

Why Gradients Must Have Zero Curl

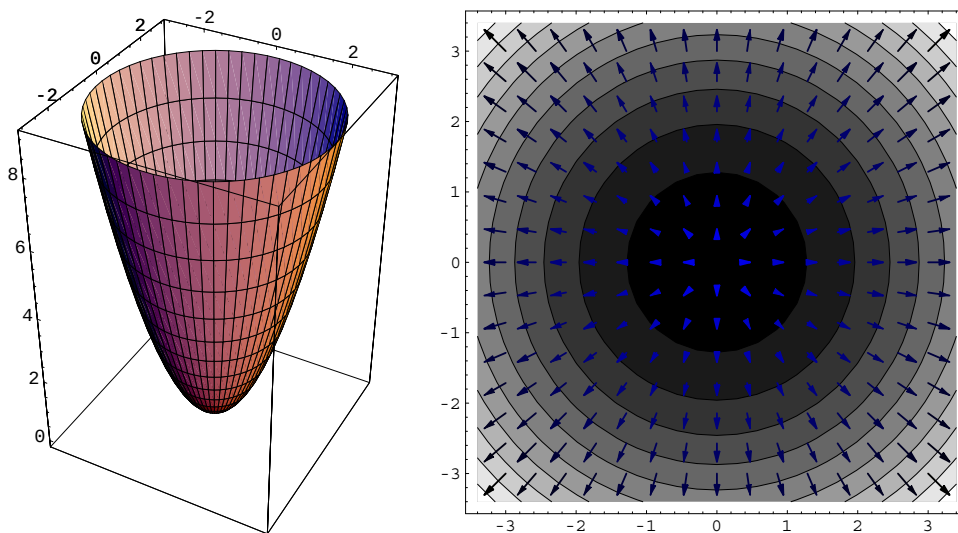
MATH 20E

CHRIS TIEE

1 Why must the gradient have zero curl?

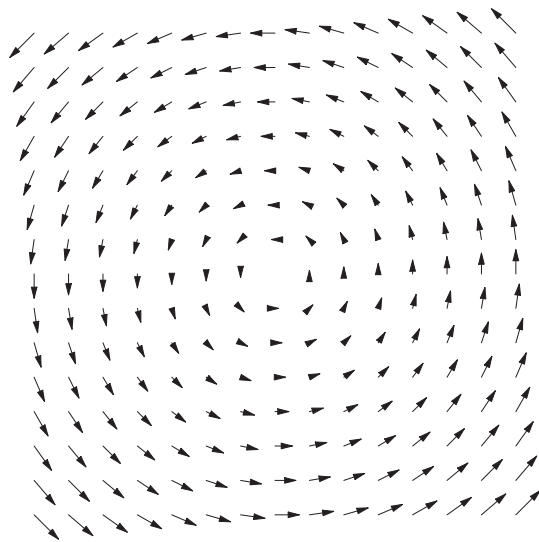
The curious reader may have asked the question “Why must the gradient have zero curl?” The answer, given in our textbook and most others is, simply “equality of mixed partials” that is, when computing the curl of the gradient, every term cancels another out due to equality of mixed partials. That’s all well and good, analytically, and the fact that mixed partials are equal at all is a deep and important result on its own.¹ But what does this mean *geometrically*?

Let’s recall what a gradient field ∇f actually is, for $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ (using 2D to assist in visualization), in terms of the scalar function f . It is a vector pointing in the direction of *increase* of f , pointing away from the level curves of f in the most direct manner possible, i.e. perpendicularly. But what are the level curve, anyway? The sets $f(x,y) = c$; this value of c is the *height* of the graph of f on that curve. Hence ∇f always points in the *uphill* direction of a graph. For example, with the function $f(x,y) = x^2 + y^2$ we have $\nabla f = (2x, 2y)$ (see pictures below, the right hand figure puts the level curves and gradient field together):

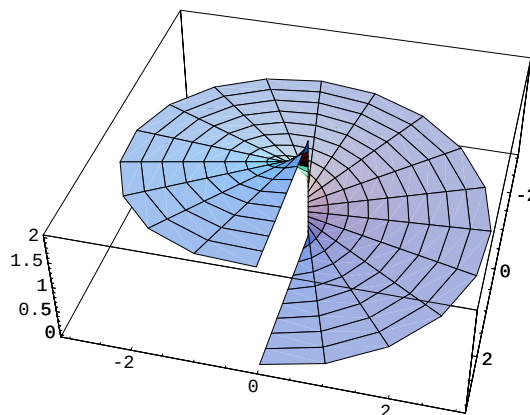
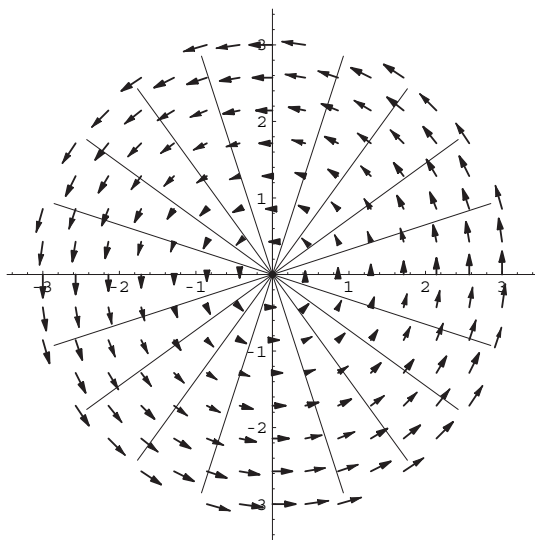


Now let’s take a look at our standard Vector Field With Nonzero curl, $\mathbf{F}(x,y) = (-y, x)$ (the curl of this guy is $(0, 0, 2)$):

¹In fact, a fellow by the name of Georg Friedrich Bernhard Riemann developed a generalization of calculus which one can apply to curved spaces, in which space is flat if and only if (the suitably generalized notion of) mixed partials are equal.



What's wrong with this picture? If all those vectors always point in the direction of increase, then if we traverse a circle counterclockwise, f goes uphill all the way around in a circle. The nostalgic reader may remember the old gag parents tell their children "When I was your age, I had to walk to school uphill both ways," and intuitively knew that this was of course impossible. For if we made such a trip and returned to our starting point, how could we be at a higher level? How can we be walking uphill the whole time? (Actually the fact that the parents' claim is wrong is precisely the fact that the gravitational field is in fact a gradient field! This is not a loosely relevant insert just to make the reader laugh, it is a direct consequence of everything we are talking about right now). But nevertheless let's examine the situation a little more closely: Suppose we could find f such that $\nabla f = \mathbf{F}$. Then looking at the "level curves" of such a function we would have something like the left hand figure below:



The straight level curves—here line segments from the origin to a point on the circle $r = 3$ —would increase in value as we go counterclockwise. To get back to the same height we would have to make a sharp drop; the right hand figure shows a potential (ha ha) candidate for a function f such that $\mathbf{F} = \nabla f$ —i.e., if there were an f such that $\nabla f = \mathbf{F}$, then it would have to look like the thing on the right above. But sharp drops are extremely bad from the standpoint of *differential* calculus. This is not

anyone's idea of a differentiable function. Even if the sharp drop were approximated by some steeply sloping, yet differentiable function (this is always possible by using what are called "smooth bump functions"), this would imply we sketched our vector field wrong. During the steep drop, the gradient field would suddenly swerve into the opposite direction and have a very large magnitude for a short time. But the field $(-y, x)$ never does anything like that, so such a function could never correspond to $(-y, x)$.

Right now the suspicious reader may be wondering, "Well, why can't we just keep going up and up instead of having to make that drop?" Good question. Recall that we are speaking of the *graph* of the function f . A function can have only one value for each point in its domain, so its graph must pass the so-called "vertical line test"—a vertical line (line parallel to the z -axis) cannot intersect the graph at two distinct points.²

The astute reader may have recalled that there is a technique to try to compute the "inverse gradient" — given a vector field $\mathbf{F}$ we seek to construct an f such that $\nabla f = \mathbf{F}$. For example for $\mathbf{F}(x, y) = (2x, 2y)$, we note that if such an f exists, $f_x = \partial f / \partial x = 2x$ implies $f(x, y) = x^2 + C(y)$ for some function C . C is a function of only y because we did a "partial integration" with respect to x and so we must have a "constant" of integration—but any function of only y is constant with respect to x . So we know partially (ha ha) what f must look like. Now we know $f_y = 2y$ and so $C'(y) = 2y$. This means $C(y) = y^2 + K$ where K is now a genuine constant (because C was only a function of y). Therefore any f such that $\nabla f = (2x, 2y)$ must have the form $f(x, y) = x^2 + y^2 + K$ for any constant K .

Now let's try it on $\mathbf{F}(x, y) = (-y, x)$ which we know isn't the gradient of anything. Suppose such an f exists. Then $f_x = -y$. Therefore $f(x, y) = -yx + C(y)$ for some function C of y only. Then $f_y = x = -x + C'(y)$. This means $C'(y) = 2x$ or that $C(y) = 2xy + K$ for some constant K . But this is Very Bad, because we assumed that C was a function of y only, whereas this shows quite blatantly it is not a function of only y . But still, the stiff-necked reader may insist upon taking $f(x, y) = -xy + 2xy + K$ anyway. This gives $f(x, y) = xy + K$. But $\nabla f = (y, x)$ which is most certainly not $(-y, x)$ so still no luck. And since its curl is not zero, further efforts along these lines will be thwarted.

2 The Converse Question

The exceedingly inquisitive reader is likely to have asked the question: Does $\nabla \times \mathbf{F} = 0$ imply $\mathbf{F} = \nabla f$ for some f ? The answer is a resounding *sometimes*. This is, in fact, a very deep question in which the most precise answer one can get requires graduate-level mathematics. Or, as the cinephilic reader may recall, mathematics that John Nash demanded of his vector calculus students. The scene is where John Nash throws a multivariable calculus text into the trash and writes up on the board the following: Given

$$V = \{\mathbf{F} : \mathbb{R}^3 \setminus X \rightarrow \mathbb{R}^3 \mid \nabla \times \mathbf{F} = 0\}$$

and

$$W = \{\mathbf{F} \mid \mathbf{F} = \nabla g\},$$

what is $\dim(V/W)$? A full answer to this requires something called the de Rham cohomology. Asking whether or not the converse hold is equivalent to asking whether or not the answer to this question is 0. It turns out that for $X = \emptyset$, that is for vector fields $\mathbf{F}$ defined on all of $\mathbb{R}^3$, the converse is in fact true, and we can always find an f such that $\nabla f = \mathbf{F}$. For vector fields that omit one point (for example

²Actually, one *can* define graphs of functions like this, called *Riemann surfaces*, yes that same Riemann in the last footnote. But the usual methods of multivariable calculus cannot be used to handle these objects.

vector fields not defined at 0, that is X in the above is the singleton $\{0\}$ this need not be the case. We can prove that the dimension is at least 1 by giving the following evil vector field:

$$\mathbf{G}(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right) = \left(\frac{-y}{r^2}, \frac{x}{r^2} \right)$$

for $(x, y) \neq \mathbf{0}$ (where $r = \sqrt{x^2 + y^2}$). This vector field looks like our canonical example of a Vector Field With Nonzero Curl, $(-y, x)$ except that the vectors grow in magnitude as they approach the origin, and it is left undefined at $\mathbf{0}$. By the same arguments above, this function is certainly not the gradient of anything as such a function would suffer the same problems as the $(-y, x)$ field (which we will call $\mathbf{F}$ as before). But a calculation gives

$$\nabla \times \mathbf{G} = \nabla \times \frac{1}{r^2} \mathbf{F} = \nabla(r^{-2}) \times \mathbf{F} + r^{-2} \nabla \times \mathbf{F}$$

The second term in this sum is just $(0, 0, 2r^{-2})$ since we know $\nabla \times \mathbf{F} = (0, 0, 2)$. Now $\nabla(r^{-2}) = -2r^{-4}\mathbf{r}$ where $\mathbf{r} = (x, y)$ by a general formula (problem 30 of 4.4). What is $\mathbf{r} \times \mathbf{F}$ (physicists: no it is not torque, sorry =) actually well if $\mathbf{F}$ is a force-field and $\mathbf{r}$ is the radius vector, it really is a torque, but this is another story). $(x, y, 0) \times (-y, x, 0) = (0, 0, x^2 + y^2) = (0, 0, r^2)$. Therefore $-2r^{-4}(0, 0, r^2) = (0, 0, -2r^{-2})$ which, when added to $(0, 0, 2r^{-2})$ gives us $\mathbf{0}$. This answers the converse in the negative. Geometrically speaking, the structure of gradient fields vs. that of curl-less (irrotational) fields is related to the number of “holes” in the space; $\mathbb{R}^2$ minus the origin has one “hole.” To answer Nash’s question in this case, the span of this single function, a “nontrivial representative of V/W ” means the space V/W is at least 1-dimensional. To prove that it is exactly 1-dimensional, however, is the hard part and that is what requires the de Rham cohomology (that sounds really cool, doesn’t it?). I am only barely just learning this material right now, whereas one Winter Quarter 5 years ago, I learned vector calculus.