Applications of Finite Element Exterior Calculus to Evolution Problems

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February 11, 2015

Introduction

- Geometry demands visual intuition, so let us use numerical methods to help actually build it.
- Many physical theories are highly geometric in nature.
- Differential forms help us focus on invariants (or "covariant" quantities in physics terminology), and in fact, reformulating things in this language, one can re-express many classical differential equations.
- Finite Element Exterior Calculus (FEEC), developed by Arnold, Falk, and Winther [3, 2] is a useful framework that allows discretization of these kinds of differential equations that respects geometrical structure. It also provides a framework for error analysis.
- Hilbert complex approach allows abstraction of the essential properties of the spaces we are concerned with: differentials, duals, Poincaré inequality, Laplacians, and their approximation.

Hilbert Complexes—Important Aspects

Definition (Hilbert Complexes [2])

A **Hilbert complex** (W, d) is a sequence of Hilbert spaces $(W^k, \langle \cdot, \cdot \rangle)$ with possibly unbounded, densely-defined closed linear maps d^k on the **domains** $V^k \subseteq W^k$), such that $d^k \circ d^{k-1} = 0$ (abbreviated $d^2 = 0$), so it is a **cochain complex**.

$$\cdots \longrightarrow V^{k-1} \xrightarrow{d^{k-1}} V^k \xrightarrow{d^k} V^{k+1} \xrightarrow{d^{k+1}} \cdots$$

- Examples include differential forms on a manifold or vector fields in $\mathbb{R}^3$; the W^k are L^2 differential forms or vector fields, and V^k are Sobolev spaces H^1, $H(\mathrm{curl})$, $H(\mathrm{div})$. d is the weak exterior derivative, or grad, curl, and div.
- The inner product $\langle \cdot, \cdot \rangle$ generalizes the definition via Hodge stars of differential forms, and Hilbert space methods for PDES.
- The domains V are also a Hilbert complex (V, d), with the **graph inner product** $\langle u, v \rangle_V := \langle u, v \rangle + \langle d^k u, d^k v \rangle$. d is *bounded* in this complex.

Cohomology in Hilbert Complexes

Definition

$\omega \in V^k$ is called **closed** if $d\omega = 0$. Write $\mathfrak{Z}^k$ for the set of all such ω. ω is called **exact** if $\omega = d\eta$ for some $\eta \in V^{k-1}$; we similarly write $\mathfrak{B}^k$ for the whole space of them. All exact elements are closed, since $d^2 = 0$. Therefore the quotients $H^k = \mathfrak{Z}^k/\mathfrak{B}^k$ are well-defined; we call it the **cohomology** of the Hilbert complex (W, d).

For example, in the case of differential forms on a manifold with boundary or open subset of Euclidean space, this corresponds to the *de Rham cohomology*, a central concept linking analysis, topology, and differential geometry.

Dual Complexes

That the d^k are closed, densely-defined operators enables, via functional analysis, generalization of the concept of adjoints:

Definition

Consider the spaces $W^*_k = W^k$ and the domains V^*_k defined by the following property: $\omega \in V^*_k$ if there exists $\sigma \in W^*_{k-1}$ such that

$$\langle \sigma, \tau \rangle = \langle \omega, d^{k-1}\tau \rangle$$

for all $\tau \in V^{k-1}$. The hypotheses on d^k ensure that σ is unique, thus is a function d^*_k of ω. This **adjoint** operator d^*_k is closed and linear with dense domains V^*_k. d^*_k maps V^k into V^{k-1}, and $d^*_{k-1} \circ d^*_k = 0$. (W^*, d^*) is called the **dual complex**.

The defining equation of the adjoint corresponds to *integration by parts* for differential forms and vector fields.

Dual Complexes

- This generalizes the Hodge codifferential of differential forms; the arrows run in the opposite (degree-decreasing) direction—it is a **chain complex**:

$$\cdots \longleftarrow V_{k-1}^* \xleftarrow{d_{k-1}^*} V_k^* \xleftarrow{d_k^*} V_{k+1}^* \xleftarrow{d_{k+1}^*} \cdots$$

- For differential forms or vector fields, the dual complex incorporates *boundary condtions* which, of course, realize the "integration by parts" mentioned above.
- Since $(d^*)^2 = 0$, we define the corresponding notions of closed, exact, and cohomology for this complex: $\mathfrak{Z}_k^* = \ker d_k^* : V_k^* \to V_{k-1}^*$ and $\mathfrak{B}_k^* = d_{k+1}^*(V_{k+1}^*) \subseteq V_k^*$.
- We define $\mathfrak{H}^k = \mathfrak{Z}^k \cap \mathfrak{Z}_k^*$, the **harmonic space**, which is also the orthogonal complement of $\mathfrak{B}^k$ in $\mathfrak{Z}^k$. It is isomorphic to $\mathfrak{Z}^k/\overline{\mathfrak{B}^k}$. For complexes that are **closed**, i.e. when $\mathfrak{B}^k$ is closed in W, $\mathfrak{H}^k$ is isomorphic to the cohomology H^k.

Hodge Decomposition

With the above, we can prove a major structural result:

Theorem (Hodge Decomposition Theorem)

Every element of V^k is uniquely the sum V-orthogonal sum of coboundary, harmonic, and boundary terms:

$$V^k = \overline{\mathfrak{B}^k} \oplus \mathfrak{H}^k \oplus \overline{\mathfrak{B}^*_k} \tag{1}$$

$$W^k = \overline{\mathfrak{B}^k} \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp W}. \tag{2}$$

(for $\mathfrak{B}^k$ and $\mathfrak{H}^k$, the V- and W-inner products coincide, so the sum is also W-orthogonal). For closed complexes, of course, we can dispense with the closures.

We write $\mathfrak{Z}^{k\perp}$ for the V-orthogonal complement (the more often-used). Using the elliptic theory below, we can give a more explicit representation of the Hodge decomposition.

An Example Harmonic Vector Field

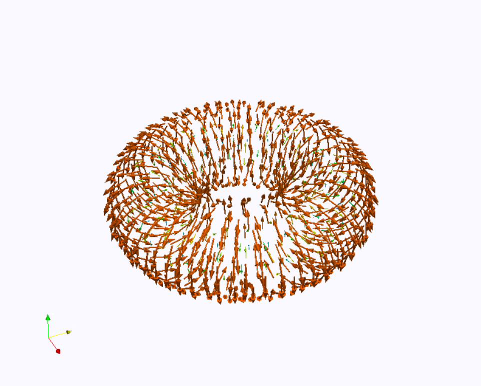

Figure: Harmonic vector field on a torus: this gives an example of an element of a harmonic space of degree $k > 0$. One can clearly see how it relates to topology, here surrounding the "tube" of the torus.

Abstract Poincaré Inequality

The most important basic result for the elliptic equations we formulate, their stability, and approximations, is the Poincaré Inequality:

Theorem (Abstract Poincaré Inequality, [2], §3.1.3)

Let (W, d) be a closed Hilbert complex and (V, d) be its domain. Then there exists $c_P > 0$ such that for all $v \in \mathfrak{Z}^{k\perp}$,

$$\|v\|_V \leq c_P \|d^k v\|. \tag{3}$$

Outline

Hodge Laplacian and Elliptic Equations

Definition

$\Delta_d := \Delta = -(dd^* + d^*d)$ is an operator on the domain $V^k \cap V_k^*$. Using the definition of the adjoints ("integration by parts"), we weakly formulate the problem $-\Delta u = f$ for $f \in W^k$:

$$\langle -\Delta u, \eta\rangle = \langle du, d\eta\rangle + \langle d^*u, d^*\eta\rangle = \langle f, \eta\rangle.$$

for all $\eta \in V^k \cap V_k^*$. A solution u satisfying the above for all η is called a **weak solution** to $-\Delta u = f$.

To make this well-posed, we W-orthogonally project f onto $\mathfrak{H}^{k\perp_W} = (\ker\Delta)^{\perp_W}$. For uniqueness, we require the solution u to be in $\mathfrak{H}^{k\perp_W}$ as well. It explicitly realizes the Hodge decomposition: if $f \in W^k$ there exist unique $u \in V^k \cap V_k^*$ and $p \in \mathfrak{H}^k$ such that:

$$f = d(d^*u) + p + d^*(du).$$

Mixed Formulation

Another weak formulation is possible, taking $\sigma = d^* u$ and rewriting it as a *system*: seek $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$ such that for all $(\tau, v, q) \in V^{k-1} \times V^k \times \mathfrak{H}^k$,

$$\begin{cases} \langle u, d\tau\rangle - \langle \sigma, \tau\rangle &= 0 \\ \langle d\sigma, v\rangle + \langle du, dv\rangle + \langle p, v\rangle &= \langle f, v\rangle \\ \langle u, q\rangle &= 0. \end{cases} \tag{4}$$

- (1) weakly expresses that $\sigma = d^* u$, (2) is the actual weak form (removing the harmonic part of f), and (3) enforces perpendicularity to the harmonics.
- This problem is well-posed, and the solution satisfies the *a priori* estimate

  $$\|\sigma\|_V + \|u\|_V + \|p\| \le c\|f\|,$$

  where the stability constant c depends only on the Poincaré constant c_P.
- As the abstraction of the elliptic problem with forms, recall the dual complex incorporates boundary conditions; here they are $\operatorname{tr} \star u = 0$ and $\operatorname{tr} \star du = 0$.

Why Mixed?

- Advantage: it is better-behaved in approximation theory: the weak formulations avoid explicitly involving the adjoint, so we have more freedom to choose subspaces (which will be finite element spaces in the more concrete settings). It is difficult to construct finite element spaces that simultaneously are in the domains of d and d^*.
- Disadvantage: the need to choose several spaces and compute and maintain some auxiliary fields (σ, p) in addition to what we really want, u, is less efficient (but that is a small price to pay for correctness).

Mixed Formulation with Nonvanishing Harmonic Part

We will need to modify things slightly for evolution problems: given a prescribed harmonic part $w \in \mathfrak{H}^k$, now we seek $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$ such that for all $(\tau, v, q) \in V^{k-1} \times V^k \times \mathfrak{H}^k$,

$$\begin{cases} \langle u, d\tau \rangle - \langle \sigma, \tau \rangle &= 0 \\ \langle d\sigma, v \rangle + \langle du, dv \rangle + \langle p, v \rangle &= \langle f, v \rangle \\ \langle u, q \rangle &= \langle w, q \rangle . \end{cases} \tag{5}$$

The last part enforces, by definition, $w = P_{\mathfrak{H}} u$, the orthogonal projection onto the harmonic space. This problem is also well-posed by the same existence theory; it satisfies the *a priori* estimate (with the same constant)

$$\|\sigma\|_V + \|u\|_V + \|p\| \leq c(\|f\| + \|w\|).$$

Outline

Approximation via Subcomplexes [2]

- We choose certain finite-dimensional subspaces of $V_h^k \subseteq V^k$ (e.g. finite element spaces associated to a triangulation of mesh size h). We refer to such spaces as **discrete**. These must satisfy the **subcomplex property** $d^k V_h^k \subseteq V_h^{k+1}$, and have **bounded cochain projections** $\pi_h^k : V \to V_h^k$ commuting with d.
- **The Finite Element Exterior Calculus (FEEC)** is the analysis of finite element methods using subspaces V_h^k with these properties.
- For differential forms, we choose polynomial differential forms on a simplex,

$$P_r\Lambda^k(T) = \{\omega \in H\Lambda^k(T) : \text{the coefficients } \omega_J \text{ in the } dx^J \text{ basis are polynomials of degree } \leq r\}.$$

  Another subspace which is dual in some sense, is $P_r^-\Lambda^k(T)$, polynomial differential forms defined using another operator (the Koszul differential).
- The cochain projections correspond to smoothed interpolation operators.

Discrete Version of the Abstract Mixed Problem

Now we approximate the abstract mixed problem $-\Delta u = f$ by solving a system in the discrete spaces: now we seek $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ such that for all $(\tau, v, q) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$,

$$\begin{cases} \langle u_h, d\tau\rangle - \langle \sigma_h, \tau\rangle &= 0 \\ \langle d\sigma_h, v\rangle + \langle du_h, dv\rangle + \langle p_h, v\rangle &= \langle f, v\rangle \\ \langle u_h, q\rangle &= \langle w, q\rangle. \end{cases} \tag{6}$$

The Poincaré constant for this complex is $c_P\|\pi_h\|$, and therefore abstract theory above shows this problem is also well-posed.

A Priori Error Estimates for Subcomplex Approximations

Arnold, Falk, and Winther [2] establish the following error estimate for approximating mixed variational problems (including nonzero harmonic part):

Theorem (Error Estimates for the Mixed Variational Problem)

Let V_h be a subcomplex of the domain complex (V, d) admitting uniformly V-bounded cochain projections. Let (σ, u, p) and (σ_h, u_h, p_h) be the solutions to the continuous, and resp. the discrete problem. Then

$$\begin{aligned} &\|\sigma - \sigma_h\|_V + \|u - u_h\|_V + \|p - p_h\| \\ &\qquad \leq C\left(\inf_{\tau \in V_h^{k-1}} \|\sigma - \tau\|_V + \inf_{v \in V_h^k} \|u - v\|_V + \inf_{q \in V_h^k} \|p - q\|_V \right. \\ &\qquad\qquad \left. + \mu \inf_{v \in V_h^k} \|P_{\mathfrak{B}} u - v\|_V \right). \qquad (7) \end{aligned}$$

with $\mu = \mu_h^k = \sup_{r \in \mathfrak{H}^k} \|(I - \pi_h^k) r\|$.

A Priori Error Estimates for Subcomplex Approximations

- This establishes that the solution is *quasi-optimal*, in terms of a constant multiple of Hilbert space best approximation errors.
- Improved error estimates are available for W-bounded cochain projections and complexes satisfying a compactness property. This is always true for differential forms, by the Rellich-Kondrachov Theorem.
- Properties of the bounded cochain operator on Euclidean spaces give a particular bound on these best approximations in terms of powers of the mesh parameter h. Specifically, for $\omega \in H^s\Omega^k(U)$, with U with sufficiently smooth boundary, we have

$$\inf_{\eta \in V_h^k} \|\omega - \eta\|_{L^2\Omega^k} \leq \|\omega - \pi_h^k \omega\|_{L^2\Omega^k} \leq Ch^s \|\omega\|_{H^s\Omega^k}.$$

Improved Estimates for the Problem in Euclidean Space

Theorem (Improved Error Estimate, [2], p. 342)

Consider the mixed variational problem for the Laplace equation in a triangulated domain in Euclidean space. Let (σ, u, p) be a solution. Suppose we choose finite element spaces Λ_h^k, (σ_h, u_h, p_h) the discrete solution, and suppose the data f allows the regularity estimate

$$\|u\|_{H^{t+2}(U)} + \|p\|_{H^{t+2}(U)} + \|du\|_{H^{t+1}(U)} + \|\sigma\|_{H^{t+1}(U)} + \|d\sigma\|_{H^{t}(U)} \leq C\|f\|_{H^t(U)}$$

for $0 \leq t \leq t_{\max}$. Then we have the following estimates for $0 \leq s \leq t_{\max}$:

$$\|d(\sigma - \sigma_h)\| \leq Ch^s\|f\|_{H^s(\Omega)}, \text{ if } s \leq r+1,$$

$$\|\sigma - \sigma_h\| \leq Ch^{s+1}\|f\|_{H^s(\Omega)}, \text{ if } \begin{cases} s \leq r+1, & \Lambda_h^{k-1} = \mathcal{P}_{r+1}\Lambda^{k-1}(\mathcal{T}), \\ s \leq r, & \Lambda_h^{k-1} = \mathcal{P}_{r+1}^{-}\Lambda^{k-1}(\mathcal{T}), \end{cases}$$

$$\|d(u - u_h)\| \leq Ch^{s+1}\|f\|_{H^s(\Omega)}, \text{ if } \begin{cases} s \leq r, & \Lambda_h^{k} = \mathcal{P}_{r+1}^{-}\Lambda^{k}(\mathcal{T}), \\ s \leq r-1, & \Lambda_h^{k} = \mathcal{P}_{r}\Lambda^{k}(\mathcal{T}), \end{cases}$$

$$\|u - u_h\| + \|p - p_h\| \leq \begin{cases} Ch\|f\|, & \Lambda_h^k = \mathcal{P}_1^{-}\Lambda^k(\mathcal{T}), \\ Ch^{s+2}\|f\|_{H^s(\Omega)} \text{ if } s \leq r-1, & \text{otherwise.} \end{cases}$$

Injective Morphisms of Complexes [5]

- For geometric problems, we need to relax the assumption that V_h^k are subspaces; instead take another complex (W_h, d_h) with domains (V_h, d_h) and injective morphisms (W-bounded, linear maps that commute with the differentials) $i_h^k : V_h^k \to V^k$ which are not necessarily inclusion.
- If i_h is *unitary*, then we may identify it with the spaces $i_h W_h$, which *are*, in fact, a subcomplex. It is the non-unitarity of the operator that requires generalization.
- We still need V-bounded projection morphisms $\pi_h : V \to V_h$ (also commuting with the differentials), which, instead of idempotency, satisfy $\pi_h \circ i_h = \mathrm{id}$.
- Holst and Stern [5] generalize FEEC to this case.

Discrete, non-Subcomplex Version of the Abstract Mixed Problem

We now seek $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ such that for all $(\tau, v, q) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$,

$$\begin{cases} \langle u_h, d_h\tau\rangle_h - \langle \sigma_h, \tau\rangle_h &= 0 \\ \langle d_h\sigma_h, v\rangle_h + \langle d_h u_h, d_h v\rangle_h + \langle p_h, v\rangle_h &= \langle f_h, v\rangle_h \\ \langle u_h, q\rangle_h &= \langle w_h, q\rangle_h, \end{cases} \tag{8}$$

where f_h and w_h are suitable interpolations of the data f and w. The Poincaré constant for this complex is $c_P\|\pi_h\|\ \|i_h\|$, and again, the abstract theory above shows this problem is well-posed.

Example Where Injective Morphisms are Needed

- For orientable hypersurfaces M of $\mathbb{R}^{n+1}$, we approximate M with a mesh of simplices in the *ambient space*: the (piecewise affine) approximating surface is *not* contained in the set of points M (although the vertices are often chosen on M), but at least lies within a tubular neighborhood.
- The polynomial spaces of forms on the simplices are pulled back via inclusion from those defined on simplices in $\mathbb{R}^{n+1}$. This works because they are affine subsets.
- Nevertheless, the normal vector to M establishes the tubular neighborhood, and provides a mapping (via projections along the normal vector) that enables us to compare true solutions defined on M to approximations on the mesh. [[needs picture]]
- These show up in the Holst and Stern [5] framework as *variational crimes*.

A Priori Error Estimates for non-Subcomplex Approximations

Holst and Stern [5] generalize the basic error estimate of Arnold, Falk, and Winther [2] (terms in red), for the case of perpendicularity to the harmonic forms.

Theorem (Error Estimates for the Problem with Variational Crimes)

Let (V_h, d_h) be a domain complex, and $i_h : V_h \to V$ be injective morphisms as above, (V, d), admitting uniformly V-bounded cochain projections. Let (σ, u, p) and (σ_h, u_h, p_h) be the solutions to the continuous, and resp. the discrete problem. Then

$$\begin{aligned} &\|\sigma - i_h\sigma_h\|_V + \|u - i_h u_h\|_V + \|p - i_h p_h\| \\ &\quad \leq C\Bigg(\inf_{\tau \in i_h V_h^{k-1}} \|\sigma - \tau\|_V + \inf_{v \in i_h V_h^k} \|u - v\|_V + \inf_{q \in i_h V_h^k} \|p - q\|_V \\ &\qquad + \|f_h - i_h^* f\| + \|I - J_h\| \, \|f\| + \mu \inf_{v \in i_h V_h^k} \|P_{\mathfrak{B}} u - v\|_V \Bigg) \end{aligned} \tag{9}$$

with $\mu = \mu_h^k = \sup_{r \in \mathfrak{H}^k} \|(I - \pi_h^k) r\|$.

Idea of the Proof for Variational Crimes

Holst and Stern add two additional *variational crimes* terms in order to prove the theorem.

- They define $J_h = i_h^* i_h$, the composition of the morphism with its adjoint with respect to the discrete inner product.
- They define an intermediate solution (σ_h', u_h', p_h') by modifying the inner product with J. This solution is considering the problem on the included spaces $i_h V_h$, and therefore AFW [2] gives estimates on the intermediate solution. We do not use this solution because it is difficult to compute; instead it is useful for the analysis.
- Comparing the intermediate solution to the computed discrete solution yields the terms $\|f_h - i_h^* f\|$ and $\|I - J_h\| \, \|f\|$; the former term is due to the need to approximate the data, and the latter measures the non-unitarity of i_h.
- The final, full error estimate is obtained by the triangle inequality.

A Priori Error Estimates for non-Subcomplex Approximations

Our first result, generalizing Holst and Stern [5] for possible nonzero harmonic part w (new terms in green). We need it for evolution problems.

Theorem (Error Estimates for the Problem with VCs and Nonvanishing Harmonic)

Let (V_h, d_h) be as before. Let (σ, u, p) and (σ_h, u_h, p_h) be as before, but with possibly nonvanishing harmonic parts w and w_h. Then

$$\begin{aligned} &\|\sigma - i_h\sigma_h\|_V + \|u - i_h u_h\|_V + \|p - i_h p_h\| \\ &\quad \le C\Bigg(\inf_{\tau \in i_h V_h^{k-1}} \|\sigma - \tau\|_V + \inf_{v \in i_h V_h^k} \|u - v\|_V + \inf_{q \in i_h V_h^k} \|p - q\|_V \\ &\qquad + \|f_h - i_h^* f\| + \|I - J_h\|(\|f\| + \|w\|) + \mu \inf_{v \in i_h V_h^k} \|P_{\mathfrak{B}} u - v\|_V \\ &\qquad + \|w_h - i_h^* w\|_V + \inf_{v \in i_h V_h^k} \|w - v\|_V \Bigg). \qquad (10) \end{aligned}$$

Idea of the Proof for Nonzero Harmonic Term Result

We rework the proof in Holst and Stern [5]:

- We define, as before, the modified complex and intermediate solution (σ_h', u_h', p_h').
- Trouble arises because comparison of the harmonic parts of the discrete and continuous solutions, because they belong in different spaces that have no reason to be preserved by the operators. Even in the case of a subcomplex, these spaces do not coincide.
- Problem is resolved by using the Hodge decomposition to project as many parts as we can to use the previous theorem, and then dealing directly with the discrete harmonic forms by using separate theorems on approximation of harmonic forms proved by Arnold, Falk, and Winther [2]. This yields both additional non-unitarity terms $\|I - J_h\| \, \|w\|$ and the best approximation term.

Outline

Adding Time Dependence

- We would like to add time dependence to our problems, so we can solve dynamical equations like the heat, wave, and Maxwell's equations. Traditionally this is done using finite differences, but the advantage of finite element methods and indeed FEEC is to provide a framework for more refined error analysis.
- One way to handle this is semidiscretization (the "Method of Lines"), which factors out the time dependence and discretizes the spatial part using these FEEC spaces, to yield a system of ODEs in the coefficients. These in turn can be numerically solved using standard methods for ODEs, like Euler, Runge-Kutta methods, and symplectic methods.

The Heat Equation

Let (W, d) be a Hilbert complex and (V, d) be its domain, and $I = [0, T]$. Consider

$$\frac{\partial u}{\partial t} = \Delta u + f(t)$$
$$u(0) = g$$

for some $u : I \to V^k$, $f : I \to (V^k)'$, $g \in V^k$, and Δ is the abstract Hodge Lapacian, i.e. we consider a heat equation as an ODE for a curve in a Hilbert space. **Semidiscretization** means we consider

$$u_h(t) = \sum_i U_{h,i}(t)\varphi_i,$$

i.e. "separation of variables" with a basis $\{\varphi_i\}_{i=1}^N$ for the spaces V_h^k. Then approximation is:

$$\sum_{i=1}^N U'_{h,i}(t)\varphi_i = \sum_{i=1}^N U_{h,i}(t)\Delta\varphi_i + f(t).$$

This only applies if φ_i is regular enough. We consider a weak formulation:

Discretization: Non-Mixed Method

Take the inner product with another φ_j, and move the operators to the other side:

$$\sum_i U'_{h,i}(t)\langle\varphi_i,\varphi_j\rangle = -\sum_i U_{h,i}(t)\left(\langle d\varphi_i, d\varphi_j\rangle + \langle d^*\varphi_i, d^*\varphi_j\rangle\right) + \langle f,\varphi_j\rangle.$$

Letting $\mathbf{u}$ be the vector $(U_{h,i})_{i=1}^N$, $M_{ij} = \langle\varphi_i,\varphi_j\rangle$ (the **mass matrix**, and $K_{ij} = \langle d\varphi_i, d\varphi_j\rangle + \langle d^*\varphi_i, d^*\varphi_j\rangle$, the **“stiffness” matrix** (terminology from hyperbolic equations), and $\mathbf{F} = (\langle f,\varphi_j\rangle)_{j=1}^N$, the **“load vector”**, we have

$$M\frac{d\mathbf{u}}{dt} = -K\mathbf{u} + \mathbf{F}$$

solvable by standard ODE methods. On manifolds, and domains in $\mathbb{R}^n$, this is the Hodge heat equation, which is the case that u is a k-form and now Δ is the Hodge Laplacian. For the ordinary scalar heat equation with Dirichlet boundary conditions, they fit into this framework *if u is regarded as an n-form*.

Here is an example of a scalar heat equation solved by the above methods with $k = 0$. (It is not a mixed method—that's what we'll get to next.). It is solved via piecewise linear elements, and applying a backward Euler method to evolve the ODES in time.

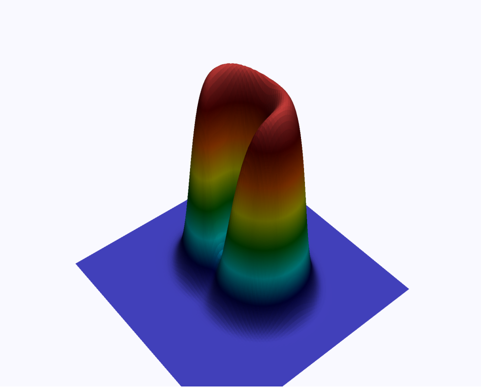

Figure: Scalar Heat Equation

Discretization: Mixed Method

In order to take advantage of the previous theory, of course, we consider the mixed problem, seeking $(\sigma, u) : I \to V^{k-1} \times V^k$ (write $\mathfrak{Y}^k = V^{k-1} \times V^k$), given $f : I \to (V^k)'$ the source and $g \in V^k$ an initial condition, such that:

$$\begin{cases} \langle u, d\tau \rangle - \langle \sigma, \tau \rangle & = 0 \\ \langle u_t, v \rangle + \langle d\sigma, v \rangle + \langle du, dv \rangle & = \langle f, v \rangle \\ u(0) & = g. \end{cases} \tag{11}$$

The harmonic forms evolve with the system; no separate equations are needed. This is why we need the extended formulations for the elliptic problem above. Standard abstract theory of evolution problems in Banach spaces yields results in the *Bochner spaces* (time-parametrized Banach spaces) $L^2(I, \mathfrak{Y}') \cap H^1(I, \mathfrak{Y}) \cap C(I, W^{k-1} \times W^k)$.

Discretization of Mixed Method

We choose spaces V_h^k with $i_h : V_h \to V$, $\pi_h : V \to V_h$ as before. Then we consider

$$\begin{cases} \langle u_h, d\tau\rangle_h - \langle \sigma_h, \tau\rangle_h &= 0 \\ \langle u_{h,t}, v\rangle_h + \langle d\sigma_h, v\rangle_h + \langle du_h, dv\rangle_h &= \langle f_h, v\rangle_h \\ u_h(0) &= g_h, \end{cases} \tag{12}$$

where g_h is suitably projected (*elliptically projected*) initial data. Just as in the discretization for the non-mixed form, this leads to an ODE in a finite-dimensional space and therefore is well-posed by the standard theory.

A Priori Error Estimates for the Parabolic Problem in Euclidean Space

Gillette and Holst [4] prove the following error estimate (for the case of *n*-forms on a domain in $\mathbb{R}^n$, taking the approximating spaces to be a subcomplex), generalizing a result from Thomée:

$$||u_h - u||_{L^2(I,L^2\Lambda^n)} \leq \begin{cases} ch\left(||\Delta u||_{L^2(I,L^2)} + \sqrt{T}\,||\Delta u_t||_{L^1(I,L^2)}\right) & \textit{if } r = 0 \\ ch^{2+s}\left(||\Delta u||_{L^2(I,H^s)} + \sqrt{T}\,||\Delta u_t||_{L^1(I,H^s)}\right) & \textit{for } r > 0, \\ & \quad \textit{if } s \leq r-1 \end{cases}$$

$$||\sigma_h - \sigma||_{L^2(I,L^2\Lambda^{n-1})} \leq \begin{cases} ch\left(||\Delta u||_{L^2(I,H^s)} + \sqrt{T}\,||\Delta u_t||_{L^2(I,L^2)}\right) \\ \qquad \textit{if } r = 0,\ s = 0,\ \Lambda_h^{n-1} = \mathcal{P}_1^-\Lambda^{n-1}(\mathcal{T}) \\ c\left(h^{1+s}\,||\Delta u||_{L^2(I,H^s)} + h\sqrt{T}\,||\Delta u_t||_{L^2(I,L^2)}\right) \\ \qquad \textit{if } r = 0,\ s \leq 1,\ \Lambda_h^{n-1} = \mathcal{P}_1\Lambda^{n-1}(\mathcal{T}) \\ c\left(h^{1+s}\,||\Delta u||_{L^2(I,H^s)} + h^{(3/2)+s}\sqrt{T}\,||\Delta u_t||_{L^2(I,H^s)}\right) \\ \qquad \textit{for } r > 0,\ \textit{if } s \leq r-1 \end{cases}$$

Idea of the Proof for the Parabolic Problem

- A generalization of the method of Thomée [6].
- Key concept: the **elliptic projection**: given the continuous solution $u(t)$, we consider, at each time, the mixed formulation approximation to the problem with data $-\Delta u$ (i.e. the continuous problem $-\Delta u = -\Delta u$): find $(\tilde{\sigma}_h, \tilde{u}_h, \tilde{p}_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ such that for all $(\tau, v, q) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ and $t_0 \in I$,

$$\begin{cases} \langle \tilde{u}_h, d\tau \rangle - \langle \tilde{\sigma}_h, \tau \rangle &= 0 \\ \langle d\tilde{\sigma}_h, v \rangle + \langle d\tilde{u}_h, dv \rangle + \langle \tilde{p}_h, v \rangle &= \langle -\Delta u(t_0), v \rangle \\ \langle \tilde{u}_h, q \rangle &= \langle P_{\mathfrak{H}} u(t_0), q \rangle, \end{cases} \tag{13}$$

- In the case $k = n$ in $\mathbb{R}^n$, the harmonic forms vanish. Similar work done by Arnold and Chen [1] for parabolic problems in more degrees treats cases in which the forms do not vanish.

Idea of the Proof for the Parabolic Problem: Elliptic Projection

- The key is to use $(\tilde{\sigma}_h, \tilde{u}_h, \tilde{p}_h)$ as an intermediate reference, given the semidiscrete (evolving) solution (σ_h, u_h, p_h) and compare using the triangle inequality:

$$\begin{array}{lclcl} \|u_h - u\| & \leq & \overbrace{\|u_h - \tilde{u}_h\|}^{\text{Estimated by generalizing Thomée [6]}} & + & \overbrace{\|\tilde{u}_h - u\|}^{\text{Estimated using AFW [2]}} \\ \|\sigma_h - \sigma\| & \leq & \|\sigma_h - \tilde{\sigma}_h\| & + & \|\tilde{\sigma}_h - \sigma\| \end{array}$$

- Second two terms are why we use elliptic projection, because the AFW [2] estimates may be immediately applied. Also, we use the same projection for the initial data, meaning that the first terms are initially 0.
- The generalization of the method of Thomée is (estimating the first terms) is to use Grönwall estimates to accumulate the norms of the time derivative of the second terms, which in turn can also be estimated by AFW [2].

Idea of the Proof for the Parabolic Problem: Error Evolution

We define the error terms $\rho = \tilde{u}_h - u$, $\theta = u_h - \tilde{u}_h$, and $\varepsilon = \sigma_h - \tilde{\sigma}_h$. Then $||u_h - u|| = ||\theta|| + ||\rho||$.Then we have Thomée's error equations:

$$\begin{aligned} \langle \varepsilon, \omega \rangle - \langle \theta, d\omega \rangle &= 0 \\ \langle \theta_t, \varphi \rangle + \langle d\varepsilon, \varphi \rangle &= \langle -\rho_t, \varphi \rangle \end{aligned}$$

We derive differential inequalities, e.g. setting $\varphi = \theta$, $\omega = \varepsilon$, and combining:

$$||\varepsilon||^2 - \langle \theta, d\varepsilon \rangle + \langle \theta_t, \theta \rangle + \langle d\varepsilon, \theta \rangle = \langle -\rho_t, \theta \rangle.$$

and therefore canceling and dropping the positive $||\varepsilon||^2$,

$$\frac{1}{2}\frac{d}{dt}||\theta||^2 \leq \langle -\rho_t, \theta \rangle \leq ||\rho_t||\,||\theta||.$$

Idea of the Proof for the Parabolic Problem: Grönwall Estimates

Writing $\frac{1}{2}\frac{d}{dt}||\theta||^2 = ||\theta||\frac{d}{dt}||\theta||$, canceling, and integrating,

$$||\theta|| \leq ||\theta(0)|| + \int_0^t ||\rho_t||\, ds$$

Elliptic projection for initial data gives $\theta(0) = 0$, and $||\rho_t||$ can be estimated in the same manner as ρ (the time derivatives also satisfy the equation). Similarly, setting $\varphi = \theta_t$ and $\omega = \varepsilon$ derives an equation for $\frac{d}{dt}||\varepsilon||^2$, but this time with squared norms:

$$||\varepsilon||^2 \leq ||\varepsilon(0)||^2 + \int_0^t ||\rho_t||^2 ds,$$

which is responsible for the estimates that look like $||\cdot||_{L^1(I,L^2)}$ on the one hand, and $||\cdot||_{L^2(I,L^2)}$ on the other.

Abstract Evolution Problem with Variational Crimes

- We modify the error estimate above generalize to non-subcomplex V_h^k (e.g. for hypersurfaces). The key strategy is again to use elliptic projection, but now with the framework of Holst and Stern [5].
- Additional complications arise due to the need for data interpolation (via operators Π_h) and the non-unitarity, beyond just that of the elliptic projection. However, they do yield the same types of error terms. The equation is then

$$\begin{cases} \langle u_h, d\tau\rangle_h - \langle \sigma_h, \tau\rangle_h & = 0 \\ \langle u_{h,t}, v\rangle_h + \langle d\sigma_h, v\rangle_h + \langle du_h, dv\rangle_h & = \langle \Pi_h f, v\rangle_h \\ u_h(0) & = g_h, \end{cases} \tag{14}$$

and the elliptic projection uses operators $\Pi_h(-\Delta u(t))$ and $\Pi_h P_{\mathfrak{H}} u$.

Abstract Evolution Problem with VCs: New Error Equations

- We define, now $\rho(t) = \tilde{u}_h(t) - i_h^* u(t)$. We then have the estimate

$$\|\rho(t)\| \leq \|J_h^{-1}\| \, \|i_h^*\| \left(\|i_h \tilde{u}_h(t) - u(t)\| + \|I - J_h\| \, \|u\| \right)$$

- This leads to the generalized Thomée error equations

$$\begin{aligned} \langle \varepsilon, \omega_h \rangle_h - \langle \theta, d\omega_h \rangle_h &= 0 \\ \langle \theta_t, \varphi_h \rangle_h + \langle d\varepsilon, \varphi_h \rangle_h + \langle d\theta, d\varphi_h \rangle_h &= \langle -\rho_t + \tilde{p}_h + (\Pi_h - i_h^*) u_t, \varphi_h \rangle_h \end{aligned} \tag{15}$$

- Then we proceed via the method of Thomée in much the same way (we do get an extra $\|d\theta\|^2$ along the way).

Abstract Evolution Problem: Collecting all the Terms

Evolution Problem for Hypersurfaces: Implementation Notes

Evolution Problem for Hypersurfaces: Demonstration

This is like the scalar equation before, but now using 2-forms for the spatial discretization and solving with the mixed formulation above.

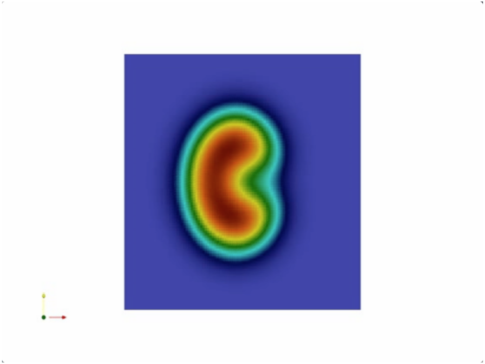

Figure: Hodge Heat Equation

Outline

Some Generalization Directions

- Semidiscretization of Ricci Flow on surfaces (using the above), a quasilinear equation for a metric conformal factor u (metric is $e^{2u}g_0$):

  $$\frac{\partial u}{\partial t} = e^{2u}(\Delta u - K_0)$$

  where K_0 is the Gaussian curvature of the initial metric g_0.
- Interesting examples are extremely hard to visualize as true geometry, due to intrinsic nature of the equation. Finding a suitable embedding, even if imperfect, is itself a very interesting problem.

The Ricci Flow for Rotationally Symmetric Data on S^2

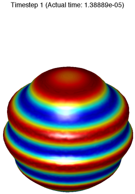

Figure: Ricci Flow on a Sphere

The Wave Equation

Now consider the wave equation,

$$\frac{\partial^2 u}{\partial t^2} = \Delta u.$$

The standard trick is to recast this as a system, by letting, say, $v = u_t$ and writing the equations down for (u, v):

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \Delta u \end{pmatrix}$$

We can actually consider this a mixed problem, an ODE on $V_h \times V_h$ where $V_h = P_1^- \Lambda^0$ as before, with the inner product $(u, v) \cdot (\varphi, \psi) = \langle u, \varphi \rangle + \langle v, \psi \rangle$.

Mixed Formulation: Recasting as a System

Given that, take the inner product of the whole above equation with test functions (φ, ψ) and use the weak form:

$$\frac{\partial}{\partial t}(u, \varphi) = (v, \varphi) \tag{16}$$

$$\frac{\partial}{\partial t}(v, \psi) = -(du, d\psi) \tag{17}$$

Mass and Stiffness Matrices for Hyperbolic Equations

Substituting, as before, basis functions φ_i and ψ_i in V_h, we get the formulation

$$\frac{d\mathbf{u}}{dt} = \mathbf{v} \tag{18}$$

$$M\frac{d\mathbf{v}}{dt} = -K\mathbf{u} \tag{19}$$

Outline

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