Applications of Finite Element Exterior Calculus to Evolution Problems

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Outline

Introduction

- Geometry is a useful tool in many physical models. We aim to use numerical methods to help actually build visual intuition, explore properties of solutions, and generate conjectures.
- Differential geometry helps us identify and focus on the invariants important to problems and formulate differential equations in an invariant way.
- Finite Element Exterior Calculus (FEEC), developed by Arnold, Falk, and Winther (AFW) [2, 3] is a useful framework that allows discretization of equations that respects those invariants. It also provides a framework for error analysis.
- Hilbert complex approach allows abstraction of the essential properties of the spaces we are concerned with, and their approximations.

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Background and History: Field Quantities

Many geometric and physical phenomena (particularly, the concepts of flux, intensity, and density) are expressed in terms of scalar and vector **fields**. For generalization to curved geometries, we use **tensor fields** or **differential forms**.

- Examples include electromagnetism (all of the field intensities, flux densities, currents, and charge densities), fluid flow, and heat flow.
- Various geometric structures on our space, such as metrics, allow an invariant generalization of common differential operators (e.g., the Laplacian) and differential equations.
- We want numerical methods that take these structures into account. For example, differential operators on our fields often are specific combinations of partial derivatives. We want to use methods that recognize and prioritize these combinations.
- The heat, wave, Poisson's, and Maxwell's equations can all be elegantly recast in this manner.

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Hilbert Complexes

Definition (Hilbert Complexes)

A **Hilbert complex** (W, d) is a sequence of Hilbert spaces $(W^k, \langle \cdot, \cdot \rangle)$ with (possibly unbounded) linear maps d^k on the **domains** $V^k \subseteq W^k$), such that $d^k \circ d^{k-1} = 0$ (abbreviated $d^2 = 0$), so it is a **cochain complex**.

$$\cdots \longrightarrow V^{k-1} \xrightarrow{d^{k-1}} V^k \xrightarrow{d^k} V^{k+1} \xrightarrow{d^{k+1}} \cdots$$

- Examples include differential forms on a compact manifold or vector fields in $\mathbb{R}^3$; the W^k are L^2 differential forms or vector fields, and V^k are Sobolev spaces H^1, $H(\mathrm{curl})$, $H(\mathrm{div})$. d is the exterior derivative, or grad, curl, and div.
- The inner product $\langle \cdot, \cdot \rangle$ generalizes the definition via Hodge stars of differential forms.
- The domains V are also a Hilbert complex (V, d), with the **graph inner product** $\langle u, v \rangle_V := \langle u, v \rangle + \langle d^k u, d^k v \rangle$. d is *bounded* in this complex.

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Cohomology in Hilbert Complexes

In analogy to differential forms on manifolds, we have the following definitions for elements of Hilbert complexes:

Definition

$\omega \in V^k$ is called **closed** if $d\omega = 0$. Write $\mathfrak{Z}^k$ for the set of all such ω. ω is called **exact** if $\omega = d\eta$ for some $\eta \in V^{k-1}$; we similarly write $\mathfrak{B}^k$ for the whole space of them. All exact elements are closed, since $d^2 = 0$. Therefore the quotients $\mathfrak{Z}^k/\mathfrak{B}^k$ are well-defined; we call it the **cohomology** of the Hilbert complex (W, d).

Dual Complexes

Functional-analytic assumptions about the maps d^k enable us to find its adjoint:

Definition

Consider the spaces $W_k^* = W^k$ and the domains V_k^* defined by the following property: $\omega \in V_k^*$ if there exists $\sigma \in W_{k-1}^*$ (write $\sigma = d_k^* \omega$) such that

$$\langle d_k^* \omega, \tau \rangle = \langle \omega, d^{k-1} \tau \rangle$$

for all $\tau \in V^{k-1}$. This **adjoint** operator d_k^* maps V_k^* into V_{k-1}^*, and $d_{k-1}^* \circ d_k^* = 0$. (W^*, d^*) is called the **dual complex**.

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- This generalizes the Hodge codifferential of differential forms; the arrows run in the opposite (degree-decreasing) direction—it is a **chain complex**:

$$\cdots \longleftarrow V_{k-1}^* \xleftarrow{d_{k-1}^*} V_k^* \xleftarrow{d_k^*} V_{k+1}^* \xleftarrow{d_{k+1}^*} \cdots$$

- For differential forms or vector fields, the dual complex incorporates *boundary conditions* which take care of boundary terms in integration by parts.
- Since $(d^*)^2 = 0$, we define the corresponding notions of closed, exact, and cohomology for this complex: $\mathfrak{Z}_k^* = \ker d_k^* : V_k^* \to V_{k-1}^*$ and $\mathfrak{B}_k^* = d_{k+1}^*(V_{k+1}^*) \subseteq V_k^*$.
- We define $\mathfrak{H}^k = \mathfrak{Z}^k \cap \mathfrak{Z}_k^*$, the **harmonic space**. It is (in our cases of interest) isomorphic to $\mathfrak{Z}^k/\mathfrak{B}^k$, the cohomology.

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Hodge Decomposition

The above generalizes a major classical result, the Hodge decomposition theorem for differential forms (or for vector fields, the Helmholtz decomposition):

Theorem (Hodge Decomposition Theorem)

Every element of V^k is uniquely the V-orthogonal sum of coboundary, harmonic, and boundary terms:

$$V^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{B}_k^* \tag{1}$$

(for $\mathfrak{B}^k$ and $\mathfrak{H}^k$, the V- and W-inner products coincide, so the sum is also W-orthogonal).

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Hodge Decomposition

- This theorem connects topology, geometry, and analysis; it was developed (for differential forms) in the 1930s and 1940s by Hodge, Kodaira, de Rham, and others.
- This helps to understand the general vector or tensor field in terms of certain canonical or conceptually simpler quantities, abstractly represented by the notion of closedness and co-closedness.
- For example, we have irrotational, or gradient vector fields (physically significant from the derivation of a potential), divergence-free fields (physically significant as incompressible flows), and harmonic fields (combining local aspects of the above) intimately related to topological features of the domain (an example on the next slide).
- This kind of decomposition has applications surprisingly far removed from its origins in geometry and topology.

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An Example Harmonic Vector Field

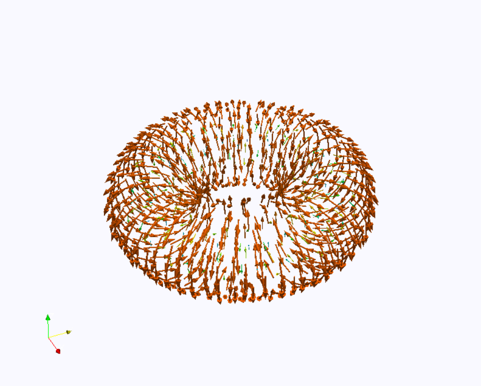

Figure: Harmonic vector field on a torus: this gives an example of an element of a harmonic space of degree $k > 0$. One can see how it relates to topology, here surrounding the "tube" of the torus.

Abstract Poincaré Inequality

The most important basic result for the elliptic equations, their stability, and approximation, is the Poincaré Inequality:

Theorem (Abstract Poincaré Inequality, AFW [3], §3.1.3)

Let (W, d) be a closed Hilbert complex and (V, d) be its domain. Then there exists $c_P > 0$ such that for all $v \in \mathfrak{Z}^{k\perp}$,

$$\|v\|_V \leq c_P \|d^k v\|. \tag{2}$$

Outline

Hodge Laplacian and Elliptic Equations

Definition

$\Delta_d := \Delta = -(dd^* + d^* d)$ is an operator with domain contained in $V^k \cap V_k^*$. Using the definition of the adjoints ("integration by parts"), we weakly formulate the problem $-\Delta u = f$ for $f \in W^k$:

$$\langle -\Delta u, \eta\rangle = \langle du, d\eta\rangle + \langle d^* u, d^*\eta\rangle = \langle f, \eta\rangle.$$

for all $\eta \in V^k \cap V_k^*$. A solution u satisfying the above for all η is called a **weak solution** to $-\Delta u = f$.

To make this well-posed, we W-orthogonally project f onto $\mathfrak{H}^{k\perp W} = (\ker \Delta)^{\perp W}$. For uniqueness, we require the solution u to be in $\mathfrak{H}^{k\perp W}$ as well. It explicitly realizes the Hodge decomposition: if $f \in W^k$ there exist unique $u \in V^k \cap V_k^*$ and $p \in \mathfrak{H}^k$ such that:

$$f = d(d^* u) + p + d^*(du).$$

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Mixed Formulation

Another weak formulation is possible, taking $\sigma = d^*u$ and rewriting it as a *system*: seek $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$ such that for all $(\tau, v, q) \in V^{k-1} \times V^k \times \mathfrak{H}^k$,

$$\begin{cases} \langle u, d\tau\rangle - \langle \sigma, \tau\rangle &= 0 \\ \langle d\sigma, v\rangle + \langle du, dv\rangle + \langle p, v\rangle &= \langle f, v\rangle \\ \langle u, q\rangle &= 0. \end{cases} \tag{3}$$

- (1) weakly expresses that $\sigma = d^*u$, (2) is the actual weak form (removing the harmonic part of f), and (3) enforces perpendicularity to the harmonics.
- This problem is well-posed, and the solution satisfies the *a priori* estimate

$$\|\sigma\|_V + \|u\|_V + \|p\| \leq c\|f\|,$$

where the stability constant c depends only on the Poincaré constant c_P.
- As the abstraction of the elliptic problem with forms, recall the dual complex incorporates boundary conditions; here they are $\operatorname{tr} \star u = 0$ and $\operatorname{tr} \star du = 0$.

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Why Mixed?

- This formulation is better-behaved in approximation theory: the weak formulations avoid explicitly involving the adjoint, so we have more freedom to choose subspaces (which will be finite element spaces in the more concrete settings).
- Well-posedness of the approximations does *not* always follow from the the continuous theory; this setup allows us to prove it for certain choices of approximation spaces.
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Mixed Formulation with Nonvanishing Harmonic Part

We will need to modify things slightly for evolution problems: given a prescribed harmonic part $w \in \mathfrak{H}^k$, now we seek $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak{H}^k$ such that for all $(\tau, v, q) \in V^{k-1} \times V^k \times \mathfrak{H}^k$,

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The last part enforces, by definition, $w = P_{\mathfrak{H}} u$, the orthogonal projection onto the harmonic space. This problem is also well-posed by the same existence theory; it satisfies the *a priori* estimate (with the same constant)

$$\|\sigma\|_V + \|u\|_V + \|p\| \le c(\|f\| + \|w\|).$$

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Outline

Approximation via Subcomplexes , AFW [3]

- We choose certain finite-dimensional subspaces of $V_h^k \subseteq V^k$ (e.g. finite element spaces associated to a triangulation of mesh size h). We refer to such spaces as **discrete**. These must satisfy the **subcomplex property** $d^k V_h^k \subseteq V_h^{k+1}$, and have **bounded cochain projections** $\pi_h^k : V \to V_h^k$ commuting with d. These are, in some sense, interpolation operators.
- **Finite Element Exterior Calculus (FEEC)** is the analysis of finite element methods using subspaces V_h^k with these properties.
- For differential forms, we choose polynomial differential forms on a simplex,

$$\begin{aligned} P_r\Lambda^k(T) = \{\omega \text{ weakly differentiable } :\\ \text{the coefficients } \omega_J \text{ in the } dx^J \text{ basis are polynomials of degree } \leq r\}. \end{aligned}$$

- We also need a related subspace which is dual in some sense, $P_r^-\Lambda^k(T)$, polynomial differential forms defined using another operator. They naturally correspond to geometric features of the domain.

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Discrete Version of the Abstract Mixed Problem

Now we approximate the abstract mixed problem $-\Delta u = f$ by solving a system in the discrete spaces: now we seek $(\sigma_h, u_h, p_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ such that for all $(\tau, v, q) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$,

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AFW [3] establish the following error estimate for approximating mixed variational problems (including nonzero harmonic part):

Theorem (Error Estimates for the Mixed Variational Problem)

Let V_h be a subcomplex of the domain complex (V, d) admitting uniformly V-bounded cochain projections. Let (σ, u, p) and (σ_h, u_h, p_h) be the solutions to the continuous, and resp. the discrete problem. Then

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- This establishes that the solution is *quasi-optimal*, in terms of a constant multiple of Hilbert space best approximation errors.
- Properties of the bounded cochain operator on Euclidean spaces give a bound on these best approximations in terms of powers of the mesh parameter h (generalizing standard approximation theory). Specifically, for $\omega \in H^s\Omega^k(U)$, with U with sufficiently smooth boundary, we have

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Injective Morphisms of Complexes, Holst and Stern [7]

- For geometric problems, we need to relax the assumption that V_h^k are subspaces; instead take another complex (W_h, d_h) with domains (V_h, d_h) and injective morphisms (W-bounded, linear maps that commute with the differentials) $i_h^k : V_h^k \to V^k$ which are not necessarily inclusion.
- If i_h is *unitary*, then we may identify it with the spaces $i_h W_h$, which *are*, in fact, a subcomplex. It is the non-unitarity of the operator that generates new terms.
- We still need V-bounded projection morphisms $\pi_h : V \to V_h$ (also commuting with the differentials), which, instead of idempotency, satisfy $\pi_h \circ i_h = \mathrm{id}$.
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Example for Injective Morphisms: Surface Finite Element Methods

For orientable hypersurfaces M of $\mathbb{R}^{n+1}$, we approximate M with a mesh M_h of simplices in the *ambient space*: Usually, $M_h \not\subseteq M$.

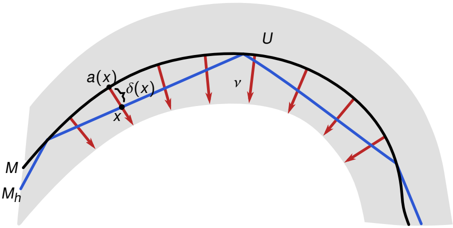

Figure: A curve M with a triangulation M_h (in blue) in a tubular neighborhood U of M. Some normal vectors ν are drawn in red; the distance function δ is measured along this normal. The intersection x of the normal with M_h defines a mapping a from x to its base point $a(x) \in M$.

- Polynomial spaces of forms on the simplices are pulled back via inclusion from those defined on simplices in $\mathbb{R}^{n+1}$. This works because they are affine subsets.
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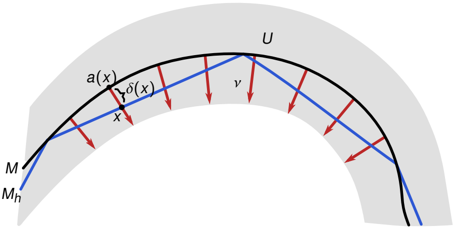

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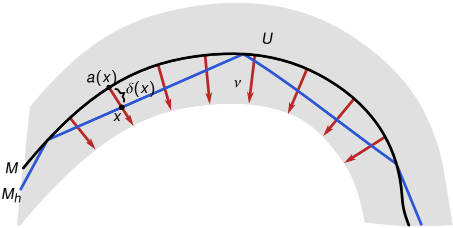

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Higher Order Hypersurface Interpolation

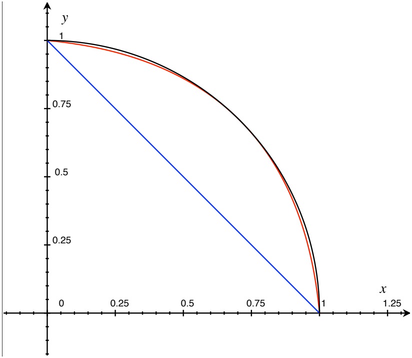

Figure: Approximation of a quarter unit circle (black) with a segment (blue) and quadratic Lagrange interpolation (red) for the normal projection, over the segment.

- Generally, the total error breaks into a PDE approximation term and a hypersurface interpolation term (consistency error).
- So, it is important to be able to interpolate hypersurfaces using higher-order Lagrange interpolation (say of degree $s > 1$).
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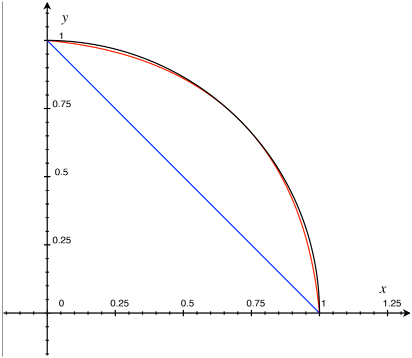

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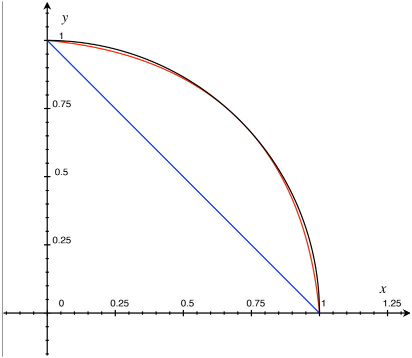

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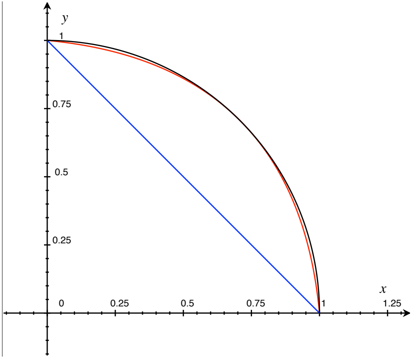

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A Priori Error Estimates for non-Subcomplex Approximations

Holst and Stern [7] generalize the estimates of AFW [3] (new terms in blue), for the case of perpendicularity to the harmonic forms.

Theorem (Error Estimates for the Problem with Variational Crimes)

Let (V_h, d_h) be a domain complex, and $i_h : V_h \to V$ be injective morphisms as above, (V, d), admitting uniformly V-bounded cochain projections. Let (σ, u, p) and (σ_h, u_h, p_h) be the solutions as above. Then

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$$\begin{aligned} &\|\sigma - i_h\sigma_h\|_V + \|u - i_h u_h\|_V + \|p - i_h p_h\| \\ &\qquad \leq C\Bigg(\inf_{\tau \in i_h V_h^{k-1}} \|\sigma - \tau\|_V + \inf_{v \in i_h V_h^k} \|u - v\|_V + \inf_{q \in i_h V_h^k} \|p - q\|_V \\ &\qquad\qquad + \mu \inf_{v \in i_h V_h^k} \|P_{\mathfrak{B}} u - v\|_V + \|f_h - i_h^* f\| + \|I - J_h\| \, \|f\| \Bigg) \end{aligned} \tag{8}$$

with $\mu = \sup_{r \in \mathfrak{H}^k} \|(I - i_h^k \pi_h^k) r\|$.

Idea of the Proof for Variational Crimes

Holst and Stern add two additional *variational crimes* in order to prove the theorem.

- They define $J_h = i_h^* i_h$, the composition of the morphism with its adjoint with respect to the discrete inner product.
- They define an intermediate solution (σ_h', u_h', p_h') by modifying the inner product with J_h. This solution is to the problem on the included spaces $i_h V_h$, and therefore AFW [3] is directly applicable. We only need this solution for the analysis (it is difficult to compute).
- Comparing the intermediate solution to the computed discrete solution yields the terms $\|f_h - i_h^* f\|$ and $\|I - J_h\| \, \|f\|$; the former term is due to the need to approximate the data, and the latter measures the non-unitarity of i_h.
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First Main Result (Result for Elliptic Equations)

Our first result, generalizing Holst and Stern [7] for possible nonzero harmonic part w (newer terms in green). We need it for evolution problems.

Theorem (Main Elliptic Result)

Let (V_h, d_h) be as before. Let (σ, u, p) and (σ_h, u_h, p_h) be as before, but with possibly nonvanishing harmonic parts w and w_h. Then

$$\|\sigma - i_h\sigma_h\|_V + \|u - i_h u_h\|_V + \|p - i_h p_h\|$$

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We rework the proof in Holst and Stern [7]:

- We define, as before, the modified complex and intermediate solution (σ'_h, u'_h, p'_h).
- Complications arise from comparison of the harmonic parts of the discrete and continuous solutions, because they belong to different spaces that may not be preserved by the operators (even in the case of a subcomplex).
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Outline

Adding Time Dependence

- We would like to add time dependence to our problems, so we can solve dynamical equations like the heat, wave, and Maxwell's equations. Traditionally this is done using finite differences, but the advantage of finite element methods and indeed FEEC is to provide a framework for more refined error analysis.
- One way to handle this is semidiscretization (the "Method of Lines"), which factors out the time dependence and discretizes the spatial part using these FEEC spaces, to yield a system of ODEs in the coefficients. These in turn can be numerically solved using standard methods for ODEs, like Euler, Runge-Kutta methods, and symplectic methods.

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The Heat Equation

We consider a heat equation as an ODE for a curve $u : I \to V^k$ ($I = [0, T]$) in one of the Hilbert spaces V^k of a Hilbert complex:

$$\frac{\partial u}{\partial t} = \Delta u + f(t)$$
$$u(0) = g$$

for some source $f : I \to (V^k)'$, and initial condition $g \in V^k$. **Semidiscretization** means we consider

$$u_h(t) = \sum_i U_{h,i}(t)\varphi_i,$$

"separation of variables" with a basis $\{\varphi_i\}_{i=1}^N$ for the spaces V_h^k. Substituting, we have

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Discretization: Non-Mixed Method

Take the inner product with another φ_j, and move the operators to the other side:

$$\sum_i U'_{h,i}(t)\langle \varphi_i, \varphi_j\rangle = -\sum_i U_{h,i}(t)\left(\langle d\varphi_i, d\varphi_j\rangle + \langle d^*\varphi_i, d^*\varphi_j\rangle\right) + \langle f, \varphi_j\rangle.$$

Let $\mathbf{u}$ be the vector $(U_{h,i})_{i=1}^N$, $M_{ij} = \langle \varphi_i, \varphi_j\rangle$ (the **mass matrix**), and $K_{ij} = \langle d\varphi_i, d\varphi_j\rangle + \langle d^*\varphi_i, d^*\varphi_j\rangle$, the **"stiffness" matrix** (solid mechanics and hyperbolic terminology), and $\mathbf{F} = (\langle f, \varphi_j\rangle)_{j=1}^N$, the **"load vector"**. This leads to the system

$$M\frac{d\mathbf{u}}{dt} = -K\mathbf{u} + \mathbf{F}$$

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Here is an example of a scalar heat equation solved by the above methods with $k = 0$. (It is not a mixed method; we cover that case next.) It is solved via piecewise linear elements, and applying a backward Euler method to evolve the ODEs in time.

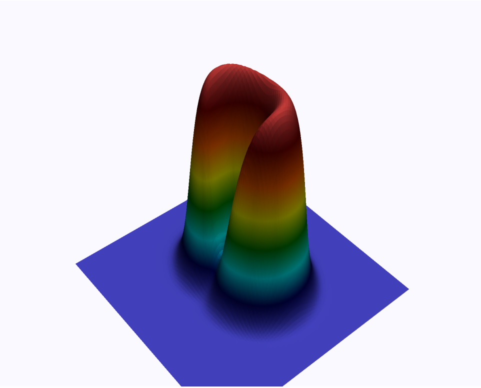

Figure: Scalar Heat Equation

Discretization: Mixed Method

In order to take advantage of the previous theory, we must consider the mixed problem, for $(\sigma, u) : I \to V^{k-1} \times V^k$, given $f : I \to (V^k)'$ the source and $g \in V^k$ an initial condition, such that:

$$\begin{cases} \langle u, d\tau \rangle - \langle \sigma, \tau \rangle & = 0 \\ \langle u_t, v \rangle + \langle d\sigma, v \rangle + \langle du, dv \rangle & = \langle f, v \rangle \\ u(0) & = g. \end{cases} \tag{10}$$

The harmonic forms evolve with the system; this is why we need the extended formulations for the elliptic problem above. Standard theory of evolution problems in Banach spaces yields results in the **Bochner spaces**, time-parametrized Banach spaces.

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Discretization: Mixed Method

We choose spaces V_h^k with $i_h : V_h \to V$, $\pi_h : V \to V_h$ as before. Then we consider the **semidiscrete evolution problem**:

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where g_h is **elliptically projected** initial data. Just as in the discretization for the non-mixed form, this leads to an ODE in a finite-dimensional space and therefore is well-posed by the standard theory.

A Priori Error Estimates for the Parabolic Problem in Euclidean Space

Gillette and Holst [6] prove the following error estimate (for the case of n-forms on a domain in $\mathbb{R}^n$, taking the approximating spaces to be a subcomplex), generalizing a result from Thomée:

Theorem

Let (σ, u) be the continuous solution and (σ_h, u_h) be the semidiscrete solution to the problem for n-forms in an open domain $U \subseteq \mathbb{R}^n$. Then we have the following error estimates:

$$\|u - u_h\|_{L^2(L^2\Omega^n)} \leq ch^{2+s}\left(\|\Delta u\|_{L^2(H^s)} + \sqrt{T}\|\Delta u_t\|_{L^1(H^s)}\right) \quad (12)$$

$$\|\sigma - \sigma_h\|_{L^2(L^2\Omega^{n-1})} \leq c\left(h^{1+s}\|\Delta u\|_{L^2(H^s)} + h^{3/2+s}\sqrt{T}\|\Delta u_t\|_{L^2(H^s)}\right). \quad (13)$$

Idea of the Proof for the Parabolic Problem

- A generalization of the method of Thomée [8].
- Key concept: the **elliptic projection**: given the true solution $u(t)$, we consider, at each time, the mixed approximation to the problem with data $-\Delta u$ (i.e. we apply the discrete solution operator to $-\Delta u$): find $(\tilde{\sigma}_h, \tilde{u}_h, \tilde{p}_h) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ such that for all $(\tau, v, q) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k$ and $t_0 \in I$,

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- In the case $k = n$ in $\mathbb{R}^n$, the harmonic forms vanish. Similar work done by Arnold and Chen [1] for parabolic problems in more degrees treats cases in which harmonic forms do not vanish.

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- The second two terms are the reason for using the elliptic projection, because the AFW [3] estimates may be immediately applied. The same projection is used for the initial data, meaning that the first terms are initially 0.
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- The key is to use $(\tilde{\sigma}_h, \tilde{u}_h, \tilde{p}_h)$ as an intermediate reference, given the semidiscrete (evolving) solution (σ_h, u_h, p_h) and compare using the triangle inequality:

$$
\begin{aligned}
\|u_h - u\| &\leq \overbrace{\|u_h - \tilde{u}_h\|}^{\text{Estimated by generalizing Thomée [8]}} + \overbrace{\|\tilde{u}_h - u\|}^{\text{Estimated using AFW [3]}} \\
\|\sigma_h - \sigma\| &\leq \|\sigma_h - \tilde{\sigma}_h\| + \|\tilde{\sigma}_h - \sigma\|
\end{aligned}
$$

- The second two terms are the reason for using the elliptic projection, because the AFW [3] estimates may be immediately applied. The same projection is used for the initial data, meaning that the first terms are initially 0.
- The generalization of the method of Thomée is (estimating the first terms) is to use Grönwall estimates to accumulate the norms of the time derivative of the second terms, which in turn can also be estimated by AFW [3].

Idea of the Proof for the Parabolic Problem: Error Evolution

- Gillette and Holst [6] (following Thomée) define the error terms $\rho = \tilde{u}_h - u$, $\theta = u_h - \tilde{u}_h$, and $\varepsilon = \sigma_h - \tilde{\sigma}_h$. Then $\|u_h - u\| \le \|\theta\| + \|\rho\|$. This yields **Thomée's error equations**:

$$\begin{aligned} \langle \varepsilon, \omega \rangle - \langle \theta, d\omega \rangle &= 0 \\ \langle \theta_t, \varphi \rangle + \langle d\varepsilon, \varphi \rangle &= \langle -\rho_t, \varphi \rangle \end{aligned}$$

- They then derive differential inequalities, e.g. setting $\varphi = \theta$, $\omega = \varepsilon$, and combining:

$$\|\varepsilon\|^2 - \langle \theta, d\varepsilon \rangle + \langle \theta_t, \theta \rangle + \langle d\varepsilon, \theta \rangle = \langle -\rho_t, \theta \rangle.$$

- Therefore canceling and dropping the positive $\|\varepsilon\|^2$,

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 \le \langle -\rho_t, \theta \rangle \le \|\rho_t\|\,\|\theta\|.$$

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Idea of the Proof for the Parabolic Problem: Grönwall Estimates

Writing $\frac{1}{2}\frac{d}{dt}||\theta||^2 = ||\theta||\frac{d}{dt}||\theta||$, canceling, and integrating,

$$||\theta|| \leq ||\theta(0)|| + \int_0^t ||\rho_t|| \, ds$$

Elliptic projection for initial data gives $\theta(0) = 0$, and $||\rho_t||$ can be estimated in the same manner as ρ (the time derivatives also satisfy the equation). Similarly, setting $\varphi = \theta_t$ and $\omega = \varepsilon$ derives an equation for $\frac{d}{dt}||\varepsilon||^2$, but this time with squared norms:

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Abstract Evolution Problem with Variational Crimes

- We modify the error estimate above generalize to non-subcomplex V_h^k (e.g. for hypersurfaces). The key strategy is again to use elliptic projection, but now with the framework of Holst and Stern [7].
- Additional complications arise due to the need for data interpolation (via operators Π_h) and the non-unitarity, beyond just that of the elliptic projection. However, they do yield the same types of error terms. The equation is then

$$\begin{cases} \langle u_h, d\tau\rangle_h - \langle \sigma_h, \tau\rangle_h &= 0 \\ \langle u_{h,t}, v\rangle_h + \langle d\sigma_h, v\rangle_h + \langle du_h, dv\rangle_h &= \langle \Pi_h f, v\rangle_h \\ u_h(0) &= g_h, \end{cases} \tag{15}$$

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Abstract Evolution Problem with VCs: New Error Equations

- We define, now $\rho(t) = \tilde{u}_h(t) - i_h^* u(t)$. We then have the estimate

$$\|\rho(t)\| \le \|J_h^{-1}\| \, \|i_h^*\| \left(\|i_h \tilde{u}_h(t) - u(t)\| + \|I - J_h\| \, \|u\|\right)$$

- This leads to a generalization of Thomée's error equations:

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- The two new terms capture additional data interpolation error for u_t, and $\tilde{p}_h$ measures precisely how the operator Π_h fails to preserve harmonicity.
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Main Result for the Abstract Evolution Problem: Summary

To summarize, we have the following:

Theorem (Main Parabolic Result)

Let (σ, u) be the true solution, (σ_h, u_h) be the semidiscrete solution, $(\tilde{\sigma}_h, \tilde{u}_h)$ the elliptic projection, and the error quantities be defined as above ($\rho = \tilde{u}_h - i_h^* u$, $\theta = u_h - \tilde{u}_h$, and $\varepsilon = \sigma_h - \tilde{\sigma}_h$; additionally we also define $\psi = \tilde{\sigma}_h - i_h^* \sigma$). We then have the following:

$$\|\theta(t)\|_h \leq \|\rho_t\|_{L^1(I,W_h)} + \|\tilde{\rho}_h\|_{L^1(I,W_h)} + \|(\Pi_h - i_h^*)u_t\|_{L^1(I,W_h)} \tag{17}$$

$$\|d\theta(t)\|_h + \|\varepsilon(t)\|_h \leq C\left(\|\rho_t\|_{L^2(I,W_h)} + \|\tilde{\rho}_h\|_{L^2(I,W_h)} + \|(\Pi_h - i_h^*)u_t\|_{L^2(I,W_h)}\right) \tag{18}$$

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with

$$\|\rho_t\|_{L^2(I,W_h)} \leq C\left(\|i_h\tilde{u}_{h,t} - u_t\|_{L^2(I,W)} + \|I - J_h\| \, \|u_t\|_{L^2(I,W)}\right) \tag{20}$$

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Evolution Problem for Hypersurfaces in Euclidean Space

As an application, we again consider the motivating case of hypersurfaces.

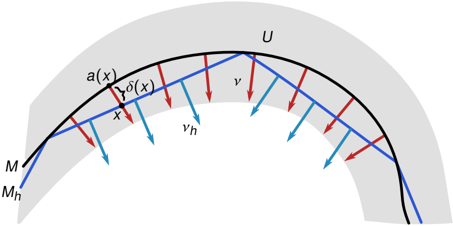

Figure: A curve M with polygonal approximation M_h and distance δ measured along the normal ν to M. The intersection x of the normal with M_h defines a mapping a from x to its base point $a(x) \in M$. Finally, ν_h is normal to M_h.

- Dziuk [5] shows that $\|\nu - \nu_h\|_\infty \le ch$, and Demlow [4] extends this result for degree-s Lagrange-interpolated surfaces, showing $\|\delta\|_{L^\infty(M_h)} \le ch^{s+1}$ and $\|\nu - \nu_h\|_{L^\infty(M_h)} \le ch^s$.
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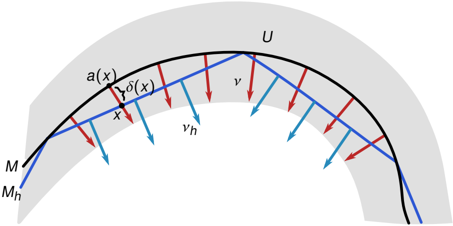

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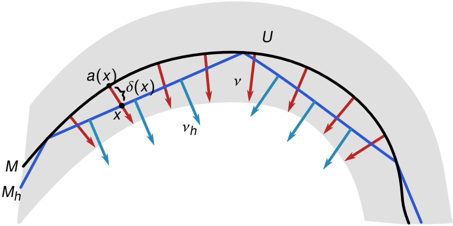

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More Explicit and Improved Estimates for the Problem in Euclidean Space

Theorem (Improved Estimates, AFW [3], p. 342)

Consider the mixed variational problem for the Laplace equation in a triangulated domain in Euclidean space. Let (σ, u, p) be a solution and (σ_h, u_h, p_h) be the discrete solution in polynomial spaces. Suppose the data f is at least H^r-regular. Then we have the following estimates for $0 \le s \le r$ (actually, many different cases here):

$$\begin{aligned}
\|d(\sigma - \sigma_h)\| &\le Ch^s \|f\|_{H^s} \\
\|\sigma - \sigma_h\| &\le Ch^{s+1} \|f\|_{H^s} \\
\|d(u - u_h)\| &\le Ch^{s+1} \|f\|_{H^s} \\
\|u - u_h\| + \|p - p_h\| &\le Ch^{s+2} \|f\|_{H^s}.
\end{aligned}$$

Applying these Estimates to the Problem on Hypersurfaces

The Improved Estimates compare the modified solution to the true solution, the Main Elliptic Result compares the elliptic projection to the modified solution, and the Main Parabolic Result compares the semidiscrete solution to the elliptic projection.

Theorem (Main Parabolic Result, applied to hypersurfaces)

For the evolving solution (σ, u) above, with harmonic part w, and elliptic projection $(\tilde{\sigma}, \tilde{u}, \tilde{p})$, we have the following error estimate:

$$\|u - i_h\tilde{u}_h\| + \|i_h\tilde{p}_h\| + h\left(\|d(u - i_h\tilde{u}_h)\| + \|\sigma - i_h\tilde{\sigma}_h\|\right) + h^2\|d(\sigma - i_h\tilde{\sigma}_h)\| \le C\left(h^{r+1}\left(\|\Delta u\|_{H^{r-1}} + \|w\|_{H^{r+1}}\right) + h^{s+1}\left(\|\Delta u\| + \|w\|\right)\right). \quad (22)$$

Notice the dependence on the degree s of the surface interpolation, vs. the degree r of the polynomial spaces used to approximate the solution. This means choosing $r = s$ (**isoparametric** elements) is best.

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Evolution Problem for Hypersurfaces: Demonstration

This is like the scalar equation before, but now using 2-forms for the spatial discretization and solving with the mixed formulation above.

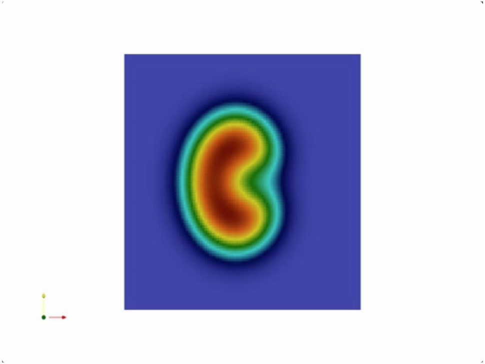

Figure: Hodge Heat Equation

A Quasilinear Equation

- Our next project is the semidiscretization of a quasilinear equation: Ricci Flow on compact surfaces M. It is equivalent to a quasilinear equation for a metric conformal factor u (the evolving metric is $e^{2u}g_b$, with g_b a fixed background metric):

$$\frac{\partial u}{\partial t} = e^{-2u}(\Delta u - K_b)$$

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A Quasilinear Equation: Semidiscretization

- We consider a similar strategy of recasting the problem into a weak form. Here, we have

$$\frac{\partial u}{\partial t} = e^{-2u}(\Delta_b u - K_b) + c =: F(u)$$

where c is a constant that makes the flow have constant volume (normalized Ricci flow).

- First, we rewrite $F(u)$ into **divergence form**, namely d^* of a (nonlinear) function of u and du:

$$F(u) = -d^*(e^{-2u}du) + 2e^{-2u}|du|^2 - e^{-2u}K_b + c.$$

- Choosing a basis φ_i as before, we write $u_h = \sum_i U^i \varphi_i$ as before, take the inner product with φ_j, and move the d^* to the other side for a weak form. This gives a nonlinear discrete operator.

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Approximating the Quasilinear Equation

- Now, given this nonlinear operator, in order to use standard (at least, implicit) ODE solvers, we must use Newton's Method to approximate the next time step from the current one.
- Newton's Method works quite well for sufficiently small timesteps, because the operator to be linearized is actually quite close to the identity for small timesteps.
- Interesting, if more difficult, error analysis for this case—the main tactic is to estimate the error in the Newton iterations, which rely on the linearization, and so the preceding theory applies. The main challenge is that the "constants" in all the preceding now *do* depend on the solution.

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Ricci Flow for Rotationally Symmetric Data on S^2

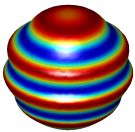

Figure: The changing geometry of the sphere as it evolves from an initial geometry with axially symmetric ridges, under (normalized) Ricci flow. The conformal factor is solved with the above; the assumption of rotational symmetry allows us to derive an embedding equation in cylindrical coordinates which realize this geometry.

Outline

Conclusion

- We have employed the abstract framework of Hilbert complexes to understand much of the fundamentals underlying certain PDEs, as well as their discretization.
- Mixed methods allow us to carry the well-posedness of the continuous problem to that of the discrete problem.
- Our main goal was to find the abstract version of error evolution equations and derive the corresponding error estimates.
- In this process, we also extended a result for elliptic problems to handle the possibility of nonzero harmonic part.
- Our main application of this theory has been to formulate and analyze surface finite element methods.

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Possible Future Directions: Hyperbolic Equations

- For method-of-lines discretizations of the wave equation, we can use the variables (u, u_t) and consider the resulting first-order system.
- Another possibility is the spacetime gradient $du = (u_t, u_x)$ (called the **velocity-stress formulation**) and derive methods based on this. Gillette and Holst [6] also work with this case, generalizing a method of Geveci.
- We would like to generalize, as we did for the parabolic case, this problem to manifolds (including curved spacetimes). An interesting feature of the hyperbolic wave operator (the "Laplacian" for a Lorentzian metric, often denoted $\Box$) in these cases is that there is a nontrivial divergence in time, i.e., rather than operators $u_{tt} - \Delta u$, we have $(au_t)_t - \Delta u$ for some function a.
- We would also like to generalize to non-scalar hyperbolic equations, e.g., Maxwell's equations.

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