Applications of Finite Element Exterior Calculus to Geometric Problems

Chris Tiee

UCSD Mathematics

March 11, 2014

Introduction

- Geometry demands visual intuition, so let us use numerical methods to help actually build it.
- Many physical theories are highly geometric in nature.
- Differential forms help us focus on invariants (or “covariant” quantities in physics terminology), and in fact, reformulating things in this language, one can re-express many classical differential equations.
- The Finite Element Exterior Calculus is a useful framework that allows discretization of these kinds of differential equations that respects various topological features of the space and solutions. It also provides a framework for error analysis.
- Applies to all 3 fundamental types of PDEs (Elliptic, Parabolic, Hyperbolic)—We will see a bunch of examples in this talk.

Differential Forms and Their Most Important Aspects

- They generalize vector fields by replacing them with a more modern, coordinate-independent, geometric representation via alternating tensors.
- $\wedge$ replaces cross products (and in some cases dot products).
- The differential, d, generalizes the classical operators div, grad, and curl, and the notion of differential.
- Differential forms can be integrated over oriented submanifolds, and we have a generalization of Stokes's Theorem.
- $d^2 = 0$, and comparison of closed (kernel of d) vs. exact (image of d) forms gives rise to cohomology theory, with deep links to topology and structures that are far from anything differentiable.
- Metric information is brought in via the Hodge Star, a kind of duality. This is used to define L^2 inner products and Sobolev spaces of differential forms, essential for the existence and uniqueness theory of PDEs.

Correspondence to $\mathbb{R}^3$

We summarize in the correspondence between forms in $\mathbb{R}^3$ and classical vector fields in the following convenient diagram:

$$\begin{array}{ccccccc}
C^\infty(U) & \xrightarrow{\text{grad}} & C^\infty(U,\mathbb{R}^3) & \xrightarrow{\text{curl}} & C^\infty(U,\mathbb{R}^3) & \xrightarrow{\text{div}} & C^\infty(U) \\
\text{id}\downarrow & & \flat\downarrow & & \downarrow *(\cdot)^\flat & & \downarrow * \\
\Omega^0(U) & \xrightarrow{d} & \Omega^1(U) & \xrightarrow{d} & \Omega^2(U) & \xrightarrow{d} & \Omega^3(U)
\end{array}$$

Here $C^\infty(U,\mathbb{R}^3)$ is the space of smooth vector fields on U, $\flat$ is the metric dual, and $*$ is the Hodge dual (defined in the following section).

- 1-forms correspond to vector fields integrated over curves to yield “work” quantities.
- 2-forms are integrated over surfaces to give “flux” integrals.
- 3-forms are integrated over volumes to give “mass.”

Cohomology Theory

Definition

$\omega \in \Omega^p$ is called *closed* if $d\omega = 0$. Write $\mathfrak{Z}^p(U)$ for all closed p-forms on U. ω is called *exact* if $\omega = d\eta$ for some $\eta \in \Omega^{p-1}$; we similarly write $\mathfrak{B}^p(U)$ for the whole space of them. All exact forms are closed, since $d^2 = 0$.

The differential forms on U thus form a **cochain complex**

$$0 \longrightarrow \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(U) \xrightarrow{d} 0$$

and $H^k(U) = \mathfrak{Z}^k(U)/\mathfrak{B}^k(U)$ are the *de Rham cohomology groups*. All closed forms are *locally* exact, that is, exact on an open neighborhood of each point, contained in U (*Poincaré's Lemma*).

Integration and Stokes's Theorem

Integration of differential forms is also defined in such a manner that it looks like surface integrals over normal vectors, and Stokes's Theorem holds:

$$\int_U d\omega = \int_{\partial U} \omega,$$

and we have integration by parts.

Hodge Duals

We bring in metric and orientational information via the Hodge duals, which help us define global inner products.

Definition

The Euclidean metric on $\mathbb{R}^n$ induces a metric on forms. The *Hodge dual* of $\omega \in \Lambda^k$ defined to be the unique form $*\omega \in \Lambda^{n-k}$ such that

$$\eta \wedge *\omega = \langle \eta, \omega \rangle dV$$

where dV is the oriented volume element ($dx^1 \wedge \cdots \wedge dx^n$ in $\mathbb{R}^n$).

On orthonormal basis covectors (with a certain index set), it acts by sending it to the orthonormal basis covector of the complementary index set (with possibly a sign). From this, we find that $** = (-1)^{p(n-p)}$, and $*$ maps Λ^k to Λ^{n-k} isometrically (unitarily).

Hodge Duals

Definition

The Exterior Coderivative δ is defined on $\Omega^k(U)$ by the relation

$$*\delta\omega = (-1)^k d * \omega.$$

or, explicitly (by taking the $*$ of both sides and multiplying the relevant sign)

$$\delta\omega = (-1)^{n(k+1)+1} * d * \omega.$$

Stokes's Theorem and the product rule also give that δ is the adjoint to d with respect to the L^2 inner product:

$$(\delta\omega, \eta)_{L^2} = (\omega, d\eta)_{L^2} = \int_U \omega \wedge *d\eta$$

for $\eta \in \Omega_c^k(U)$. We use this to extend the domains of d and δ.

L^2 Inner Product and Norm

Definition

Let

$$(\eta, \omega)_{L^2\Omega^k} = \int_U \langle \eta, \omega \rangle \, dV = \int_U \eta \wedge *\omega,$$

called the L^2 inner product, and define its associated norm,

$$\|\omega\|_{L^2\Omega^k} := \left(\int_U \omega \wedge *\omega \right)^{1/2}.$$

We call this the L^2 norm. Let $W^k := L^2\Omega^k(U)$ be the completion of $\Omega^k(U)$ in this norm.

Weak Exterior Derivative

Definition (Weak Exterior Derivatives and Sobolev Spaces)

Let $\omega \in L^2\Omega^k(U)$. We can extend d as follows: $d\omega$ is the unique (up to Lebesgue a.e. equivalence) form such that $\eta \in \Omega_c^{k+1}(U)$,

$$(d\omega, \eta) = (\omega, \delta\eta).$$

if such a form exists (called the **weak exterior derivative**). We define

$$V^k = H\Omega^k(U) := \{\omega \in L^2\Omega^k(U) : \omega \text{ has a weak derivative in } L^2\Omega^{k+1}(U)\}.$$

Analogously, we have the space $H^*\Omega^k(U) = *H\Omega^{n-k}(U))$ for weak δ. These are called **Sobolev Spaces** of k-forms.

Note that for the $\mathbb{R}^3$ correspondence, $H\Omega^0 \leftrightarrow H^1$, $H\Omega^1 \leftrightarrow H(\mathrm{curl})$, $H\Omega^2 \leftrightarrow H(\mathrm{div})$, and $H\Omega^3 \leftrightarrow L^2$.

Outline

Hodge Laplacian and Elliptic Equations

Definition

$\Delta_d = \Delta = -(d\delta + \delta d)$ is an operator defined on smooth k-forms. Using integration by parts, we weakly formulate the problem $-\Delta\omega = f$: we want

$$(-\Delta\omega, \eta)_{L^2} = (d\omega, d\eta)_{L^2} + (\delta\omega, \delta\eta)_{L^2} = (f, \eta)_{L^2}.$$

for all η. The second expression allows us to extend the bilinear form to all of $H\Omega^k \cap H^*\Omega^k$. A solution ω satisfying the above for all η (with the appropriate restrictions on support) is called a **weak solution** to $-\Delta\omega = f$.

We need to account for a kernel (harmonic forms) for the solution to exist. We solve this by subtracting off the L^2-orthogonal projection of f onto the harmonic space $\mathfrak{H}^k = \ker(\Delta)$. For uniqueness we require the solution ω to be orthogonal to the harmonics. With that, this formulation is well-posed.

Hodge Decomposition

The major structural result, coming from the above, is the following

Theorem (Hodge Decomposition Theorem)

Every form is uniquely the sum of a boundary, coboundary, and harmonic term:

$$H\Omega^k(U) = \mathfrak{B}^k(U) \oplus \mathfrak{B}^{*k}(U) \oplus \mathfrak{H}^k(U).$$

Moreover, the harmonic forms are isomorphic to the de Rham cohomology groups.

Indeed, if f is any form, and p its orthogonal projection onto the harmonics, a weak solution $-\Delta\omega = f - p$ gives $f = d(\delta u) + \delta(du) + p$. This generalizes the Helmholtz Decomposition Theorem for vector fields.

An Example Harmonic Vector Field

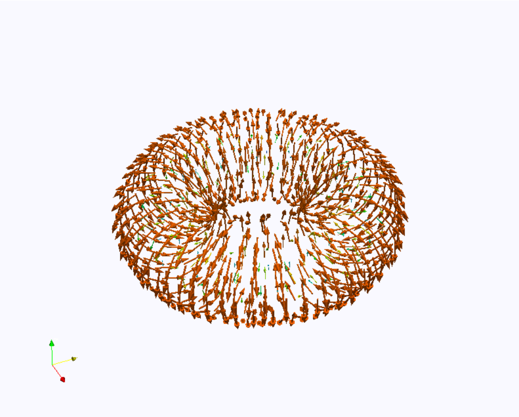

Figure : Harmonic Vector Field on a Torus

Mixed Formulation

Another weak formulation is possible, taking $\sigma = \delta\omega$ and rewriting it as a *system*: seek $(\sigma, \omega, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$ such that for all $(\tau, v, q) \in V^{k-1} \times V^k \times \mathfrak{H}^k$,

$$(\omega, d\tau) - (\sigma, \tau) = 0 \tag{1}$$
$$(d\sigma, v) + (d\omega, dv) + (p, v) = (f, v) \tag{2}$$
$$(\omega, q) = 0. \tag{3}$$

The natural boundary conditions are $\mathrm{tr}(*\omega) = 0$ and $\mathrm{tr}(*d\omega) = 0$. (1) weakly expresses that $\sigma = \delta\omega$, (2) is the actual weak form (with the harmonic part of f removed), and (3) enforces perpendicularity to the harmonics.

Why Mixed?

Advantages:

- We seek ω in larger function spaces, making existence easier (although perhaps less regular).
- Better-behaved when discretized: the weak formulations actually avoid explicitly involving the weak codifferential, and gives us more freedom to choose good finite element spaces. It is difficult to construct finite element spaces that simultaneously are in the domains of d and δ.

Disadvantages:

- The bilinear form corresponding to the mixed formulation is *not* coercive (positive-definite) and in fact corresponds to a saddle-point problem when treated variationally. Though still well-posed, it is an additional complication.
- The need to choose several different spaces and compute and maintain some auxiliary fields (σ and p) in addition to what we really want, ω, is obviously less efficient.

Outline

Discretization Generalities

- In order to discretize, we choose certain finite-dimensional subspaces of $V_h^k \subseteq H\Omega^k(U)$, associated to a triangulation of U of mesh size h.
- The Finite Element Exterior Calculus is the analysis of finite element methods using these subspaces V_h^k.
- Our choice of subspaces: polynomial differential forms on a simplex:

$$P_r\Lambda^k(T) = \{\omega \in H\Lambda^k(T) :$$

$$\text{the coefficients } \omega_J \text{ in the } dx^J \text{ basis are polynomials of degree } \le r\}.$$

We also need another subspace which is dual in some sense, $P_r^-\Lambda^k(T)$, but we will only use the $r = 1$ case and characterize it below.[1]

[1] The full definition involves another operator, called the Koszul operator which acts, in some sense, oppositely to d. For those familiar with topology, it is related to the cone operator used in the proof of Poincaré's lemma for constructing potentials.

Polynomial Differential Forms

Now we approximate the problem $-\Delta\omega = f$ by solving a system in the subspaces: now we seek $(\sigma_h, \omega_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ such that for all $(\tau, v, q) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$,

$$(\omega_h, d\tau) - (\sigma_h, \tau) = 0 \tag{4}$$

$$(d\sigma_h, v) + (d\omega_h, dv) + (p_h, v) = (f, v) \tag{5}$$

$$(\omega_h, q) = 0. \tag{6}$$

Special Case: The Whitney Forms, $P_1^- \Lambda^k$

- Natural choice of degrees of freedom for piecewise linear polynomials: integrating over the subsimplices.
- In some sense the forms correspond to the faces and edges themselves, emphasizing the geometric nature, and so capture more information in discretizing than just "an n-tuple of functions which we approximate individually."
- We characterize a special subspace of piecewise linear forms by the following (a special case of the Geometic Decomposition given in AFW):

$$P_1^- \Lambda^k(T)^* \cong \bigoplus_{f \in \Delta_k} P_0 \Lambda^0(f).$$

It reduces to saying that our degrees of freedom are precisely integration over the k-simplices. The forms in the dual basis corresponding to this are called **Whitney Forms**.

Piecewise Polynomial Forms on a Triangulated Domain

Let $\mathcal{T}$ be a triangulation of a polyhedral domain U. We assemble the finite element spaces in each triangle to a full finite element space:

- We define

$$P_r\Lambda^k(\mathcal{T}) = \left\{\omega \in H\Omega^k(\mathbb{R}^n) : \omega|_T \in P_r\Lambda^k(T) \text{ for all } T \in \mathcal{T}\right\}$$

- Interelement continuity generalizes classical electrostatic boundary conditions $(B_2 - B_1) \times n = 0$ and $(E_2 - E_1) \cdot n = 0$

Outline

Adding Time Dependence

- We would like to add time dependence to our problems, so we can solve dynamical equations like the heat, wave, and Maxwell's equations. Traditionally this is done using finite differences, but the advantage of FEM and indeed FEEC is to provide a framework for more refined error analysis.
- One way to handle this is semidiscretization (the "Method of Lines"), which literally factors out the time dependence and discretizes the spatial part using these FEEC spaces, to yield a system of ODEs in the coefficients. These in turn can be numerically solved using standard methods for ODEs, like Euler, Runge-Kutta methods, and symplectic methods.

The Heat Equation

Consider

$$\frac{\partial u}{\partial t} = \Delta u.$$

in some domain, satisfying some boundary conditions. *Semidiscretization* means we consider

$$u_h(x,t) = \sum_i U_{h,i}(t)\varphi_i(x),$$

essentially the method of separation of variables with interesting basis functions φ_i which are to be in one of the spatial finite element spaces. Then the equation for the approximation becomes:

$$\sum_i U'_{h,i}(t)\varphi_i(x) = \sum_i U_{h,i}(t)\Delta\varphi_i(x).$$

Discretization

Now if we take the inner product with another $\varphi_j(x)$, and use the weak form, we have:

$$\sum_i U'_{h,i}(t)(\varphi_i, \varphi_j)_{L^2} = -\sum_i U_{h,i}(t)(d\varphi_i, d\varphi_j)_{L^2}.$$

Letting $\mathbf{u}$ be the vector $(U_{h,i})$, $M_{ij} = (\varphi_i, \varphi_j)$ (the *mass* matrix), and $K_{ij} = (d\varphi_i, d\varphi_j)$, the “stiffness” matrix (terminology from hyperbolic equations, actually), we have

$$M\frac{d\mathbf{u}}{dt} = -K\mathbf{u}.$$

solvable by standard ODE methods.

We can actually consider the Hodge heat equation, which is the case that u is a k-form and now Δ is the Hodge Laplacian. The boundary conditions for a Dirichlet problem can be considered naturally if u is an n-form.

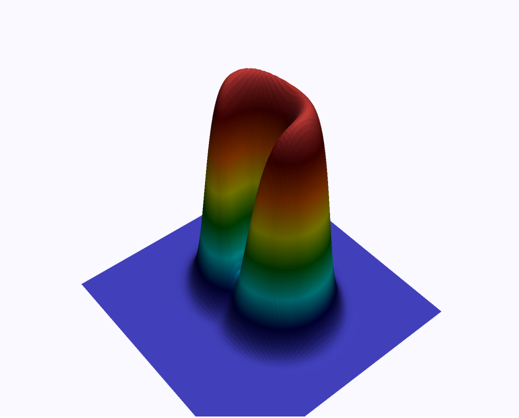

Figure : Heat Equation

Some Generalization Directions

- We can consider differential forms and Hodge Theory on surfaces, with spaces like $H\Omega^k(M)$ for M a smooth manifold. This leads to finite elements on curved or triangulated approximating surfaces (example will be shown for hyperbolic equations).
- Semidiscretization of Ricci Flow on surfaces (using the above), a quasilinear equation for a metric conformal factor u (metric is $e^{2u}g_0$): (Joint work with M. Holst)

$$\frac{\partial u}{\partial t} = e^{2u}(\Delta u - K_0)$$

  where K_0 is the Gaussian curvature of the initial metric g_0.
- Interesting examples are extremely hard to visualize as true geometry, due to intrinsic nature of the equation. Finding a suitable embedding, even if imperfect, is itself a very interesting problem.

The Ricci Flow for Rotationally Symmetric Data on S^2

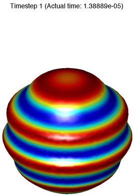

Figure : Ricci Flow on a Sphere

Outline

Mixed Formulation: Recasting as a System

Now consider the wave equation,

$$\frac{\partial^2 u}{\partial t^2} = \Delta u.$$

The standard trick is to recast this as a system, by letting, say, $v = u_t$ and writing the equations down for (u, v):

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \Delta u \end{pmatrix}$$

We can actually consider this a mixed problem, an ODE on $V_h \times V_h$ where $V_h = P_1^- \Lambda^0$ as before, with the inner product $(u, v) \cdot (\varphi, \psi) = \langle u, \varphi \rangle + \langle v, \psi \rangle$.

Mixed Formulation: Recasting as a System

Given that, take the inner product of the whole above equation with test functions (φ, ψ) and use the weak form:

$$\frac{\partial}{\partial t}(u, \varphi) = (v, \varphi) \tag{7}$$

$$\frac{\partial}{\partial t}(v, \psi) = -(du, d\psi) \tag{8}$$

Mass and Stiffness Matrices for Hyperbolic Equations

Substituting, as before, basis functions φ_i and ψ_i in V_h, we get the formulation

$$\frac{d\mathbf{u}}{dt} = \mathbf{v} \tag{9}$$

$$M\frac{d\mathbf{v}}{dt} = -K\mathbf{u} \tag{10}$$

Example using Whitney Forms for Maxwell

Now for an example that actually uses forms of degree $k > 0$.
Maxwell's equations ($c = 1$ as mathematicians like it):

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{11}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \tag{12}$$

$$\nabla \cdot \mathbf{D} = \rho \tag{13}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{14}$$

along with the constitutive equations $\mathbf{B} = \mu \mathbf{H}$ and $\mathbf{D} = \varepsilon \mathbf{E}$. We recast these into equations involving time-dependent 1-forms and 2-forms.

Recasting into Forms notation

We consider E, H to be 1-forms corresponding to **E**, **H** respectively, via the flat-correspondence as mentioned way above, and similarly, B, D, and J to be the corresponding 2-forms, via the flat-and-star, and finally ρ should be a 3-form:

$$dE = -\frac{\partial B}{\partial t} \tag{15}$$

$$dH = \frac{\partial D}{\partial t} + J \tag{16}$$

$$dD = \rho \tag{17}$$

$$dB = 0 \tag{18}$$

The constitutive relations are now $D = \varepsilon * E$ and $H = \mu * B$. Gone are all the strangely different ways that ∇ interacts.

Discretization

We choose for our finite element spaces $P_1^-\Lambda^1(U)$ for E and H, and $P_1^-\Lambda^2(U)$ for B and D (the Whitney form complexes). These spaces correspond to Nédélec elements and Raviart-Thomas elements, respectively. We transform into an ODE by taking the inner product with a test form in the same spaces, leading to a very natural mixed formulation:

$$\frac{\partial}{\partial t}\langle B, B'\rangle = -\langle dE, B'\rangle \tag{19}$$

$$\frac{\partial}{\partial t}\langle \varepsilon^{-1} * E, dE'\rangle = \langle \mu^{-1} * B, E'\rangle - \langle J, dE'\rangle \tag{20}$$

for all $B' \in P_1^-\Lambda^2$ and $E' \in P_1^-\Lambda^1$ (the names of these test functions are chosen for mnemonic purposes and don't correspond to additional fields). Notice the similarity to the wave equation discretization as above (but now we are working on $P_1^-\Lambda^1 \times P_1^-\Lambda^2$ instead).

Example using Whitney Forms for Maxwell

The other equations hold automatically with our data ($\rho = 0$ and J divergenceless).

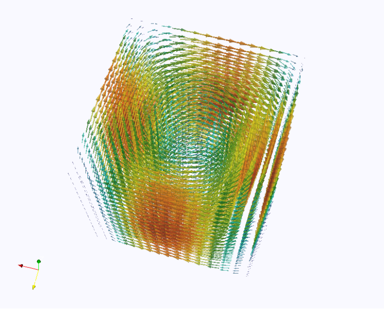

Figure : Magnetic Field on a Cube

Notice the propagational delay in the movie.

Outline

Possible Extensions and Current Projects

Major Motivation: Unifying Space and Time

- Useful as semidiscretization may be, it treats time as a completely separate entity, or assumes the product geometry $\mathbb{R} \times U$ or with universal time coordinate. We want to see if we can have a more integrated approach—spacetime (there is even some parabolic theory that benefits from this, cf. Chow, Lu, Ni).
- We want good qualitative behavior—preservation of various physical invariants. This is possible in semidiscretization, as we saw, by considering symplectic methods.
- We would like to eliminate spurious coordinate dependencies, in order to help improve stability.
- Good error analysis—inherited from the framework of FEEC rather than restarting from scratch.

Maxwell's Equations in Spacetime

The approach using separate magnetic and electric fields and their duals is admittedly still a little clumsy. In Minkowski spacetime, with an extended spacetime Hodge star, we can regard the electromagnetic field as one single 2-form:

$$F = E \wedge dt + B.$$

$$J_4 = \rho + J \wedge dt.$$

Then Maxwell's Equations imply[2]:

$$dF = 0 \tag{21}$$

$$\delta F = \mu J_4. \tag{22}$$

Unification no longer forces split into E and B fields, and thus this can carry over to curved spacetimes. It is also conceptually simpler.

[2] Different unit systems and sign conventions exist, so beware.

Classical Field-Theoretic Formulation

- The relation $dF = 0$ allows us, for Minkowski space ($\mathbb{R}^4$), to find a *potential* A such that $dA = F$. (All closed forms are exact on all of $\mathbb{R}^4$.
- Maxwell's Equations for a potential become

$$\delta dA = J_4.$$

They arise as the Euler-Lagrange equations for the action

$$S[A] = \int_{\mathbb{R}^4} dA \wedge *dA = \int_{\mathbb{R}^4} \langle dA, dA \rangle_{\text{Minkowski}}$$

The Minkowski metric is Lorentzian $-dt^2 + dx^2 + dy^2 + dz^2$ (or its negative, preferred by field theorists).

Possible Methods

- Hyperbolic equations are very different—how can spacetime finite elements work? Possible objection—non-coercive bilinear form—the mixed formulation already deals with saddle-point type problems, so is an inf-sup condition possible?
- We can extend the Hodge star to Lorentzian metrics, which could lead to fruitful methods.
- We can operate on a given spacetime mesh, or “make the mesh as we go” via tent-pitching: constructing spacetime meshes from initial spacelike meshes. It frees us from the constraint of using the same rigid timestep for *every* element as in the above case.
- Determining best points to pitch tents is itself a very interesting evolutionary problem.

Evolution of Data on Spacetime Mesh

- Given a mesh either by fiat, or constructed by tent-pitching, how do we evolve the data?
- We can solve for nodal values using specified data on a triangulated Cauchy surface (which need not lie in a preferred time slice!) and extend in a timelike direction. There is the discontinuous Galerkin method, and symplectic methods (working with J. Salamon, J. Moody, and M. Leok)

Comparison

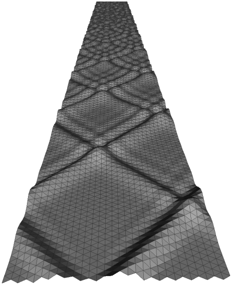

Figure : Solution of Wave Equation in $(1+1)$ Minkowski Spacetime Using Method of J. Moody

The obvious comparison should be with semidiscretized wave equations timestepped with a symplectic method.