Math 10C
Practice Midterm #2 Solutions*

February 23, 2016

1. (6 points) Let F be the function defined by $F(x,y) = e^{(x-1)^2+y}$.

   (a) Compute algebraically the partial derivatives F_x and F_y.
   SOLUTION:

$$\begin{aligned} F_x &= \frac{\partial}{\partial x} F \\ &= e^{(x-1)^2+y} \cdot \frac{\partial}{\partial x}((x-1)^2 + y) \\ &= e^{(x-1)^2+y} \cdot 2(x-1) \end{aligned}$$

$$\begin{aligned} F_y &= \frac{\partial}{\partial y} F \\ &= e^{(x-1)^2+y} \cdot \frac{\partial}{\partial y}((x-1)^2 + y) \\ &= e^{(x-1)^2+y} \cdot 1 \end{aligned}$$

   ANSWER: $\boxed{F_x = 2e^{(x-1)^2+y}(x-1),\ F_y = e^{(x-1)^2+y}}$

   (b) What is the equation of the plane tangent to F at the point $(1,0)$?
   SOLUTION:
   Recall that the equation of the tangent plane to the surface $z = f(x,y)$ at the point (x_0, y_0, z_0) is given by

$$z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

*If you find typos (which is likely) or errors in logic (hopefully less likely) in these solutions, please let me know either on Piazza or by email (jbonthiu@ucsd.edu)

Then

$$
\begin{aligned}
F_x(1,0) &= e^{(1-1)^2+y} \cdot 2(1-1) \\
&= 0 \\
F_y(1,0) &= e^{(1-1)^2+0} \\
&= 1 \\
z_0 &= F(1,0) \\
&= e^{(1-1)^2+0} \\
&= 1
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
z - 1 &= 0(x-1) + 1(y-0) \\
\implies z &= y + 1
\end{aligned}
$$

ANSWER: $\boxed{z = y + 1}$

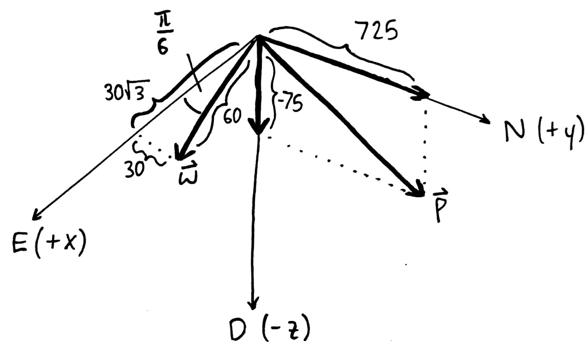


Figure 1:

2. (6 points) A plane is traveling due north with an airspeed of 725 km/hr while descending at a rate of 75 km/hr. There is a 60 km/hr wind blowing from 30 degrees south of due west. What is the ground speed of the airplane?

SOLUTION:

Although you guys have not added vectors in $\mathbb{R}^3$ yet (I believe?) the process is the same. We break the vectors into their respective x, y, and z ($\vec{i}$, $\vec{j}$, and $\vec{k}$) components, and add them all together.

So let $\vec{P}$ represent the plane's motion, and $\vec{w}$ represent the wind forcing. Then

$$\vec{P} = (0, 725, -75)$$
$$\vec{w} = (30\sqrt{3}, 30, 0)$$

Then the speed relative to the ground is given by $\left\|\vec{P} + \vec{w}\right\| = \sqrt{(30\sqrt{3})^2 + (755)^2 + (75)^2} = \sqrt{578350} \approx 760.5$

ANSWER: $\boxed{\sqrt{2700 + 755^2 + 75^2}}$ is fine.

3. (6 points) Let $f(x,y) = 2x^2 + 3xy + 5y^2$. At the point $(-2,1)$:

(a) Find a unit vector $\vec{u}$ so that the directional derivative $f_u(-2,1)$ is maximum.
SOLUTION: Recall the the gradient of a function at a point, $\nabla f(x_0, y_0)$, gives the direction of maximum rate of change at a point. In other words, it maximizes the direction derivative, so is exactly what we need.

$$\begin{aligned}\nabla f(-2,1) &= f_x(-2,1)\vec{i} + f_y(-2,1)\vec{j}\\ &= (4x+3y)\vec{i} + (3x+10y)\vec{j}\,\Big|_{(-2,1)}\\ &= -5\vec{i} + 4\vec{j}\end{aligned}$$

So $\nabla f(-2,1) = -5\vec{i} + 4\vec{j}$ points in the direction of the maximum rate of change at $(-2,1)$. Now we need to make this vector unit length:

$$\begin{aligned}\frac{\nabla f(-2,1)}{\|\nabla f(-2,1)\|} &= \frac{-5\vec{i}+4\vec{j}}{\sqrt{25+16}}\\ &= -\frac{5}{\sqrt{41}}\vec{i} + \frac{4}{\sqrt{41}}\vec{j}\end{aligned}$$

ANSWER: $\boxed{\vec{u} = -\frac{5}{\sqrt{41}}\vec{i} + \frac{4}{\sqrt{41}}\vec{j}}$

(b) Find a unit vector $\vec{u}$ so that the directional derivative $f_u(-2,1)$ is minimum.
SOLUTION: While the gradient gives the direction of the maximum rate of change at a point, the negative of the gradient gives the minimum rate of change at a point. So we just take the negative of the unit vector found in part (a).

ANSWER: $\boxed{\vec{u} = \frac{5}{\sqrt{41}}\vec{i} - \frac{4}{\sqrt{41}}\vec{j}}$

(c) Find a unit vector $\vec{u}$ so that the directional derivative $f_u(-2,1)$ is zero.
SOLUTION: For the directional derivative of a vector $\vec{u}$ to be zero means that $\vec{u}$ is parallel to the contours of the function. Since the gradient at the same point is perpendicular to the contours, then we just need to find a vector that is perpendicular to the gradient, i.e., $\vec{u}\cdot\nabla f(-2,1) = 0$

$$\begin{aligned}0 &= \vec{u}\cdot\nabla f(-2,1)\\ &= (u_1, u_2, u_3)\cdot(-5,4,0)\\ &= -5u_1 + 4u_2\\ \implies 5u_1 &= 4u_2\end{aligned}$$

The choice $u_1 = 4$, $u_2 = 5$ works fine, so $\vec{u} = 4\vec{i} + 5\vec{j}$ It doesn't really matter what you choose, because we are going to force it to be unit-length.

$$\frac{\vec{u}}{\|\vec{u}\|} = \frac{4\vec{i} + 5\vec{j}}{\sqrt{16+25}} = \frac{4}{\sqrt{41}}\vec{i} + \frac{5}{\sqrt{41}}\vec{j}$$

ANSWER: $\boxed{\vec{u} = \frac{4}{\sqrt{41}}\vec{i} + \frac{5}{\sqrt{41}}\vec{j}}$

4. (6 points) Let $f(x,y) = x^3 + y^2 - 3x^2 - 2y + 10$. Find and critical points and classify each as a local maximum, local minimum, or saddle point.
SOLUTION:
Recall that critical points occur when $\nabla f = \vec{0}$. Then

$$\begin{aligned}\nabla f &= f_x\vec{i} + f_y\vec{j} \\ &= (3x^2 - 6x)\vec{i} + (2y-2)\vec{j} = \vec{0}\end{aligned}$$

Then by considering each indivudl component, we get

$$\begin{aligned} 3x^2 - 6x &= 0 \\ \implies 3x^2 &= 6x \\ \implies x &= 0 \text{ or} \\ x &= 2 \end{aligned}$$

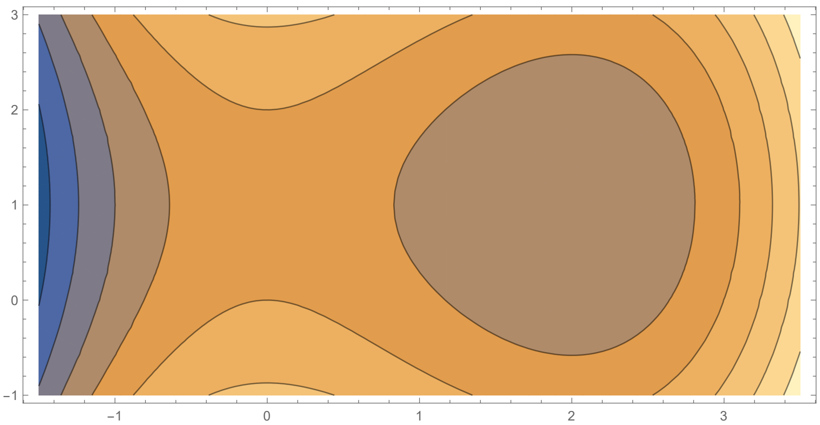
$$\begin{aligned} 2y - 2 &= 0 \\ \implies 2y &= 2 \\ \implies y &= 1 \end{aligned}$$

Thus, our critical points are $(0,1)$ and $(2,1)$. In order to classify these critical points, we use the second derivative test for functions of two variables (d-test)

$$\begin{aligned} f_{xx} &= 6x - 6 \\ f_{yy} &= 2 \\ f_{xy} &= 0 \end{aligned}$$

$$\begin{aligned} D &= f_{xx}f_{yy} - (f_{xy})^2 \\ &= 12x - 12 \end{aligned}$$

$$\begin{aligned} D(0,1) &= -12 < 0 \\ &\implies \text{saddle} \end{aligned}$$

Figure 2: 

$$
\begin{aligned}
D(2,1) &= 24 - 12 \\
&= 12 > 0 \\
f_{xx}(2,1) &= 12 - 6 \\
&= 6 > 0 \\
&\implies \text{local min}
\end{aligned}
$$

ANSWER: $\boxed{(0,1): \text{ saddle}, (2,1): \text{ local min}}$

As an aside, this is the contour plot of the function. Can you see the critical points?