & -1 & 3 \\ 6 & 5 & 1 \end{bmatrix}.$$

(5 pts)

$$2 \equiv \begin{bmatrix} -3 & -1 & 3 \\ 0 & 1 & 2 \\ 6 & 5 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -3 & -1 & 3 \\ 0 & 1 & 2 \\ -2 & 3 & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} -3 & -1 & 3 \\ 0 & 1 & 2 \\ -2 & 3 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 3 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

I checked $PA = LU$
2. Find all solutions of \[
\begin{bmatrix}
1 & 2 & -1 \\
2 & 1 & 1 \\
1 & 3 & -2
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} =
\begin{bmatrix}
4 \\
2 \\
6
\end{bmatrix}.
\]

(5 pts)

First find the reduction:
\[
\begin{bmatrix}
1 & 2 & -1 \\
2 & 1 & 1 \\
1 & 3 & -2
\end{bmatrix} \Rightarrow \begin{bmatrix}
1 & 2 & -1 \\
0 & -3 & 3 \\
0 & 1 & -2
\end{bmatrix} \Rightarrow \begin{bmatrix}
1 & 2 & -1 & 4 \\
0 & -3 & 3 & -6 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

The free variable is \( z \), the pivot ones are \( x, y \).

Get \( R \) to find \( \text{Ker} \, A \):
\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & -1 \\
0 & 0 & 0
\end{bmatrix}
\Rightarrow \begin{bmatrix}
0 & 1 & -1 \\
0 & 0 & 0
\end{bmatrix}
\]

so \( x + z = 0 \) is the kernel, or \( x = -z \) and \( y - z = 0 \), so kernel is
\[
\begin{bmatrix}
-x \\
z \\
z
\end{bmatrix} = z \begin{bmatrix}
-1 \\
1 \\
1
\end{bmatrix}
\]

A particular soln is given by
\[
\begin{bmatrix}
1 & 2 & -1 \\
0 & -3 & 3 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \begin{bmatrix}
4 \\
2 \\
6
\end{bmatrix},
\]

so \( x + 2y = 4 \) and \( -3y = -6 \), so \( y = 2, \ x = 0 \).

So all solns are \( \begin{bmatrix}
0 \\
2 \\
0
\end{bmatrix} + z \begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix} \).
3. Prove that any two solutions of $Ax = b$ differ by an element of $N(A)$, the null space of $A$. (5 pts)

Suppose $x_1$ and $x_2$ are solutions of $Ax = b$, i.e., that $Ax_1 = b$ and $Ax_2 = b$.

Then $A(x_1 - x_2) = Ax_1 - Ax_2 = b - b = 0$.

Thus $x_1 - x_2$ is in the kernel of $A$, which is what it means for two solutions to differ by an element of $N(A)$.

(Since $x_1, x_2$ were any solutions, we proved this for all solutions.)
4. Explain what a subspace of a vector space is. Give both an example and a non-example for subspaces of the vector space \( \mathbb{R}^\infty \), (i.e., real valued sequences), explaining why they are and aren't subspaces, respectively. You may assume the reader knows what a vector space is. (5 pts)

A subspace of a vector space is a subset of the vector space that is itself a vector space. Since the subset automatically inherits the same addition and scalar multiplication rules, the main thing to check is that those operations leave vectors within the subspace.

**EX:** In \( \mathbb{R}^\infty \), an example is the set of all vectors with a zero in the first spot. Since if I multiply by a scalar, zero will stay the same, and since adding two zeros is still zero, the set is closed under scalar multiplication and addition, and so is a subset.

**non-EX:** One non-example would be positive sequences, like \( \{1, 2, 3, \ldots \} \). This is not a subspace since \(-1 \cdot \{1, 2, 3, \ldots \} = \{-1, -2, -3, \ldots \}\) which is no longer a positive sequence.
5. Give the definition of linear independence for vectors, then explain why the column vectors of an invertible matrix $A$ are linearly independent. (5 pts)

A set of vectors $\{v_i\}$ are linearly independent if the only linear combination that gives zero is the trivial one, i.e., if

$$c_1v_1 + \cdots + c_nv_n = 0 \iff c_i = 0.$$ 

If $A$ is invertible, that means $Ax = 0$ has only the trivial solution $x = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$. If $A_i$ are the columns of $A$, $Ax$ is given by $x_1A_1 + \cdots + x_nA_n = \vec{0}$. Thus, the only solution is $x_i = 0$, which is the definition of linear independence.
6. Suppose that \( V \) and \( W \) are three-dimensional subspaces of \( \mathbb{R}^5 \). Prove that \( V \) and \( W \) must share (at least) one nonzero vector. (Hint: Use bases for \( V \) and \( W \).) (5 pts)

Let \( \{v_1, v_2, v_3\} \) be a basis for \( V \), and
let \( \{w_1, w_2, w_3\} \) be a basis for \( W \). They contain 3 vectors since \( V, W \) are 3-dimensional.
The set \( \{v_1, v_2, v_3, w_1, w_2, w_3\} \) is linearly dependent, because
there are 6 vectors in a 5-dimensional space.
Thus, there are some \( c_i, d_i \) s.t.
\[
\sum c_i v_i + \sum d_i w_i = 0,
\]
where not all of the \( c_i \) and \( d_i \) are zero. Indeed, if all the \( d_i \) were zero,
\[
\sum c_i v_i = 0,
\]
with not all the \( c_i \) zero, which contradicts the \( v_i \) being linearly independent.
Thus, at least one \( d_i \neq 0 \), and at least one \( c_i \neq 0 \), similarly.

So
\[
\sum c_i v_i = -\sum d_i w_i.
\]
Both sides are nonzero, and, clearly, the left side is in \( V \)
and the right side is in \( W \). Thus \( V, W \) share the
nonzero vector \( \sum c_i v_i \).