Some Results on the Conformal Method for Compact Manifolds with Boundary

MSRI Lecture

Michael Holst

Department of Mathematics
Department of Physics
University of California, San Diego

Collaborators:
C. Meier, G. Nagy, G. Tsogtgerel

Funding:
NSF: DMS/FRG 1065972, DMS/FRG 1262982, DMS/CM 1217175

Relevant Manuscripts and References

The two talks this week at MSRI (Holst) and in Evans Hall (Meier):

[HNT08] MH, G. Nagy, and G. Tsogtgerel, *Far-from-constant mean curvature solutions of Einstein's constraint equations with positive Yamabe metrics*, Phys. Rev. Lett. **100** (2008), no. 16, 161101.1–161101.4, Available as arXiv:0802.1031[gr-qc].

[HNT09] MH, G. Nagy, and G. Tsogtgerel, *Rough solutions of the Einstein constraints on closed manifolds without near-CMC conditions*, Comm. Math. Phys. **288** (2009), no. 2, 547–613, Available as arXiv:0712.0798[gr-qc].

[HT13] MH and G. Tsogtgerel, *The Lichnerowicz equation on compact manifolds with boundary*, Class. Quantum Grav., 30 (2013), p. 205011. Available as arXiv:1306.1801 [gr-qc].

[HMT] MH, C. Meier, and G. Tsogtgerel, *Non-CMC solutions of the Einstein constraint equations on compact manifolds with apparent horizon boundaries*. Submitted for publication. Available as arXiv:1310.2302 [gr-qc].

The talk at MSRI in November (Meier):

[HMb] MH and C. Meier, *Non-uniqueness of solutions to the conformal formulation*. Submitted for publication. Available as arXiv:1210.2156 [gr-qc].

[HMa] MH and C. Meier, *An alternative between non-unique and negative Yamabe solutions to the conformal formulation of the Einstein constraint equations*. Submitted for publication. Available as arXiv:1306.1210 [gr-qc].

Other work complementing ours (more related to Meier talk tomorrow):

[Dil] J. Dilts. The Einstein constraint equations on compact manifolds with boundary. Preprint.

Key Articles for Purposes of this Lecture

Our work builds on a very large literature, but in particular:

[Ise95] J. Isenberg. Constant mean curvature solutions of the Einstein constraint equations on closed manifolds. *Classical Quantum Gravity*, 12(9):2249–2274, 1995.

[IM96] J. Isenberg and V. Moncrief. A set of nonconstant mean curvature solution of the Einstein constraint equations on closed manifolds. *Class. Quantum Grav.*, 13:1819–1847, 1996.

[Max04] D. Maxwell. *Initial Data for Black Holes and Rough Spacetimes*. PhD thesis, University of Washington, 2004.

[Max05a] D. Maxwell. Rough solutions of the Einstein constraint equations on compact manifolds. *J. Hyp. Diff. Eqs.*, 2(2):521–546, 2005.

[Max05b] D. Maxwell. Solutions of the Einstein constraint equations with apparent horizon boundaries. *Comm. Math. Phys.*, 253(3):561–583, 2005.

[Max06] D. Maxwell. Rough solutions of the Einstein constraint equations. *J. Reine Angew. Math.*, 590:1–29, 2006.

[Max09] D. Maxwell. A class of solutions of the vacuum Einstein constraint equations with freely specified mean curvature. *Math. Res. Lett.*, 16(4):627–645, 2009.

[Dai04] S. Dain. Trapped surfaces as boundaries for the constraint equations. *Classical Quantum Gravity*, 21(2):555–573, 2004.

[Dai06] S. Dain. Generalized Korn's inequality and conformal Killing vectors. *Calc. Var.*, 25(4):535–540, 2006.

[YP82] J. York and T. Piran. The initial value problem and beyond. In R. A. Matzner and L. C. Shepley, editors, *Spacetime and Geometry: The Alfred Schild Lectures*, pages 147–176. University of Texas Press, Austin (Texas), 1982.

[Esc92] J. F. Escobar. The Yamabe problem on manifolds with boundary. *J. Differential Geom.*, 35(1):21–84, 1992.

[Esc96] J. F. Escobar. Conformal deformation of a Riemannian metric to a constant scalar curvature metric with constant mean curvature on the boundary. *Indiana Univ. Math. J.*, 45(4):917–943, 1996.

Twelve-component Einstein evolution system for $(\hat{h}_{ab}, \hat{k}_{ab})$ on a foliation.

Constrained by coupled eqns on spacelike $\mathcal{M} = \mathcal{M}_t$, with $\hat{\tau} = \hat{k}_{ab}\hat{h}^{ab}$,

$$ {}^3\hat{R} + \hat{\tau}^2 - \hat{k}_{ab}\hat{k}^{ab} - 2\kappa\hat{\rho} = 0, \qquad \hat{\nabla}^a\hat{\tau} - \hat{\nabla}_b\hat{k}^{ab} - \kappa\hat{j}^a = 0. $$

York conformal decomposition: split initial data into 8 freely specifiable pieces plus 4 determined via: $\hat{h}_{ab} = \phi^4 h_{ab}$, $\hat{\tau} = \hat{k}_{ab}\hat{h}^{ab} = \tau$, and

$$ \hat{k}_{ab} = \phi^{-10}[\sigma^{ab} + (\mathcal{L}w)^{ab}] + \frac{1}{4}\phi^{-4}\tau h^{ab}, \quad \hat{j}^a = \phi^{-10}j^a, \quad \hat{\rho} = \phi^{-8}\rho. $$

Produces coupled elliptic system for conformal factor ϕ and a w^a:

$$ -8\Delta\phi + R\phi + \frac{2}{3}\tau^2\phi^5 - (\sigma_{ab} + (\mathcal{L}w)_{ab})(\sigma^{ab} + (\mathcal{L}w)^{ab})\phi^{-7} - 2\kappa\rho\phi^{-3} = 0, $$

$$ -\nabla_a(\mathcal{L}w)^{ab} + \frac{2}{3}\phi^6\nabla^b\tau + \kappa j^b = 0. $$

Differential structure on $\mathcal{M}$ defined through background 3-metric h_{ab}:

$$ (\mathcal{L}w)^{ab} = \nabla^a w^b + \nabla^b w^a - \frac{2}{3}(\nabla_c w^c)h^{ab}, \qquad \nabla_b V^a = V^a_{\ ;b} = V^a_{\ ,b} + \Gamma^a_{bc}V^c, $$

$$ V^a_{\ ,b} = \frac{\partial V^a}{\partial x^b}, \qquad \Gamma^a_{bc} = \frac{1}{2}h^{ad}\left(\frac{\partial h_{db}}{\partial x^c} + \frac{\partial h_{dc}}{\partial x^b} - \frac{\partial h_{bc}}{\partial x^d}\right). \qquad (\Gamma^a_{bc} = \Gamma^a_{cb}) $$

The Conformal Method as an Elliptic System

Let $\mathcal{M}$ be a space-like Riemannian 3-manifold with (possibly empty) boundary submanifold $\partial\mathcal{M}$, split into disjoint submanifolds satisfying:

$$\partial_D\mathcal{M} \cup \partial_N\mathcal{M} = \partial\mathcal{M}, \quad \partial_D\mathcal{M} \cap \partial_N\mathcal{M} = \emptyset. \quad (\overline{\partial_D\mathcal{M}} \cap \overline{\partial_N\mathcal{M}} = \emptyset)$$

Metric h_{ab} associated with $\mathcal{M}$ induces boundary metric σ_{ab}, giving boundary value formulation of conformal method for ϕ and w^a:

$$\begin{aligned}
L\phi + F(\phi, w) &= 0, \ \text{in } \mathcal{M}, \\
\mathbb{L}w + \mathbb{F}(\phi) &= 0, \ \text{in } \mathcal{M}, \\
(\mathcal{L}w)^{ab}\nu_b + C^a{}_b w^b &= V^a_\phi \ \text{on } \partial_N\mathcal{M}, \quad \textit{and} \quad w^a = w^a_D \ \text{on } \partial_D\mathcal{M}, \\
(\nabla^a\phi)\nu_a + k_w(\phi) &= g \ \text{on } \partial_N\mathcal{M}, \quad \textit{and} \quad \phi = \phi_D \ \text{on } \partial_D\mathcal{M},
\end{aligned}$$

where:

$$L\phi = -\Delta\phi, \quad (\mathbb{L}w)^a = -\nabla_b(\mathcal{L}w)^{ab},$$

$$F(\phi, w) = a_R\phi + a_\tau\phi^5 - a_w\phi^{-7} - a_\rho\phi^{-3}, \quad \mathbb{F}(\phi) = b^b_\tau\phi^6 + b^b_j,$$

with:

$$a_R = \tfrac{R}{8}, \ a_\tau = \tfrac{\tau^2}{12}, \ a_w = \tfrac{1}{8}[\sigma_{ab} + (\mathcal{L}w)_{ab}]^2, \ a_\rho = \tfrac{\kappa\rho}{4}, \ b^b_\tau = \tfrac{2}{3}\nabla^b\tau, \ b^b_j = \kappa j^b,$$

$$(\mathcal{L}w)^{ab} = \nabla^a w^b + \nabla^b w^a - \tfrac{2}{3}(\nabla_c w^c)h^{ab}, \qquad \nabla_b V^a = V^a{}_{;b} = V^a{}_{,b} + \Gamma^a_{bc}V^c,$$

$$V^a{}_{,b} = \frac{\partial V^a}{\partial x^b}, \qquad \Gamma^a_{bc} = \frac{1}{2}h^{ad}\left(\frac{\partial h_{db}}{\partial x^c} + \frac{\partial h_{dc}}{\partial x^b} - \frac{\partial h_{bc}}{\partial x^d}\right). \quad (\Gamma^a_{bc} = \Gamma^a_{cb})$$

Well-posedness, estimates, approximation, ...

This problem has the form:

$$\text{Find } u \in \bar{u} + X \text{ such that } \langle \mathcal{F}(u), v \rangle = 0, \ \forall v \in Y, \tag{1}$$

where X and Y are B-spaces and $\mathcal{F} : X \to Y^*$. With G-derivative

$$\langle \mathcal{F}'(u)w, v \rangle = \frac{d}{d\epsilon} \langle \mathcal{F}(u + \epsilon w), v \rangle \Big|_{\epsilon=0},$$

one solves for u using Newton iteration given approximation $u^0 \approx u$:

(a) Find $w \in X$ such that: $\langle \mathcal{F}'(u^k)w, v \rangle = -\langle \mathcal{F}(u^k), v \rangle + r, \quad \forall v \in Y$
(b) Set: $u^{k+1} = u^k + \lambda w$

One discretizes (a)-(b) at “last moment”, giving matrix equations.

Many questions about constraints open until recently, many remain:

1. Is there existence, uniqueness, stability?
2. How *smooth* is X?
3. Can one build approximation spaces $X_h \subset X$?
4. Performance of linear approximation for (1)?
5. Performance of nonlinear approximation for (1)?
6. Can we produce such (linear and nonlinear) approximations with optimal (linear) space and time complexity?

Known Results for Closed Manifolds

$\nabla^b \tau = 0$: Constant Mean Curvature (CMC): $\Rightarrow$ constraints de-couple.

Results: O'Murchadha-York ('73-74), Isenberg-Marsden ('82-83), Choquet-Bruhat-Isenberg-Moncrief ('92), Isenberg ('95), Maxwell ('04,'06), Choquet-Bruhat ('04), others.

$\nabla^b \tau \neq 0$: Non-CMC case: $\Rightarrow$ constraints couple.

Limited results: Isenberg-Moncrief ('96), Choquet-Bruhat-Isenberg-York ('01), and others; all based on Isenberg-Moncrief, all requiring a near-CMC condition (made precise below).

Question #1: Do non-CMC solutions to conformal equations for 3-manifolds with arbitrary mean extrinsic curvature τ (near-CMC violated, or "Far-CMC")? $\rightarrow$ Yes [HNT08, HNT09, Max09].

Also now vacuum case: Maxwell ('09), and limit equation approach: Dahl-Gicquaud-Humbert ('11), Gicquaud-Sakovich ('11), ...

Question #2: Do "rough" non-CMC conformal solutions exist for 3-manifolds with "rough" background metrics h_{ab}? $\rightarrow$ Yes [HNT09]

Fixed-point arguments involve composition $G(\phi) = T(\phi, S(\phi))$, where:

1. Given ϕ, solve MC for w: $w = S(\phi)$
2. Given w, solve HC for ϕ: $\phi = T(\phi, w)$

Map $S : X \to \mathcal{R}(S) \subset Y$ is MC solution map;
Map $T : X \times \mathcal{R}(S) \to X$ is some fixed-point map for HC.

Yamabe Classes: Rough/Closed

Yamabe classification of smooth metrics: Let $u > 0$ solve: $-8\Delta u + Ru = R_u u^5$. Then:

$$R_u > 0 \Rightarrow h_{ab} \in \mathcal{Y}^+, \quad R_u < 0 \Rightarrow h_{ab} \in \mathcal{Y}^-, \quad R_u = 0 \Rightarrow h_{ab} \in \mathcal{Y}^0.$$

Yamabe classification of rough metrics: The Yamabe problem on closed manifolds for rough metrics is still open; however, one can still get the following result [HNT09] which is all we need here:

Theorem 1 (Yamabe Classification of Rough Metrics)

Let $(\mathcal{M}, h)$ be a smooth, closed, connected Riemannian manifold with dimension $n \geqslant 3$ and with a metric $h \in W^{s,p}$, where we assume $sp > n$ and $s \geqslant 1$. Then, the followings hold:

- *$\mu_{2^\star} > 0$ iff there is a metric in $[h]$ with continuous positive scalar curvature.*
- *$\mu_{2^\star} = 0$ iff there is a metric in $[h]$ with vanishing scalar curvature.*
- *$\mu_{2^\star} < 0$ iff there is a metric in $[h]$ with continuous negative scalar curvature.*

In particular, two conformally equivalent metrics cannot have scalar curvatures with distinct signs.

Near-CMC condition and Contractions

Theorem: *(Isenberg-Moncrief) For case $R = -1$ on a closed manifold ($h_{ab} \in \mathcal{Y}^-$), strong smoothness assumptions, and near-CMC conditions, Isenberg-Moncrief show this is a contraction in Hölder spaces:*

$$[\phi^{(k+1)}, w^{(k+1)}] = G([\phi^{(k)}, w^{(k)}]).$$

Proof Outline: *Maximum principles, barriers, Banach algebra properties, plus contraction-mapping argument.* $\square$

Recall now the:

Theorem 2 (Contraction Mapping Theorem)

Let X be Banach and $U \subset X$ nonempty & closed. If $G : X \to X$ is a k-contraction on U:

$$\|G(u) - G(v)\|_X \leq k\|u - v\|_X, \quad 0 \leqslant k < 1, \quad \forall u, v \in U,$$

then there exists a (unique) fixed-point $u \in U \subset X$ satisfying $u = G(u)$.

Impact of the near-CMC restriction

To establish contraction properties for coupled PDE systems gives coupling restrictions; for the constraints, the restriction that results is the near-CMC condition:

$$\|\nabla\tau\|_r < C \inf_{\mathcal{M}} |\tau|, \tag{2}$$

where particular L^r norm depends on context. Condition appears in two distinct places:

(1) Construction of the contraction G,
(2) Construction of the set U on which G is a contraction.

The near-CMC condition is basically a condition that ensures the coupling between the two equations is weak.

Partial answer to Open Question #1: In [HNT08, HNT09, Max09], new analysis framework is developed that is free of the near-CMC condition.

No limit on strength of coupling between equations, therefore establishing existence for broader set of physical situations.

Partial answer to Open Question #2: The results in [HNT09] also extend CMC results in [Max05a] to the non-CMC case, allowing for “roughest” possible solutions to the constraints in the non-CMC case.

Building New Framework: Mappings S and T

We now build a different near-CMC-free fixed-point argument.

We first make precise the definitions of the maps S and T.

To deal with the non-trivial kernel that exists for L on closed manifolds, fix an arbitrary positive *shift* $s > 0$. Now write the constraints as

$$L_s\phi + F_s(\phi, w) = 0, \tag{3}$$

$$(\mathbb{L}w)^a + \mathbb{F}(\phi)^a = 0, \tag{4}$$

where $L_s : W^{2,p} \to L^p$ and $\mathbb{L} : W^{2,p} \to L^p$ are defined as

$$L_s\phi := [-\Delta + s]\phi, \qquad (\mathbb{L}w)^a := -\nabla_b(\mathcal{L}w)^{ab},$$

and where $F_s : [\phi_-, \phi_+] \times W^{2,p} \to L^p$ and $\mathbb{F} : [\phi_-, \phi_+] \to L^p$ are

$$F_s(\phi, w) := [a_R - s]\phi + a_\tau\phi^5 - a_w\phi^{-7} - a_\rho\phi^{-3}, \qquad \mathbb{F}(\phi)^a := b_\tau^a\phi^6 + b_j^a.$$

Introduce the operators $S : [\phi_-, \phi_+] \to W^{2,p}$ and $T : [\phi_-, \phi_+] \times W^{2,p} \to W^{2,p}$ as

$$S(\phi) := -\mathbb{L}^{-1}\mathbb{F}(\phi), \tag{5}$$

$$T(\phi, w) := -L_s^{-1}F_s(\phi, w). \tag{6}$$

Both maps are well-defined when $s > 0$ (L_s is invertible) and when there are no conformal Killing vectors ($\mathbb{L}$ is invertible).

Schauder Approach to get at non-CMC

Alternatives to Contraction Mapping Theorem that are more topological:

Theorem 3 (Schauder Theorem)

Let X be a Banach space, and let $U \subset X$ be a non-empty, convex, closed, bounded subset. If $G : U \to U$ is a compact operator, then there exists a fixed-point $u \in U$ such that $u = G(u)$.

Here is a variation of Schauder tuned for the constraints.

Theorem 4 (Coupled Schauder Theorem)

Let X and Y be Banach spaces, and let Z be a Banach space with compact embedding $X \hookrightarrow Z$. Let $U \subset Z$ be non-empty, convex, closed, bounded, and let $S : U \to \mathcal{R}(S) \subset Y$ and $T : U \times \mathcal{R}(S) \to U \cap X$ be continuous maps. Then, there exist $w \in \mathcal{R}(S)$ and $\phi \in U \cap X$ such that

$$\phi = T(\phi, w) \quad \text{and} \quad w = S(\phi). \tag{7}$$

Proof Outline: *Show $G(\phi) = i \circ T(\phi, S(\phi)) : U \subset Z \to U \subset Z$ is compact and then use Schauder, where $i : X \to Z$ is (compact) canonical injection.* $\square$

Global barriers and *a priori* L^∞-bounds

To remove the near-CMC condition we will use the following approach:

- Compactness-type fixed-point arguments (Coupled Schauder).
- Identifying the non-empty, convex, closed, bounded set U.
- Establishing properties of the constraint maps S and T.

Note: Establishing continuity of maps S and T, identifying the set U, and establishing convergence/optimality of numerical methods, will ALL depend on construction of compatible global barriers ϕ_- and ϕ_+ that are free of the near-CMC condition. (Compatibility: $0 \leqslant \phi_- \leqslant \phi_+$)

Sub- and super-solutions, or barriers to HC satisfy:

$$-\Delta\phi_- + a_R\phi_- + a_\tau\,\phi_-^5 - a_w\,\phi_-^{-7} - a_\rho\,\phi_-^{-3} \leqslant 0,$$
$$-\Delta\phi_+ + a_R\phi_+ + a_\tau\,\phi_+^5 - a_w\,\phi_+^{-7} - a_\rho\,\phi_+^{-3} \geqslant 0.$$

Barriers related to *a priori* L^∞-bounds on any solution (if one exists):

$$0 < \alpha \leqslant \phi \leqslant \beta < \infty.$$

When nonlinearity monotone decreasing, can show barriers also *a priori* L^∞-bounds. (One can establish bounds directly; see [HNT09, Max09].)

Working in ordered Banach spaces; need for non-empty order-cone interval $U = [\phi_-, \phi_+]$ leads to concept of global barriers: Barriers for HC for any a_w generated from solutions w to MC with source $\phi \in [\phi_-, \phi_+]$.

Existence/estimates for momentum constraint

Assume for the moment we have global barriers (must still find them), and they give us (must verify) a non-empty, convex, closed, bounded subset $U \subset Z$ of the Banach space Z, and that in addition, we can show (must verify) that T is invariant on U.

To use the Coupled Schauder Theorem to establish existence, it would remain to establish continuity properties of momentum and Hamiltonian constraint mappings S and T. First consider S (see [HNT09, Max09]).

Theorem 5 (MC – Existence and Estimates)

Let $(\mathcal{M}, h_{ab})$ be a 3-dimensional, closed, C^2, Riemannian manifold, with h_{ab} having no conformal Killing vectors, and let $b^a_\tau, b^a_j \in L^p$ with $p \geqslant 2$ and $\phi \in L^\infty$; Then, equation (4) has a unique solution $w^a \in W^{2,p}$ with

$$c \, \|w\|_{2,p} \leqslant \|\phi\|^6_\infty \, \|b_\tau\|_p + \|b_j\|_p, \tag{8}$$

where $c > 0$ is a constant.

Proof Outline: *Korn inequalities (Gårding) + Riesz-Schauder theory.* □

Generalizations appear in [HNT09], allowing rougher metric and coefficients, giving existence down to $w^a \in W^{1,p}$, with real $p \geqslant 2$.

Key inequalities for momentum constraint

Under the assumption that any $\phi \in L^\infty$ appearing as the source in the momentum constraint equation (4) satisfies for some compatible barriers $0 < \phi_- \leqslant \phi_+ < \infty$

$$\phi \in U = [\phi_-, \phi_+] \subset L^\infty,$$

then one can establish continuity of S (see [HNT09, Max09]). One can also show stronger boundedness and Lipschitz properties:

$$\|S(\phi)\|_Y \leqslant C_{SB}, \qquad \|S(\phi_1) - S(\phi_2)\|_Y \leqslant C_{SL}\|\phi_1 - \phi_2\|_Z,$$
$$Y = W^{2,p}, \qquad Z = L^\infty.$$

The inequality in equation (8) also gives for $p > 3$ the following estimate:

$$a_{\mathbf{w}} \leqslant K_1 \|\phi\|_\infty^{12} + K_2, \tag{9}$$

with $K_1 = (\frac{c_s c_{\mathcal{L}}}{\sqrt{2c}})^2 \|b_\tau\|_p^2$, $K_2 = \frac{1}{4}\|\sigma\|_\infty^2 + (\frac{c_s c_{\mathcal{L}}}{\sqrt{2c}})^2 \|b_j\|_p^2$, where c_s is the constant in the embedding $W^{1,p} \hookrightarrow L^\infty$, and $c_{\mathcal{L}}$ is a bound on the norm of $\mathcal{L} : W^{2,p} \to W^{1,p}$.

Inequality (9) will appear in a critical part of the analysis of the coupling between the two equations. Note that there is no smallness assumption on $\|b_\tau\|_p$, so the near-CMC condition is not required for these results.

Existence/estimates for Hamiltonian constraint

Turn now to Hamiltonian map T. From e.g. [HNT09, Max09] we have

Theorem 6 (HC – Existence and Estimates)

Let $(\mathcal{M}, h_{ab})$ be a 3-dimensional, C^2, closed Riemannian manifold. Let free data τ^2, σ^2 and ρ be in L^p, with $p \geqslant 2$. Let ϕ_- and ϕ_+ be barriers to (3) for particular vector $w^a \in W^{1,2p}$. Then, there exists solution $\phi \in [\phi_-, \phi_+] \cap W^{2,p}$ of HC (3). Furthermore, if metric h_{ab} in positive Yamabe class, then ϕ is unique.

Proof Outline: *Barriers plus monotone increasing maps.* □

Generalizations appear in [HNT09], allowing rougher metric and coefficients, giving existence down to $\phi \in W^{1,p}$, with real $p \geqslant 2$.

This result, together MC results above and barrier results below, give required continuity properties for map T (see [HNT09, Max09] for details). One can show stronger boundedness and Lipschitz conditions:

$$\|T(\phi, w)\|_X \leqslant C_{TB}, \qquad \|T(\phi_1, w) - T(\phi_2, w)\|_X \leqslant C_{TL}\|\phi_1 - \phi_2\|_Z,$$

$$\|T(\phi, w_1) - T(\phi, w_2)\|_X \leqslant C_{TLW}\|w_1 - w_2\|_Y,$$

$$X = W^{2,p}, Y = W^{2,p}, \qquad Z = L^\infty.$$

Construction of the nonempty closed set U

Remaining assumptions for use of the Coupled Schauder Theorem are

(A) Let $U \subset Z$ be non-empty, convex, closed, and bounded (w.r.t. vector space, topological space, normed space structure of Z).

(B) T is invariant on U.

We take $U = [\phi_-, \phi_+]_{t,q} \cap \overline{B}_R(0)$, for appropriate $t \geqslant 0$, $1 \leqslant q \leqslant \infty$, where $\overline{B}_R(0)$ is closed ball in Z of radius R about 0, and verify (A).

For brevity denote $[\phi_-, \phi_+]_q = [\phi_-, \phi_+]_{0,q}$, and $[\phi_-, \phi_+] = [\phi_-, \phi_+]_{0,\infty}$.

Lemma 7 (Order cone intervals in $W^{t,q}$)

For $t \geqslant 0$, $1 \leqslant p \leqslant \infty$, the set

$$U = [\phi_-, \phi_+]_{t,q} = \{\phi \in W^{t,q} : \phi_- \leqslant \phi \leqslant \phi_+\} \subset W^{t,q}$$

is convex with respect to the vector space structure of $W^{t,q}$ and closed in the topology of $W^{t,q}$. For $t = 0$, $1 \leqslant p \leqslant \infty$, the set U is also bounded with respect to the metric space structure of $L^q = W^{0,q}$.

Proof Outline: *Convexity straightforward; closedness follows since norm convergence in L^q, $1 \leqslant q \leqslant \infty$, implies pointwise subsequential convergence a.e., and from continuous embedding $W^{t,q} \hookrightarrow L^q$ for $t > 0$; boundedness when $t = 0$ since order cone L^q_+ is normal.* $\square$

Invariance of T on U

"Global" property of barriers ensures T invariant on $[\phi_-, \phi_+]_{\tilde{s},\tilde{p}}$. Barrier compatibility ensures interval non-empty, convex, and closed.

In smooth case can take $s = 0$, then $U = [\phi_-, \phi_+]_{0,\tilde{p}}$ bounded, since order cone structure on $L^{\tilde{p}}$ is normal.

In weak metric case $h_{ab} \in W^{s,p}$, S and T not continuous for $Z = L^\infty$, and must take $Z = W^{\tilde{s},\tilde{p}}$ to get continuity of S and T, then deal with non-normal order structure on Z. (closed intervals not bounded).

For $\tilde{s} > 0$, must then take $U = [\phi_-, \phi_+]_{\tilde{s},\tilde{p}} \cap \overline{B}_R$ to ensure U is bounded, where $\overline{B}_R$ is the closed ball in Z of radius R.

It remains then only to establish invariance of T on $\overline{B}_R$.

Lemma 8 (Invariance of T on $\overline{B}_R$.)

Assume $p \in (\frac{3}{2}, \infty)$, $s \in (\frac{3}{p}, \infty)$, that $a_{\mathbf{w}} \in W^{s-2,p}$, and that "suitable conditions" on the other data hold. Then, for any $\tilde{s} \in (\frac{3}{p}, s]$ and for some $t \in (\frac{3}{p}, \tilde{s})$ there exists a closed ball $\overline{B}_R \subset W^{\tilde{s},p}$ of radius $R = \mathcal{O}\left([1 + \|a_{\mathbf{w}}\|_{s-2,p}]^{\tilde{s}/(\tilde{s}-t)}\right)$, such that

$$\phi \in [\phi_-, \phi_+]_{\tilde{s},p} \cap \overline{B}_M \quad \Rightarrow \quad T^s(\phi, a_{\mathbf{w}}) \in \overline{B}_M.$$

Main Result 0: Far-CMC $W^{2,p}$ solutions, $p > 3$

Except barrier construction (must still find them), all results in place for applying Coupled Schauder Theorem to constraints. Next (smooth) result from [HNT08]; more general result from [HNT09] after.

Theorem 9 (Non-CMC existence without near-CMC)

Let $(\mathcal{M}, h_{ab})$ be a 3-dimensional, smooth, closed Riemannian manifold with metric h_{ab} in positive Yamabe class with no conformal Killing vectors. Let $\tau \in W^{1,p}$, with σ^2, j^a and ρ in L^p, with $p > 3$ and small enough norms as given in Global Super-Solution Lemma so global barriers ϕ_- and ϕ_+ exist for HC (3), with $\rho \not\equiv 0$. Then, there exists $\phi \in [\phi_-, \phi_+] \cap W^{2,p}$ and $w^a \in W^{2,p}$ solving constraint equations (3)-(4).

Proof Outline: We have the operators $S : [\phi_-, \phi_+] \to W^{2,p}$ and $T : [\phi_-, \phi_+] \times W^{2,p} \to W^{2,p}$ which are again given by

$$S(\phi) := -\mathbb{L}^{-1}\mathbb{F}(\phi), \quad T(\phi, w) := -L_s^{-1} F_s(\phi, w).$$

Note the mapping S is well-defined due to absence of conformal Killing vectors, ensuring $\mathbb{L}$ is invertible. Mapping T well-defined by use of positive shift $s > 0$, ensuring L_s also invertible (see [HNT09]).

Proof outline (continued)

The constraint equations in (3)–(4) thus have precisely the form (7) for use of the Coupled Schauder Theorem.

We have the reflexive Banach spaces $X = W^{2,p}$ and $Y = W^{2,p}$, and ordered Banach space $Z = L^\infty$ with normal order cone and compact embedding $W^{2,p} \hookrightarrow L^\infty$.

With our compatible barriers forming the L^∞-interval $U = [\phi_-, \phi_+]$, we have by construction that U is non-empty as a subset of L^p, for $1 \leqslant p \leqslant \infty$. As noted earlier, the interval $[\phi_-, \phi_+] \subset L^p$ is convex with respect to the vector space structure of L^p, closed in the topology of L^p, and bounded in the norm on L^p, for $1 \leqslant p \leqslant \infty$ (see [HNT09]).

It remains to show that S and T are continuous maps from their respective domains to their respective ranges, and that T is invariant on U. These properties follow from equation (8) and from the Hamiltonian constraint theorem, with global barriers from the Global barriers theorem, using standard inequalities. The result now follows from the Coupled Schauder Theorem. $\square$

Main Result 1: Far-CMC $W^{s,p}$ weak solutions

Main results in [HNT09] consist of following three more general theorems for weak solutions. In [Max09] Maxwell extends this result to the vacuum case.

Theorem 10 (Far-CMC $W^{s,p}$ solutions)

Let $(\mathcal{M}, h_{ab})$ be a 3-dimensional closed Riemannian manifold. Let $h_{ab} \in W^{s,p}$ admit no conformal Killing field and be in $\mathcal{Y}^+(\mathcal{M})$, where $p \in (1, \infty)$ and $s \in (1 + \frac{3}{p}, \infty)$ are given. Select q and e to satisfy:

- $\frac{1}{q} \in (0,1) \cap (0, \frac{s-1}{3}) \cap [\frac{3-p}{3p}, \frac{3+p}{3p}]$,
- $e \in (1 + \frac{3}{q}, \infty) \cap [s-1, s] \cap [\frac{3}{q} + s - \frac{3}{p} - 1, \frac{3}{q} + s - \frac{3}{p}]$.

Assume that the data satisfies:

- $\tau \in W^{e-1,q}$ *if* $e \geqslant 2$, *and* $\tau \in W^{1,z}$ *otherwise, with* $z = \frac{3q}{3+\max\{0,2-e\}q}$,
- $\sigma \in W^{e-1,q}$, *with* $\|\sigma^2\|_\infty$ *sufficiently small,*
- $\rho \in W^{s-2,p} \cap L^\infty_+ \setminus \{0\}$, *with* $\|\rho\|_\infty$ *sufficiently small,*
- $\boldsymbol{j} \in \boldsymbol{W}^{e-2,q}$, *with* $\|\boldsymbol{j}\|_{e-2,q}$ *sufficiently small.*

Then there exists $\phi \in W^{s,p}$ *with* $\phi > 0$ *and* $\boldsymbol{w} \in \boldsymbol{W}^{e,q}$ *solving the constraints.*

Remark: Weak metric $h_{ab} \in W^{s,p}$ requires verifying usual relationships for $W^{s,p}$ available; gives conditions on exponents s and p to ensure e.g. Laplace-Beltrami bilinear form is continuous. The construction in appendix of [HNT09] based on Besov spaces and partitions of unity.

Theorem 11 (Near-CMC $W^{s,p}$ solutions)

Let $(\mathcal{M}, h_{ab})$ be a 3-dimensional closed Riemannian manifold. Let $h_{ab} \in W^{s,p}$ admit no conformal Killing field, where $p \in (1,\infty)$ and $s \in (1+\frac{3}{p},\infty)$ are given. Select q, e and z to satisfy:

- $\frac{1}{q} \in (0,1) \cap (0, \frac{s-1}{3}) \cap [\frac{3-p}{3p}, \frac{3+p}{3p}]$,
- $e \in (1+\frac{3}{q},\infty) \cap [s-1,s] \cap [\frac{3}{q}+s-\frac{3}{p}-1, \frac{3}{q}+s-\frac{3}{p}]$.
- $z = \frac{3q}{3+\max\{0,2-e\}q}$.

Assume τ satisfies near-CMC condition (2) with z above, and data satisfies:

- $\tau \in W^{e-1,q}$ *if* $e > 2$, *and* $\tau \in W^{1,z}$ *if* $e \leqslant 2$,
- $\sigma \in W^{e-1,q}$,
- $\rho \in W_+^{s-2,p}$,
- $\boldsymbol{j} \in \boldsymbol{W}^{e-2,q}$.

In addition, let one of the following sets of conditions hold:

(a) h_{ab} *in* $\mathcal{Y}^-(\mathcal{M})$; h_{ab} *conformally equiv to metric w/ scalar curvature* $(-\tau^2)$;

(b) h_{ab} *in* $\mathcal{Y}^0(\mathcal{M})$ *or* $\mathcal{Y}^+(\mathcal{M})$; *either* $\rho \not\equiv 0$ *and* $\tau \not\equiv 0$ *or* $\tau \in L^\infty$ *and* $\inf_{\mathcal{M}} \sigma^2$ *suff. large.*

Then there exists $\phi \in W^{s,p}$ with $\phi > 0$ and $\boldsymbol{w} \in \boldsymbol{W}^{e,q}$ solving the constraints.

Theorem 12 (CMC $W^{s,p}$ solutions)

Let $(\mathcal{M}, h_{ab})$ be a 3-dimensional closed Riemannian manifold. Let $h_{ab} \in W^{s,p}$ where $p \in (1,\infty)$ and $s \in (\frac{3}{p},\infty) \cap [1,\infty)$ are given. With $d := s - \frac{3}{p}$, select q and e to satisfy:

- $\frac{1}{q} \in (0,1) \cap [\frac{3-p}{3p}, \frac{3+p}{3p}] \cap [\frac{1-d}{3}, \frac{3+sp}{6p})$,
- $e \in [1,\infty) \cap [s-1, s] \cap [\frac{3}{q} + d - 1, \frac{3}{q} + d] \cap (\frac{3}{q} + \frac{d}{2}, \infty)$.

Assume $\tau = \mathrm{const}$ (CMC) and that the data satisfies:

- $\sigma \in W^{e-1,q}$,
- $\rho \in W_{+}^{s-2,p}$,
- $\boldsymbol{j} \in \boldsymbol{W}^{e-2,q}$.

In addition, let one of the following sets of conditions hold:

(a) h_{ab} is in $\mathcal{Y}^{-}(\mathcal{M})$; $\tau \neq 0$;

(b) h_{ab} is in $\mathcal{Y}^{0}(\mathcal{M})$; $\rho \not\equiv 0$;

(c) h_{ab} is in $\mathcal{Y}^{+}(\mathcal{M})$; $\tau \neq 0$; $\rho \not\equiv 0$.

Then there exists $\phi \in W^{s,p}$ with $\phi > 0$ and $\boldsymbol{w} \in \boldsymbol{W}^{e,q}$ solving the Einstein constraints.

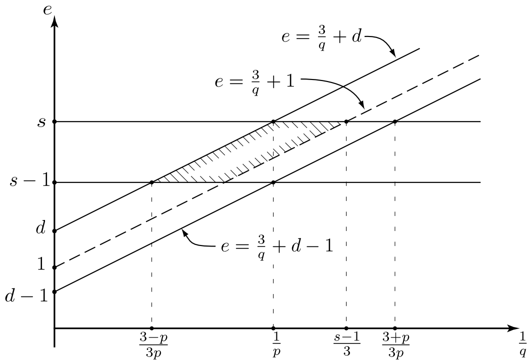

Figure : Range of e and q in Main Results 1 and 2, with $d = s - \frac{3}{p} > 1$.

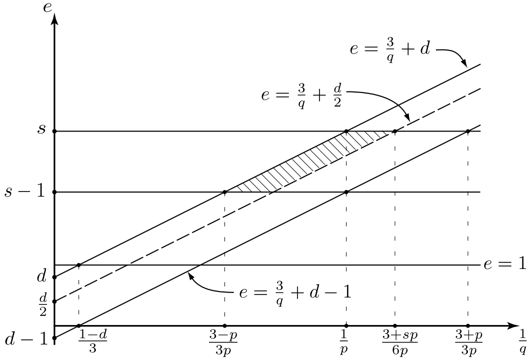

Figure : Range of e and q in Main Result 3. Recall that $d = s - \frac{3}{p} > 0$.

Sub-/super-solutions and *a priori* L^∞-bounds

Proofs of the results existence results were based on:

- Compactness-type fixed-point arguments (Coupled Schauder).
- Identifying a non-empty, convex, closed, bounded set U.
- Establishing continuity properties of constraint maps S and T.

Establishing continuity of maps S and T, identifying the set U, and establishing convergence/optimality of numerical methods, all depend on construction of compatible global barriers ϕ_- and ϕ_+ that are free of the near-CMC condition. (Compatibility: $0 \leqslant \phi_- \leqslant \phi_+$)

Sub- and super-solutions, or barriers to HC satisfy:

$$-\Delta\phi_- + a_R\phi_- + a_\tau\,\phi_-^5 - a_w\,\phi_-^{-7} - a_\rho\,\phi_-^{-3} \leqslant 0,$$
$$-\Delta\phi_+ + a_R\phi_+ + a_\tau\,\phi_+^5 - a_w\,\phi_+^{-7} - a_\rho\,\phi_+^{-3} \geqslant 0.$$

Barriers related to *a priori* L^∞-bounds on any solution (if one exists):

$$0 < \alpha \leqslant \phi \leqslant \beta < \infty.$$

When nonlinearity monotone decreasing, can show barriers also *a priori* L^∞-bounds. (One can establish bounds directly; see [HNT09, Max09].)

Working in ordered Banach spaces; need for non-empty order-cone interval $U = [\phi_-, \phi_+]$ leads to concept of global barriers: Barriers for HC for any a_w generated from solutions w to MC with source $\phi \in [\phi_-, \phi_+]$.

Near-CMC-free global barrier construction

Can one build Far-CMC barriers? $\rightarrow$ Yes [HNT08, HNT09, Max09].

Lemma 13 (Near-CMC-Free Global Super-Solution)

Let $(\mathcal{M}, h_{ab})$ be a 3-dimensional, smooth, closed Riemannian manifold with metric h_{ab} in the positive Yamabe class with no conformal Killing vectors. Let u be a smooth positive solution of the Yamabe problem

$$-\Delta u + a_R u - u^5 = 0, \tag{10}$$

and define the Harnack-type constant $k = u^\wedge / u^\vee$. If the function τ is non-constant and the rescaled matter sources j^a, ρ, and traceless transverse tensor σ^{ab} are sufficiently small, then

$$\phi_+ = \epsilon u, \quad \epsilon = \left[\frac{1}{2K_1 k^{12}}\right]^{\frac{1}{4}} \tag{11}$$

is a global super-solution of the Hamiltonian constraint.

Proof Outline: Using the notation

$$E(\phi_+) = -\Delta\phi_+ + a_R\phi_+ + a_\tau\phi_+^5 - a_w\phi_+^{-7} - a_\rho\phi_+^{-3},$$

we have to show $E(\phi_+) \geqslant 0$. The definition of $\phi_+ = \epsilon u$ implies $-\Delta\phi_+ + a_R\phi_+ = \epsilon\, u^5$.

Proof outline (continued)

Using an estimate for a_w (see [HNT09]), we have then

$$E(\phi_+) \geqslant -\Delta\phi_+ + a_R\phi_+ - \frac{K_1(\phi_+^\wedge)^{12} + K_2}{\phi_+^7} - \frac{a_\rho^\wedge}{\phi_+^3}$$
$$\geqslant \epsilon\, u^5 - K_1 \left[\frac{\phi_+^\wedge}{\phi_+^\vee}\right]^{12} \phi_+^5 - \frac{K_2}{\phi_+^7} - \frac{a_\rho^\wedge}{\phi_+^3}.$$

Notice that $\phi_+^\wedge/\phi_+^\vee = u^\wedge/u^\vee = k$, therefore we have

$$E(\phi_+) \geqslant \epsilon u^5 \left[1 - K_1\, k^{12}\epsilon^4 - \frac{K_2}{\epsilon^8 u^{12}} - \frac{a_\rho^\wedge}{\epsilon^4 u^8}\right].$$

Choice of ϵ made in (11) is equivalent to condition $1/2 = 1 - K_1\, k^{12}\epsilon^4$. For this ϵ, impose on the free data σ^{ab}, ρ and j^a the condition

$$\frac{1}{2} - \frac{K_2}{\epsilon^8 (u^\vee)^{12}} - \frac{a_\rho^\wedge}{\epsilon^4 (u^\vee)^8} \geqslant 0.$$

Thus for any $K_1 > 0$, we can guarantee $E(\phi_+) \geqslant 0$. □

Remarks:

Thus global super-solutions can be built by rescaling solutions to (10).
Existence of k related to Harnack inequality for Yamabe.
Compatible global sub-solutions available so that $0 < \phi_- \leqslant \phi_+$.

Now on to Compact Manifolds with Boundary

OK, let us now consider in more detail the case of boundaries, focusing (for now) primarily on the Lichnerowicz equation.

To allow my lecture to follow closely the paper [HT13], I change notation slightly from here on and refer to the spatial metric as g and $\hat{g}$ rather than h and $\hat{h}$; hatted quantities maintain their role.

To allow for a general discussion, assume the spatial dimension is $n \geqslant 3$; later we restrict to $n = 3$.

Let M be a compact manifold with boundary.
Let ϕ be a positive scalar field on M.
Decompose extrinsic curvature as $\hat{K} = \hat{S} + \tau\hat{g}$.
Here $\tau = \frac{1}{n}\mathrm{tr}_{\hat{g}}\hat{K}$ is (averaged) trace, so $\hat{S}$ is the traceless part.
With $\bar{q} = \frac{n}{n-2}$, conformal metric g and symmetric traceless S come via

$$\hat{g} = \phi^{2\bar{q}-2}g, \qquad \hat{S} = \phi^{-2}S. \tag{12}$$

Chosen powers give Lichnerowicz equation and momentum constraint:

$$-\frac{4(n-1)}{n-2}\Delta\phi + R\phi + n(n-1)\tau^2\phi^{2\bar{q}-1} - |S|_g^2\phi^{-2\bar{q}-1} = 0, \tag{13}$$

$$\mathrm{div}_g S - (n-1)\phi^{2\bar{q}}\mathrm{d}\tau = 0, \tag{14}$$

where $\Delta \equiv \Delta_g$ is the Laplace-Beltrami operator with respect to the metric g, and $R \equiv \mathrm{scal}_g$ is the scalar curvature of g.

Lichnerowicz, Compact with Boundary

Interpret (13)–(14) as PDE for ϕ and (part of) traceless symmetric S.

Metric g is considered as given.

To rephrase, given ϕ and S fulfilling (13)–(14), $\hat{g}$ and $\hat{K}$ given by

$$\hat{g} = \phi^{2\bar{q}-2} g, \qquad \hat{K} = \phi^{-2} S + \phi^{2\bar{q}-2} \tau g,$$

satisfy the Einstein constraint system.

$\hat{g}$ = *physical metric*
g = *conformal metric* (only specifies conformal class of $\hat{g}$, other info lost)

Assume now that traceless symmetric bilinear form S given.

Consider Lichnerowicz (13) on a compact manifold with boundary.

Boundaries emerge when one eliminates asymptotic ends or singularities from the manifold.

Need to impose appropriate boundary conditions for ϕ.

Approximating Asymptotically Flat Manifolds

On asymptotically flat manifolds, one has [YP82]

$$\phi = 1 + Ar^{2-n} + \varepsilon, \qquad \text{with} \quad \varepsilon = O(r^{1-n}), \quad \text{and} \quad \partial_r \varepsilon = O(r^{-n}), \tag{15}$$

where A is multiple total energy, r is the flat-space radial coordinate.

Idea is: cut out asymptotically Euclidean end along the sphere with large radius r and impose Dirichlet condition $\phi \equiv 1$ at boundary.

Improvement via differentiating (15) with respect to r and eliminating A:

$$\partial_r \phi + \frac{n-2}{r}(\phi - 1) = O(r^{-n}). \tag{16}$$

Equating right hand side to zero gives inhomogeneous Robin condition known to give accurate values for total energy.

Approximating Black Hole Data

Main approach: excise region around singularities and solve in exterior.

Such are "inner"-boundaries; again need boundary conditions.

In [YP82] they introduce

$$\partial_r \phi + \frac{n-2}{2a}\phi = 0, \qquad \text{for } r = a. \tag{17}$$

Means $r = a$ is a minimal surface; under appropriate data conditions minimal surface is a *trapped surface*.

Trapped surface important since implies existence of event horizon outside surface.

Various trapped surface conditions more general than minimal surface. in literature.

Make clear what we mean by a trapped surface.

Suppose all necessary regions (singularities, asymptotic ends) excised from initial slice,

Assume boundary $\Sigma := \partial M$ has finitely many components $\Sigma_1, \Sigma_2, \ldots$.

Trapped Surfaces

Let $\hat{\nu} \in \Gamma(T\Sigma^{\perp})$ be outward pointing unit normal (wrt $\hat{g}$).

Expansion scalars corresponding to outgoing and ingoing future directed null geodesics orthogonal to Σ are given by

$$\hat{\theta}_{\pm} = \mp(n-1)\hat{H} + \mathrm{tr}_{\hat{g}}\hat{K} - \hat{K}(\hat{\nu},\hat{\nu}), \tag{18}$$

where $(n-1)\hat{H} = \mathrm{div}_{\hat{g}}\hat{\nu}$ is the mean extrinsic curvature of Σ.

Surface Σ_i is called *trapped surface* if $\hat{\theta}_{\pm} < 0$ on Σ_i.
Called *marginally trapped surface* if $\hat{\theta}_{\pm} \leqslant 0$ on Σ_i.

In terms of the conformal quantities:

$$\hat{\theta}_{\pm} = \mp(n-1)\phi^{-\bar{q}}(\tfrac{2}{n-2}\partial_\nu\phi + H\phi) + (n-1)\tau - \phi^{-2\bar{q}}S(\nu,\nu), \tag{19}$$

where $\nu = \phi^{\bar{q}-1}\hat{\nu}$ is the unit normal with respect to g, and $\partial_\nu\phi$ is the derivative of ϕ along ν.

The mean curvature H with respect to g is related to $\hat{H}$ by

$$\hat{H} = \phi^{-\bar{q}}(\tfrac{2}{n-2}\partial_\nu\phi + H\phi). \tag{20}$$

Trapped Surfaces: Maxwell Approach

In [Max05b, Dai04], authors studied boundary conditions leading to trapped surfaces in the asymptotically flat and constant mean curvature ($\tau = \mathrm{const}$) setting.

Decay condition on $\hat{K}$ gives automatically $\tau \equiv 0$.

In [Max05b], boundary conditions obtained via setting $\hat{\theta}_+ \equiv 0$.

More generally, if one specifies *scaled expansion scalar* $\theta_+ := \phi^{\bar{q}-e}\hat{\theta}_+$ for some $e \in \mathbb{R}$, and poses no restriction on τ, then the (inner) boundary condition for the Lichnerowicz equation (13) can be given by

$$\frac{2(n-1)}{n-2}\partial_\nu \phi + (n-1)H\phi - (n-1)\tau\phi^{\bar{q}} + S(\nu,\nu)\phi^{-\bar{q}} + \theta_+\phi^e = 0. \quad (21)$$

Trapped Surfaces: Dain Approach

In [Dai04], boundary conditions obtained via specifying $\hat{\theta}_-$.

Similarly to Maxwell case, if generalize approach so that $\theta_- := \phi^{\bar{q}-e}\hat{\theta}_-$ is specified, then we get the (inner) boundary condition

$$\frac{2(n-1)}{n-2}\partial_\nu\phi + (n-1)H\phi + (n-1)\tau\phi^{\bar{q}} - S(\nu,\nu)\phi^{-\bar{q}} - \theta_-\phi^e = 0. \quad (22)$$

Note that in above , one of $\theta_\pm$ remains unspecified, so in order to guarantee that both $\theta_\pm \leqslant 0$, one has to impose some conditions on the data, e.g., on τ or on S.

Another option: rigidly specify both $\theta_\pm$; can eliminate S from (19) and get boundary condition

$$\frac{4(n-1)}{n-2}\partial_\nu\phi + 2(n-1)H\phi + (\theta_+ - \theta_-)\phi^e = 0. \quad (23)$$

At the same time, eliminating the term involving $\partial_\nu\phi$ from (19) we get a boundary condition on S that reads as

$$2S(\nu,\nu) = 2(n-1)\tau\phi^{2\bar{q}} - (\theta_+ + \theta_-)\phi^{e+\bar{q}}. \quad (24)$$

We see something interesting: the Lichnerowicz equation couples to the momentum constraint (14) through the boundary conditions.

Even in constant mean curvature setting (where $\tau \equiv \mathrm{const}$), constraint equations (13)–(14) generally do *not* decouple.

The only reasonable way to decouple the constraints is to consider $\tau \equiv 0$ and $e = -\bar{q}$.
Note that all boundary conditions considered above (except Dirichlet) are of form:

$$\partial_\nu \phi + b_H \phi + b_\theta \phi^e + b_\tau \phi^{\bar{q}} + b_w \phi^{-\bar{q}} = 0. \tag{25}$$

Eg., in (21) and (22), one has $b_H = \frac{n-2}{2} H$, $b_\theta = \pm \frac{n-2}{2(n-1)} \theta_\pm$, $b_\tau = \mp \frac{n-2}{2} \tau$, and $b_w = \pm \frac{n-2}{2(n-1)} S(\nu, \nu)$.

Minimal surface condition (17) corresponds to the choice $b_\theta = b_\tau = b_w = 0$, and $b_H = \frac{n-2}{2} H$.

The outer Robin condition (16) is $b_H = (n-2)H$, $b_\theta = -(n-2)H$ with $e = 0$, and $b_\tau = b_w = 0$.

The Setup: General BVP for Lichnerowicz

Here we suppose each boundary component Σ_i has either Dirichlet condition $\phi \equiv 1$ or the Robin condition (25) enforced.

In particular, we allow the situation where no Dirichlet condition is imposed anywhere.

Also, to allow linear Robin condition (16) and a nonlinear condition like (21) at same time, must allow exponent e in (25) to be only locally constant.

Main tools used in paper are order-preserving maps iteration together with maximum principles and some results from conformal geometry.

These techniques sensitive to signs of coefficients in (25).

Defocusing case (preferred signs): $(e-1)b_\theta \geqslant 0$, $b_\tau \geqslant 0$, and $b_w \leqslant 0$.

Non-Defocusing case: Otherwise.

Results for defocusing case (terminology motivated by dispersive equations) more or less complete (see below).

Summary of Main Results in [HT13]

The main results and supporting tools appearing in [HT13] are:

- Justification of Yamabe classification of rough metrics on compact manifolds with boundary.
- A basic result on conformal invariance of the Lichnerowicz equation.
- A uniqueness result for the Lichnerowicz equation.
- An order-preserving maps theorem for manifolds with boundary.
- Construction of upper and lower barriers that respect the trapped surface conditions.
- Combination of the results above to produce a fairly complete existence and uniqueness theory for the defocusing case.
- Combination of the results above to produce some partial results for the non-defocusing case.
- Some perturbation results (looking ahead to the asymptotically Euclidean case).

Yamabe Classes: Rough/Compact/Boundary

Yamabe classification of rough metrics: The Yamabe problem for rough metrics on compact manifolds with boundary is again still open; the work [Esc92, Esc96] was for smooth metrics. However, as in the closed case, one can still get the following result [HT13] which is all we need:

Theorem 14 (Yamabe Classification of Rough Metrics)

Let (M, g) be a smooth, compact, connected Riemannian manifold with boundary, where we assume that the components of the metric g are (locally) in $W^{s,p}$, with $sp > n$ and $s \geqslant 1$. Let the dimension of M be $n \geqslant 3$. Then, the following are equivalent:

a) $\mathcal{Y}_g > 0$ ($\mathcal{Y}_g = 0$ or $\mathcal{Y}_g < 0$).

b) $\mathcal{Y}_g(q, r, b) > 0$ (resp. $\mathcal{Y}_g(q, r, b) = 0$ or $\mathcal{Y}_g(q, r, b) < 0$) for any $q \in [2, 2\bar{q})$, $r \in [2, \bar{q} + 1)$ with $q > r$, and any $b \in \mathbb{R}$.

c) *There is a metric in $[g]$ whose scalar curvature is continuous and positive (resp. zero or negative), and boundary mean curvature is continuous and has any given sign (resp. is identically zero, has any given sign).*

In particular, two conformally equivalent metrics cannot have scalar curvatures with distinct signs.

Conformal Invariance

Let M be smooth, compact, connected n-dimensional manifold with boundary, equipped with a Riemannian metric $g \in W^{s,p}$, $n \geqslant 3$, $p \in (1, \infty)$, and that $s \in (\frac{n}{p}, \infty) \cap [1, \infty)$.

We consider following model for Lichnerowicz problem

$$F(\phi) := \begin{pmatrix} -\Delta\phi + \frac{n-2}{4(n-1)} R\phi + a\phi^t \\ \gamma_N \partial_\nu \phi + \frac{n-2}{2} H \gamma_N \phi + b(\gamma_N \phi)^e \\ \gamma_D \phi - c \end{pmatrix} = 0,$$

where $t, e \in \mathbb{R}$ constants, $R \in W^{s-2,p}(M)$ and $H \in W^{s-1-\frac{1}{p},p}(\Sigma)$ are scalar and mean curvatures of metric g, and other coefficients satisfy $a \in W^{s-2,p}(M)$, $b \in W^{s-1-\frac{1}{p},p}(\Sigma_N)$, and $c \in W^{s-\frac{1}{p},p}(\Sigma_D)$.

Setting $\bar{q} = \frac{n}{n-2}$, interested in transformation properties of F under conformal change $\tilde{g} = \theta^{2\bar{q}-2} g$ with factor $\theta \in W^{s,p}(M)$ satisfying $\theta > 0$.

Conformal Invariance

To this end, we consider

$$\tilde{F}(\psi) := \begin{pmatrix} -\tilde{\Delta}\psi + \frac{n-2}{4(n-1)}\tilde{R}\psi + \tilde{a}\psi^t \\ \gamma_N \partial_{\tilde{\nu}}\psi + \frac{n-2}{2}\tilde{H}\gamma_N\psi + \tilde{b}(\gamma_N\psi)^e \\ \gamma_D\psi - \tilde{c} \end{pmatrix} = 0,$$

where $\tilde{\Delta}$ is Laplace-Beltrami operator associated to metric $\tilde{g}$, $\tilde{\nu}$ is the outer normal to Σ with respect to $\tilde{g}$, $\tilde{R} \in W^{s-2,p}(M)$ and $\tilde{H} \in W^{s-1-\frac{1}{p},p}(\Sigma)$ are respectively the scalar and mean curvatures of $\tilde{g}$, and $\tilde{a} \in W^{s-2,p}(M)$, $\tilde{b} \in W^{s-1-\frac{1}{p},p}(\Sigma_N)$, and $\tilde{c} \in W^{s-\frac{1}{p},p}(\Sigma_D)$.

The result we need in this direction is the following [HT13].

Lemma 15 (Conformal Invariance)

Let $\tilde{a} = \theta^{t+1-2\bar{q}}a$, $\tilde{b} = \theta^{e-\bar{q}}b$, and $\tilde{c} = \theta^{-1}c$. Then we have

$$\begin{aligned} \tilde{F}(\psi) = 0 &\quad\Leftrightarrow\quad F(\theta\psi) = 0, \\ \tilde{F}(\psi) \geqslant 0 &\quad\Leftrightarrow\quad F(\theta\psi) \geqslant 0, \\ \tilde{F}(\psi) \leqslant 0 &\quad\Leftrightarrow\quad F(\theta\psi) \leqslant 0. \end{aligned}$$

Uniqueness Results

The conformal invariance result implies the following uniqueness result for the model Lichnerowicz problem [HT13].

Lemma 16 (Uniqueness 1)

Let the coefficients of the model Lichnerowicz problem satisfy $(t-1)a \geqslant 0$, $(e-1)b \geqslant 0$, and $c > 0$. If the positive functions $\theta, \phi \in W^{s,p}(M)$ are distinct solutions of the constraint, i.e., $F(\theta) = F(\phi) = 0$, and $\theta \neq \phi$, then $(t-1)a = 0$, $(e-1)b = 0$, $\Sigma_D = \varnothing$, and the ratio θ/ϕ is constant. If in addition, $t \neq 1$, then $\mathcal{Y}_g = 0$.

The following theorem essentially says that in order to have multiple positive solutions the Lichnerowicz problem must be a linear pure Robin boundary value problem on a conformally flat manifold [HT13].

Theorem 17 (Uniqueness 2)

Let the coefficients of the Lichnerowicz problem satisfy $a_\tau \geqslant 0$, $a_w \geqslant 0$, $(e-1)b_\theta \geqslant 0$, $b_\tau \geqslant 0$, $b_w \leqslant 0$, and $\phi_D > 0$. Let the positive functions $\theta, \phi \in W^{s,p}(M)$ be solutions of the Lichnerowicz problem, with $\theta \neq \phi$. Then $a_\tau = a_w = 0$, $(e-1)b_\theta = b_\tau = b_w = 0$, $\Sigma_D = \varnothing$, the ratio θ/ϕ is constant, and $\mathcal{Y}_g = 0$.

Order-Preserving Maps Theorem

Let us write our problem in the form:

$$F(\phi) := \begin{pmatrix} -\Delta\phi + f(\phi) \\ \gamma_N \partial_\nu \phi + h(\phi) \\ \gamma_D \phi - \phi_D \end{pmatrix} = 0.$$

Say ψ is *super-solution* if $F(\psi) \geqslant 0$, and *sub-solution* if $F(\psi) \leqslant 0$, component-wise.

The following theorem from [HT13] extends the standard argument used for closed manifolds (cf. [Ise95, Max05a]) to manifolds with boundary; note that the required sub- and super-solutions need only satisfy inequalities in both the interior and on the boundary.

Theorem 18 (Order-Preserving Maps w/ Boundaries)

Suppose that the signs of the coefficients a_τ, a_w, b_θ, b_τ, b_w, and $b_H - \frac{n-2}{2}H$ are locally constant, and let $\phi_D > 0$. Let $\phi_-, \phi_+ \in W^{s,p}(M)$ be respectively sub- and super-solutions satisfying $0 < \phi_- \leqslant \phi_+$. Then there exists a positive solution $\phi \in [\phi_-, \phi_+]_{s,p}$ to the Lichnerowicz problem.

We start with metrics with nonnegative Yamabe invariant. In the following theorem from [HT13], the symbol $\vee$ denotes the logical disjunction (or logical OR).

Theorem 19 (Existence - Defocusing and $\mathcal{Y}_g \geqslant 0$)

Let $\mathcal{Y}_g \geqslant 0$. Let the coefficients of the Lichnerowicz problem satisfy $a_\tau \geqslant 0$, $a_w \geqslant 0$, $b_H \geqslant \frac{n-2}{2}H$, $(e-1)b_\theta \geqslant 0$ with $e \neq 1$, $b_\tau \geqslant 0$, $b_w \leqslant 0$, and $\phi_D > 0$. Then there exists a positive solution $\phi \in W^{s,p}(M)$ of the Lichnerowicz problem if and only if one of the following conditions holds:

a) $\Sigma_D \neq \varnothing$;

b) $\Sigma_D = \varnothing$, $b_\theta = 0$, $\left(\mathcal{Y}_g > 0 \vee a_\tau \neq 0 \vee b_H \neq \frac{n-2}{2}H \vee b_\tau \neq 0\right)$, and $(a_w \neq 0 \vee b_w \neq 0)$;

c) $\Sigma_D = \varnothing$, $b_\theta \neq 0$, $b_\theta \geqslant 0$, and $(a_w \neq 0 \vee b_w \neq 0)$;

d) $\Sigma_D = \varnothing$, $b_\theta \neq 0$, $b_\theta \leqslant 0$, and $\left(\mathcal{Y}_g > 0 \vee a_\tau \neq 0 \vee b_H \neq \frac{n-2}{2}H \vee b_\tau \neq 0\right)$;

e) $\Sigma_D = \varnothing$, $b_\theta = b_\tau = b_w = 0$, $b_H = \frac{n-2}{2}H$, $a_\tau = a_w = 0$, and $\mathcal{Y}_g = 0$.

The next theorem from [HT13] treats metrics with negative Yamabe invariant, and reduces the Lichnerowicz problem into a prescribed scalar curvature problem.

Theorem 20 (Existence - Defocusing and $\mathcal{Y}_g < 0$)

Let $\mathcal{Y}_g < 0$. Let the coefficients of the Lichnerowicz problem satisfy $a_\tau \geqslant 0$, $a_w \geqslant 0$, $b_H \leqslant \frac{n-2}{2}H$, $(e-1)b_\theta \geqslant 0$ with $e \neq 1$, $b_\tau \geqslant 0$, $b_w \leqslant 0$, and $\phi_D > 0$. Then there exists a positive solution $\phi \in W^{s,p}(M)$ of the Lichnerowicz problem if and only if there exists a positive solution $u \in W^{s,p}(M)$ to the following problem

$$\begin{aligned} -\Delta u + a_R u + a_\tau u^{2\bar{q}-1} &= 0, \\ \gamma_N \partial_\nu u + b_H u + b_\tau u^{\bar{q}} + b_\theta^+ u^e &= 0, \\ \gamma_D u &= 1, \end{aligned} \tag{26}$$

where $b_\theta^+ = \max\{0, b_\theta\}$.

There are also partial results in [HT13] for the non-defocusing case, but will not be outlined in this talk.

Non-CMC case: Main Results in [HMT]

What about the non-CMC case?

In fact, even the CMC case was not yet discussed; this is because the CMC assumption does not actually decouple the constraints due to the boundary coupling, and we have only solved the Lichnerowicz equation.

The extension of the results in [HT13] to the non-CMC (far, near, and also CMC itself) is considered in [HMT].

Some of the main results appearing in [HMT] are:

- Number of necessary supporting results for momentum constraint that were not needed for pure Lichnerowicz case in [HT13].
- Construction of upper and lower barriers that respect trapped surface conditions in coupled setting (delicate boundary coupling).
- Combination of Schauder argument from [HNT09] with results for Lichnerowicz equation from [HT13] to give existence results for near-CMC and far-CMC data, analogous to known results for closed manifolds.
- CMC case comes as (still coupled) special case of near-CMC result.

For details: Please come to Caleb Meier's talk tomorrow (Friday Oct. 11) down the Hill in Evans Hall (3pm?)

Relevant Manuscripts and References

The two talks this week at MSRI (Holst) and in Evans Hall (Meier):

[HNT08] MH, G. Nagy, and G. Tsogtgerel, *Far-from-constant mean curvature solutions of Einstein's constraint equations with positive Yamabe metrics*, Phys. Rev. Lett. **100** (2008), no. 16, 161101.1–161101.4, Available as arXiv:0802.1031[gr-qc].

[HNT09] MH, G. Nagy, and G. Tsogtgerel, *Rough solutions of the Einstein constraints on closed manifolds without near-CMC conditions*, Comm. Math. Phys. **288** (2009), no. 2, 547–613, Available as arXiv:0712.0798[gr-qc].

[HT13] MH and G. Tsogtgerel, *The Lichnerowicz equation on compact manifolds with boundary*, Class. Quantum Grav., 30 (2013), p. 205011. Available as arXiv:1306.1801 [gr-qc].

[HMT] MH, C. Meier, and G. Tsogtgerel, *Non-CMC solutions of the Einstein constraint equations on compact manifolds with apparent horizon boundaries.* Submitted for publication. Available as arXiv:1310.2302 [gr-qc].

The talk at MSRI in November (Meier):

[HMb] MH and C. Meier, *Non-uniqueness of solutions to the conformal formulation.* Submitted for publication. Available as arXiv:1210.2156 [gr-qc].

[HMa] MH and C. Meier, *An alternative between non-unique and negative Yamabe solutions to the conformal formulation of the Einstein constraint equations.* Submitted for publication. Available as arXiv:1306.1210 [gr-qc].

Other work complementing ours (more related to Meier talk tomorrow):

[Dil] J. Dilts. The Einstein constraint equations on compact manifolds with boundary. Preprint.

Our work builds on a very large literature, but in particular:

[Ise95] J. Isenberg. Constant mean curvature solutions of the Einstein constraint equations on closed manifolds. *Classical Quantum Gravity*, 12(9):2249–2274, 1995.

[IM96] J. Isenberg and V. Moncrief. A set of nonconstant mean curvature solution of the Einstein constraint equations on closed manifolds. *Class. Quantum Grav.*, 13:1819–1847, 1996.

[Max04] D. Maxwell. *Initial Data for Black Holes and Rough Spacetimes*. PhD thesis, University of Washington, 2004.

[Max05a] D. Maxwell. Rough solutions of the Einstein constraint equations on compact manifolds. *J. Hyp. Diff. Eqs.*, 2(2):521–546, 2005.

[Max05b] D. Maxwell. Solutions of the Einstein constraint equations with apparent horizon boundaries. *Comm. Math. Phys.*, 253(3):561–583, 2005.

[Max06] D. Maxwell. Rough solutions of the Einstein constraint equations. *J. Reine Angew. Math.*, 590:1–29, 2006.

[Max09] D. Maxwell. A class of solutions of the vacuum Einstein constraint equations with freely specified mean curvature. *Math. Res. Lett.*, 16(4):627–645, 2009.

[Dai04] S. Dain. Trapped surfaces as boundaries for the constraint equations. *Classical Quantum Gravity*, 21(2):555–573, 2004.

[Dai06] S. Dain. Generalized Korn's inequality and conformal Killing vectors. *Calc. Var.*, 25(4):535–540, 2006.

[YP82] J. York and T. Piran. The initial value problem and beyond. In R. A. Matzner and L. C. Shepley, editors, *Spacetime and Geometry: The Alfred Schild Lectures*, pages 147–176. University of Texas Press, Austin (Texas), 1982.

[Esc92] J. F. Escobar. The Yamabe problem on manifolds with boundary. *J. Differential Geom.*, 35(1):21–84, 1992.

[Esc96] J. F. Escobar. Conformal deformation of a Riemannian metric to a constant scalar curvature metric with constant mean curvature on the boundary. *Indiana Univ. Math. J.*, 45(4):917–943, 1996.

References in Alphabetical Order

[Dai04] S. Dain, *Trapped surfaces as boundaries for the constraint equations*, Classical Quantum Gravity **21** (2004), no. 2, 555–573. MR MR2030884 (2004m:83060)

[Dai06] S. Dain, *Generalized Korn's inequality and conformal Killing vectors*, Calc. Var. **25** (2006), no. 4, 535–540, Available as gr-qc/0505022.

[Dil] J. Dilts, *The Einstein constraint equations on compact manifolds with boundary*, Preprint.

[Esc92] J. F. Escobar, *The Yamabe problem on manifolds with boundary*, J. Differential Geom. **35** (1992), no. 1, 21–84. MR 1152225 (93b:53030)

[Esc96] J. F. Escobar, *Conformal deformation of a Riemannian metric to a constant scalar curvature metric with constant mean curvature on the boundary*, Indiana Univ. Math. J. **45** (1996), no. 4, 917–943. MR MR1444473 (98d:53051)

[HMa] M. Holst and C. Meier, *An alternative between non-unique and negative yamabe solutions to the conformal formulation of the einstein constraint equations*, Submitted for publication. Available as arXiv:1306.1210 [gr-qc].

[HMb] ______, *Non-uniqueness of solutions to the conformal formulation*, Submitted for publication. Available as arXiv:1210.2156 [gr-qc].

[HMT] M. Holst, C. Meier, and G. Tsogtgerel, *Non-CMC solutions of the Einstein constraint equations on compact manifolds with apparent horizon boundaries*, Submitted for publication. Available as arXiv:1310.2302 [gr-qc].

[HNT08] M. Holst, G. Nagy, and G. Tsogtgerel, *Far-from-constant mean curvature solutions of Einstein's constraint equations with positive Yamabe metrics*, Phys. Rev. Lett. **100** (2008), no. 16, 161101.1–161101.4, Available as arXiv:0802.1031 [gr-qc].

[HNT09] ______, *Rough solutions of the Einstein constraints on closed manifolds without near-CMC conditions*, Comm. Math. Phys. **288** (2009), no. 2, 547–613, Available as arXiv:0712.0798 [gr-qc].

[HT13] M. Holst and G. Tsogtgerel, *The Lichnerowicz equation on compact manifolds with boundary*, Class. Quantum Grav. **30** (2013), 205011, Available as arXiv:1306.1801 [gr-qc].

[IM96] J. Isenberg and V. Moncrief, *A set of nonconstant mean curvature solution of the Einstein constraint equations on closed manifolds*, Class. Quantum Grav. **13** (1996), 1819–1847.