1. (6 points) Let $F$ be the function defined by $F(x, y) = e^{(x-1)^2+y}$.

(a) Compute algebraically the partial derivatives $F_x$ and $F_y$.

**SOLUTION:**

$$F_x = \frac{\partial}{\partial x} F = e^{(x-1)^2+y} \cdot \frac{\partial}{\partial x} ((x-1)^2 + y) = e^{(x-1)^2+y} \cdot (2x - 1)$$

$$F_y = \frac{\partial}{\partial y} F = e^{(x-1)^2+y} \cdot \frac{\partial}{\partial y} ((x-1)^2 + y) = e^{(x-1)^2+y} \cdot 1$$

**ANSWER:** $F_x = 2e^{(x-1)^2+y}(x - 1)$, $F_y = e^{(x-1)^2+y}$

(b) What is the equation of the plane tangent to $F$ at the point $(1, 0)$?

**SOLUTION:**

Recall that the equation of the tangent plane to the surface $z = f(x, y)$ at the point $(x_0, y_0, z_0)$ is given by

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

*If you find typos (which is likely) or errors in logic (hopefully less likely) in these solutions, please let me know either on Piazza or by email (jbonthiu@ucsd.edu)
Then

\[ F_x(1, 0) = e^{(1-1)^2 + y} \cdot 2(1 - 1) \]
\[ = 0 \]
\[ F_y(1, 0) = e^{(1-1)^2 + 0} \]
\[ = 1 \]
\[ z_0 = F(1, 0) \]
\[ = e^{(1-1)^2 + 0} \]
\[ = 1 \]

Thus, we have

\[ z - 1 = 0(x - 1) + 1(y - 0) \]
\[ \implies z = y + 1 \]

**ANSWER:** \[ z = y + 1 \]
2. (6 points) A plane is traveling due north with an airspeed of 725 km/hr while descending at a rate of 75 km/hr. There is a 60 km/hr wind blowing from 30 degrees south of due west. What is the ground speed of the airplane?

**SOLUTION:**
Although you guys have not added vectors in $\mathbb{R}^3$ yet (I believe?) the process is the same. We break the vectors into their respective $x$, $y$, and $z$ ($\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$) components, and add them all together.
So let $\vec{P}$ represent the plane’s motion, and $\vec{w}$ represent the wind forcing. Then

\[
\vec{P} = (0, 725, -75)
\]
\[
\vec{w} = (30\sqrt{3}, 30, 0)
\]

Then the speed relative to the ground is given by $\left\|\vec{P} + \vec{w}\right\| = \sqrt{(30\sqrt{3})^2 + (755)^2 + (75)^2} = \sqrt{578350} \approx 760.5$

**ANSWER:** $\sqrt{2700 + 755^2 + 75^2}$ is fine.
3. (6 points) Let \( f(x, y) = 2x^2 + 3xy + 5y^2 \). At the point \((-2, 1)\):

(a) Find a unit vector \( \vec{u} \) so that the directional derivative \( f_u(-2, 1) \) is maximum.

**SOLUTION:** Recall the the gradient of a function at a point, \( \nabla f(x_0, y_0) \), gives the direction of maximum rate of change at a point. In other words, it maximizes the direction derivative, so is exactly what we need.

\[
\nabla f(-2, 1) = f_x(-2, 1)\hat{i} + f_y(-2, 1)\hat{j} = (4x + 3y)\hat{i} + (3x + 10y)\hat{j}\big|_{(-2, 1)} = -5\hat{i} + 4\hat{j}
\]

So \( \nabla f(-2, 1) = -5\hat{i} + 4\hat{j} \) points in the direction of the maximum rate of change at \((-2, 1)\). Now we need to make this vector unit length:

\[
\frac{\nabla f(-2, 1)}{\|\nabla f(-2, 1)\|} = \frac{-5\hat{i} + 4\hat{j}}{\sqrt{25 + 16}} = \frac{-5}{\sqrt{41}}\hat{i} + \frac{4}{\sqrt{41}}\hat{j}
\]

**ANSWER:** \( \vec{u} = \frac{-5}{\sqrt{41}}\hat{i} + \frac{4}{\sqrt{41}}\hat{j} \)

(b) Find a unit vector \( \vec{u} \) so that the directional derivative \( f_u(-2, 1) \) is minimum.

**SOLUTION:** While the gradient gives the direction of the maximum rate of change at a point, the negative of the gradient gives the minimum rate of change at a point. So we just take the negative of the unit vector found in part (a).

**ANSWER:** \( \vec{u} = \frac{5}{\sqrt{41}}\hat{i} - \frac{4}{\sqrt{41}}\hat{j} \)

(c) Find a unit vector \( \vec{u} \) so that the directional derivative \( f_u(-2, 1) \) is zero.

**SOLUTION:** For the directional derivative of a vector \( \vec{u} \) to be zero means that \( \vec{u} \) is parallel to the contours of the function. Since the gradient at the same point is perpendicular to the contours, then we just need to find a vector that is perpendicular to the gradient, i.e., \( \vec{u} \cdot \nabla f(-2, 1) = 0 \)

\[
0 = \vec{u} \cdot \nabla f(-2, 1) = (u_1, u_2, u_3) \cdot (-5, 4, 0) = -5u_1 + 4u_2 \\
\implies 5u_1 = 4u_2
\]

The choice \( u_1 = 4, u_2 = 5 \) works fine, so \( \vec{u} = 4\hat{i} + 5\hat{j} \). It doesn’t really matter what
you choose, because we are going to force it to be unit-length.

\[
\vec{u} = \frac{4\vec{i} + 5\vec{j}}{\sqrt{16 + 25}} = \frac{4}{\sqrt{41}}\vec{i} + \frac{5}{\sqrt{41}}\vec{j}
\]

ANSWER: \[
\vec{u} = \frac{4}{\sqrt{41}}\vec{i} + \frac{5}{\sqrt{41}}\vec{j}
\]
4. (6 points) Let \( f(x, y) = x^3 + y^2 - 3x^2 - 2y + 10 \). Find and critical points and classify each as a local maximum, local minimum, or saddle point.

**SOLUTION:**
Recall that critical points occur when \( \nabla f = \vec{0} \). Then

\[
\nabla f = f_x \vec{i} + f_y \vec{j} \\
= (3x^2 - 6x) \vec{i} + (2y - 2) \vec{j} = \vec{0}
\]

Then by considering each individual component, we get

\[
3x^2 - 6x = 0 \\
\implies 3x^2 = 6x \\
\implies x = 0 \text{ or } x = 2
\]

\[
2y - 2 = 0 \\
\implies 2y = 2 \\
\implies y = 1
\]

Thus, our critical points are \((0, 1)\) and \((2, 1)\). In order to classify these critical points, we use the second derivative test for functions of two variables (d-test)

\[
f_{xx} = 6x - 6 \\
f_{yy} = 2 \\
f_{xy} = 0
\]

\[
D = f_{xx}f_{yy} - (f_{xy})^2 \\
= 12x - 12
\]

\[
D(0, 1) = -12 < 0 \\
\implies \text{saddle}
\]
\[ D(2, 1) = 24 - 12 \]
\[ = 12 > 0 \]
\[ f_{xx}(2, 1) = 12 - 6 \]
\[ = 6 > 0 \]
\[ \Rightarrow \text{local min} \]

**ANSWER:** \((0, 1) : \text{saddle}, (2, 1) : \text{local min}\)

As an aside, this is the contour plot of the function. Can you see the critical points?