

# NOTES ON THE SLIDES OF THE DD20 TUTORIAL

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**1. Introduction.** This tutorial was offered on the UCSD campus on February 6, 2011, the day before the first day of the 20th International Conference on Domain Decomposition Methods; see <http://ccom.ucsd.edu/dd20>. These notes provide references and some comments on the relevant literature, in particular, to the sections of the monograph [18] and articles relevant to the material covered. The questions formulated on the slides are also discussed.

We note that the presentation is strongly colored by the research of the author, his students, and collaborators. The topics chosen were developed while the author taught a short course on domain decomposition algorithms at the Chinese University of Hong Kong in Fall 2010 while enjoying the great hospitality of its Mathematics Department.

A number of slides have been rewritten following suggestions by Clark Dohrmann, Axel Klawonn, Luca Pavarino, Oliver Rheinbach, and Daniel Szyld; the author wishes to thank them all for their assistance.

**2. Comments on the slides and topics covered.** The tutorial begins with a discussion of the structure of finite element problems on how to build a full finite element model from those for two subdomains into which the domain has been subdivided. The Schur complements of the subproblems play an important role in particular in the definition and analysis of a number of domain decomposition algorithms. Several important results on Sobolev spaces are also introduced in this first part, which ends with a discussion of what is now often called the multiplicative Schwarz methods.

The alternative additive Schwarz algorithms are also introduced and what is called the abstract Schwarz theory is developed although not in its general form. The next important topic is the two-level additive Schwarz method for which the main part of the theory is discussed at some length.

The next main part concerns linear elasticity problems and how coarse spaces can be developed and analyzed for quite general subdomains such as those obtained from mesh partitioning tools. This part ends with a discussion of almost incompressible elasticity problems solved by using mixed finite element methods; that discussion is confined to the case when the pressure approximation is discontinuous.

The tutorial ends with an introduction of the FETI-DP and BDDC algorithms. It is shown that, given the same set of primal constraints, the two algorithms have essentially the same spectra. The central role of a certain operator,  $P_D$ , for the FETI-DP algorithm is established and an outline is given of how a certain norm of this operator can be estimated; this leads directly to a good bound for the condition number of the algorithm. We note that the technical tools, previously developed, play a central role in this analysis.

What follows are some detailed comments on the slides:

Slide 2: The articles cited are [16, 4, 5].

Slide 3: The preconditioned conjugate gradient method is described in many places including [18, Appendix C.5] were additional references can also be found.

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Slides 5–8: Poincaré’s and Friedrichs’ inequalities are discussed in [18, Appendix A.4] and in many texts on partial differential equations and also in the Brenner-Scott text [2].

Slides 9–22: This presentation follows [18, Sections 1.1-1.3] quite closely. We note that the extension theorem used in formula (2) on slide 20 can be found in [18, Lemma 4.6]; see also [18, Section 4.4]. The idea of an alternative proof of a finite element extension theorem, as outlined on slide 20, is worked out in [14, Lemma 4.5] for *uniform domains*. As shown in [8] uniform domains allow for a bounded extension of  $H^1(\Omega)$  to  $H^1(\mathbb{R}^n)$ . The reference to the work by Scott and Zhang is [17]. For other work on domain decomposition theory for irregular subdomains in the plane, see also [19, 3, 6].

Slides 23–24: This is standard material and is discussed in [18, Appendix A.1-A.2] and in texts on Sobolev spaces.

Slides 25–29: This can be viewed as a two-dimensional counterpart of the material developed, in a different way, in [18, Section 4.6.3] for three dimensions.

Slides 30–32: This follows [2, Lemma 4.9.1] closely. A different method is used in the proof of [18, Lemma 4.15].

Slides 33–34: For a discussion of the results of formulas (6) and (7), see [18, Sections 4.6.1 and 4.6.2].

Slides 35–38: See [18, Section 1.3.6].

Slides 39–40: See [16].

Slides 41–44: See [18, Section 1.4].

Slides 45–46: See [18, Section 1.6.1].

Slide 47: See [18, Section 1.6.2] for a proof of the result given at the bottom of the slide.

Slides 48–49: See [18, Section 1.5].

Slides 50–52: This is a much abbreviated form of [18, Sections 2.2 and 2.3]. This theory is also developed in [2]. We note that the implementation of a particular hybrid Schwarz method is described in [18, Fig. 2.1].

Slides 53–56: The standard two-level overlapping Schwarz methods are covered in considerable detail in [18, Sections 3.1-3.8].

Slides 57–58: This topic was first discussed in [7]. See also a discussion of related questions in the next section of these notes. With a proper choice of a coarse subspace, the algorithm is identical to that of classical paper [1].

Slides 59–73 and 75–81: This presentation follows [4, 5] quite closely. We note that the effects of non-straight edges, mentioned on slide 69 are discussed in [12].

Slide 82, which is meant to provide an overview: Briefly, we can say that arbitrary jumps in material properties across the interface between the subdomains can be handled very well in domain decomposition theory and practice.

The use of approximate solvers presents some unresolved technical issues for iterative substructuring algorithms of the FETI or BDD families but essentially there are no outstanding questions on approximate solvers for the different subproblems of overlapping Schwarz methods.

Assembled stiffness matrices present no difficulties for methods based on overlapping subdomains but present a real problems for algorithms of the FETI and BDD families of [18, Chapter 6], since we cannot recover the matrices for local problems with Neumann boundary conditions from the assembled stiffness matrix.

Theory and practice are well developed for overlapping Schwarz methods for compressible and almost incompressible elasticity and finite element problems based on

mixed methods with discontinuous pressure approximations. There has also been interesting work on FETI-DP and BDDC algorithms for the almost incompressible case, but it appears that some important issues are not yet fully settled. The case of continuous pressure approximation also poses open questions.

Some relatively recent work for problems in the plane and irregular subdomains has already been mentioned. For three dimensions there are a number of open questions.

There is recent work on the effects of jumps inside the subdomains, in particular, by Dryja, Pechstein, Sarkis, and Scheichl. This now appears to be developing into a quite active area of research.

Slides 83–97: This all basically follows [16]. We also note that the choice of a small and effective set of primal constraints has been a challenging problem especially for elasticity; see [15, 10, 9, 11, 13]. By now these methods have been well developed and been successfully tested on massively parallel computing systems.

### 3. The questions, Q1–Q11.

Q1: Starting out with a domain of diameter 1, make the substitution  $\mathbf{x} \leftarrow \mathbf{x}/H$ , where  $\mathbf{x} \in \mathbb{R}^n$  with  $n$  the dimension of the space, and use elementary calculus.

Q2: We first note that an upper bound for the eigenvalues can again be obtained by Gershgorin’s theorem. This question can then be reformulated as providing a sharp lower bound of  $|u_h|_{H^1(\Omega)}^2 / \|u_h\|_{L^2(\Omega)}^2$  subject to the condition that the finite element function  $u_h(x)$  vanishes at  $x = 0$ . The key is to use formula (5) on slide 30 and formula (6) on slide 34 for two and three dimensions, respectively. Since these bounds are sharp, we can then, in three dimensions, select a function with its maximum equal to 1 at  $x = 0$  and with  $H^1$ -energy on the order of  $h$ . We then subtract the constant function 1 from this function and note that the constant has no energy. We can then conclude that the condition number is on the order of  $h^{-3}$  for three dimensions. In two dimensions, a sharp estimate on the order of  $h^{-2}/\log(h)$  can similarly be obtained.

Q3 on slide 12: We can view the Schur complement as a restriction of  $A$  to a subspace and by using Courant–Fisher’s theorem, we find that the smallest eigenvalue of the Schur complement must be at least as large as that of  $A$ . A similar argument applies to the largest eigenvalues of the two matrices.

Q4 and Q5 on slide 45: The quadratic forms related to the Schur complements are intimately related to the energy of discrete harmonic extensions of the data given on the boundary of the subdomains and to the minimal energy extension. We then see that the energy for such a minimal extension will decrease if a domain is made larger since the subspace of finite element functions, which provide candidates for the minimizing element, will be enlarged.

Q6 on slide 49: We should look for the worst  $u$  to be a smooth function of modest energy, which can be made much larger by bringing the function to zero over a region with a thickness on the order of the mesh size  $h$ .

Q7: We cannot decompose all functions into such local components since any linear combination of such functions would vanish at all the subdomain vertices. We have to conclude that the method will not converge.

Q8: So far there is no mechanism to global communication of information in each step; see the discussion [18, Page 17].

Q9: If we use the nodal basis functions, we will get a convergent Schwarz method but it fails to have a mechanism for the global communication of information.

Q10: The number of necessary colors is three for the case when we have regular subdomains, which are all equilateral triangles. If standard nodal basis are used for

the subdomain vertices, the parameter  $C_0^2$  must be on the order of  $H^{-2}$ ; see discussion on [18, Page 17].

Q11: An effective coarse subspace is given by  $V^H$ , the space of piecewise linear functions on the coarse triangles defined by the subdomains. A  $C(1 + \log(H/h))^2$  condition number bound can be developed using only tools that have been developed so far on these slides.

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