

Equations for a Projected-Search Path-Following Method for Nonlinear Optimization

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Abstract

In [2], Gill and Zhang propose a primal-dual path-following method for general nonlinearly constrained optimization that combines a shifted primal-dual path-following method with a projected-search method for bound-constrained optimization. The method involves the computation of an approximate Newton direction for a primal-dual penalty-barrier function that incorporates shifts on both the primal and dual variables. This note concerns the formulation of approximate Newton equations for a nonlinear optimization problem in general form. These equations may be used in conjunction with a projected-search method to force convergence from an arbitrary starting point. It is shown that under certain conditions, the approximate Newton equations are equivalent to a regularized form of the conventional primal-dual path-following equations.

Key words. Nonlinearly constrained optimization, path-following methods, primal-dual methods, shifted penalty and barrier methods, projected-search methods, Armijo line search, augmented Lagrangian methods, regularized methods.

AMS subject classifications. 49J20, 49J15, 49M37, 49D37, 65F05, 65K05, 90C30

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1. Introduction

This note concerns that derivation of the primal-dual equations for a shifted primal-dual penalty-barrier merit method for constrained optimization. These methods are intended for the minimization of a twice-continuously differentiable function subject to both equality and inequality constraints that may include a set of twice-continuously differentiable constraint functions. A description of the projected-search method for a problem with nonlinear inequality constraints is given by Gill and Zhang [2]. The equations are formulated for problems written in the general form:

$$\underset{x \in \mathbb{R}^n, s \in \mathbb{R}^m}{\text{minimize}} \quad f(x) \quad \text{subject to} \quad \begin{cases} c(x) - s = 0, & L_X s = h_X, & \ell^s \leq L_L s, & L_U s \leq u^s, \\ Ax - b = 0, & E_X x = b_X, & \ell^x \leq E_L x, & E_U x \leq u^x, \end{cases} \quad (\text{NLP})$$

where A denotes a constant $m_A \times n$ matrix, and $b, h_X, b_X, \ell^s, u^s, \ell^x$ and u^x are fixed vectors of dimension $m_A, m_X, n_X, m_L, m_U, n_L$ and n_U , respectively. Similarly, L_X, L_L and L_U denote fixed matrices of dimension $m_X \times m, m_L \times m$ and $m_U \times m$, respectively, and E_X, E_L and E_U are fixed matrices of dimension $n_X \times n, n_L \times n$ and $n_U \times n$, respectively. Throughout the discussion, the functions $c : \mathbb{R}^n \mapsto \mathbb{R}^m$ and $f : \mathbb{R}^n \mapsto \mathbb{R}$ are assumed to be twice-continuously differentiable. The components of s may be interpreted as slack variables associated with the nonlinear constraints.

The quantity E_X denotes an $n_X \times n$ matrix formed from n_X independent rows of I_n , the identity matrix of order n . This implies that the equality constraints $E_X x = b_X$ fix n_X components of x at the corresponding values of b_X . Similarly, E_L and E_U denote $n_L \times n$ and $n_U \times n$ matrices formed from subsets of rows of I_n such that $E_X^T E_L = 0, E_X^T E_U = 0$, i.e., a variable is either fixed or free to move, possibly bounded by an upper or lower bound. Note that an x_j may be an unrestricted variable in the sense that it is neither fixed nor subject to an upper or lower bound, in which case e_j^T is not a row of E_X, E_L or E_U . Analogous definitions hold for L_X, L_L and L_U as subsets of rows of I_m . However, we impose the restriction that a given s_j must be either fixed or restricted by an upper or lower bound, i.e., there are no unrestricted slacks¹. Let E_F denote the matrix of rows of I_n that are not rows of E_X , and let L_F denote the matrix of rows of I_m that are not rows of L_X . If $n_F = n - n_X$ and $m_F = m - m_X$, then E_F and L_F are $n_F \times n$ and $m_F \times m$ respectively. Note that $n_L + n_U$ may be less than n_F , but m_F must equal $m_L + m_U$. The matrices $\begin{pmatrix} E_X^T & E_F^T \end{pmatrix}$ and $\begin{pmatrix} L_X^T & L_F^T \end{pmatrix}$ are column permutations of I_n and I_m . Moreover, there are $n \times n$ and $m \times m$ permutation matrices P_x and P_s such that

$$P_x = \begin{pmatrix} E_F \\ E_X \end{pmatrix} \quad \text{and} \quad P_s = \begin{pmatrix} L_F \\ L_X \end{pmatrix},$$

with $E_F E_F^T = I_F^x, E_X E_X^T = I_X^x$, and $E_F E_X^T = 0$, and $L_F L_F^T = I_F^s, L_X L_X^T = I_X^s$, and $L_F L_X^T = 0$.

All general inequality constraints are imposed indirectly using a shifted primal-dual barrier function. The general equality constraints $c(x) - s = 0$ and $Ax = b$ are enforced using an primal-dual augmented Lagrangian algorithm, which implies that the equalities are satisfied in the limit. The exception to this is when the constraints $E_X x = b_X$, and $L_X s = h_X$ are used to fix a subset of the variables and slacks. These bounds are enforced at every iterate.

¹This is not a significant restriction because a ‘‘free’’ slack is equivalent to a unrestricted nonlinear constraint, which may be discarded from the problem. The shifted primal-dual penalty-barrier equations can be derived without this restriction, but the derivation is beyond the scope of this note.

An equality constraint $c_i(x) = 0$ may be handled by introducing the slack variable s_i and writing the constraint as the two constraints $c_i(x) - s_i = 0$ and $s_i = 0$. In this case the i th coordinate vector e_i can be included as a row of L_x . Linear *inequality* constraints must be included as part of c . A linear equality constraint can be either included with the nonlinear equality constraints or the matrix A . The constraints involving A may be used to temporarily fix a subset of the variables at their bounds without altering the underlying structure of the approximate Newton equations. In this case, the associated rows of A are rows of the identity matrix.

The optimality conditions for problem (NLP) are given in Section 2. The shifted path-following equations are formulated in Section 3. The shifted primal-dual penalty-barrier function associated with problem is discussed in Section 4. This function serves as a merit function for the projected-search method. The equations are formulated in Sections 5 and 6, and summarized in Section 7. The analogous equations for the trust-region method are derived in Section 8 and summarized in Section 9.

Notation. Given vectors x and y , the vector consisting of x augmented by y is denoted by (x, y) . The subscript i is appended to vectors to denote the i th component of that vector, whereas the subscript k is appended to a vector to denote its value during the k th iteration of an algorithm, e.g., x_k represents the value for x during the k th iteration, whereas $[x_k]_i$ denotes the i th component of the vector x_k . Given vectors a and b with the same dimension, the vector with i th component $a_i b_i$ is denoted by $a \cdot b$. Similarly, $\min(a, b)$ is a vector with components $\min(a_i, b_i)$. The vector e denotes the column vector of ones, and I denotes the identity matrix. The dimensions of e and I are defined by the context. The vector two-norm or its induced matrix norm are denoted by $\|\cdot\|$. For brevity, in some equations the vector $g(x)$ is used to denote $\nabla f(x)$, the gradient of $f(x)$. The matrix $J(x)$ denotes the $m \times n$ constraint Jacobian, which has i th row $\nabla c_i(x)^T$. Given a Lagrangian function $L(x, y) = f(x) - c(x)^T y$ with y a m -vector of dual variables, the Hessian of the Lagrangian with respect to x is denoted by $H(x, y) = \nabla^2 f(x) - \sum_{i=1}^m y_i \nabla^2 c_i(x)$. The equations utilize the Moore-Penrose pseudoinverse of a diagonal matrix. In particular, if $D = \text{diag}(d_1, d_2, \dots, d_n)$, then the pseudoinverse D^\dagger is diagonal with $D_{ii}^\dagger = 0$ for $d_i = 0$ and $D_{ii}^\dagger = 1/d_i$ for $d_i \neq 0$.

2. Optimality conditions

The first-order KKT conditions for problem (NLP) are

$$\left. \begin{aligned}
 \nabla f(x^*) - J(x^*)^\top y^* - A^\top v^* - E_x^\top z_x^* - E_L^\top z_1^* + E_U^\top z_2^* &= 0, & z_1^* &\geq 0, & z_2^* &\geq 0, \\
 y^* - L_x^\top w_x^* - L_L^\top w_1^* + L_U^\top w_2^* &= 0, & w_1^* &\geq 0, & w_2^* &\geq 0, \\
 c(x^*) - s^* &= 0, & & & L_x s^* - h_x &= 0, \\
 & & Ax^* - b &= 0, & E_x x^* - b_x &= 0, \\
 E_L x^* - \ell^x &\geq 0, & u^x - E_U x^* &\geq 0, & & \\
 L_L s^* - \ell^s &\geq 0, & u^s - L_U s^* &\geq 0, & & \\
 z_1^* \cdot (E_L x^* - \ell^x) &= 0, & z_2^* \cdot (u^x - E_U x^*) &= 0, & & \\
 w_1^* \cdot (L_L s^* - \ell^s) &= 0, & w_2^* \cdot (u^s - L_U s^*) &= 0, & &
 \end{aligned} \right\} \quad (2.1)$$

where y^* , w_x^* , and z_x^* are the multipliers for the equality constraints $c(x) - s = 0$, $L_x s^* = h_x$ and $E_x x^* = b_x$, and z_1^* , z_2^* , w_1^* and w_2^* may be interpreted as the Lagrange multipliers for the inequality constraints $E_L x - \ell^x \geq 0$, $u^x - E_U x \geq 0$, $L_L s - \ell^s \geq 0$ and $u^s - L_U s \geq 0$, respectively. The components of v^* are the multipliers for the linear equality constraints $Ax = b$.

The discussion that follows makes extensive use of the auxiliary quantities

$$x_1 = E_L x - \ell^x, \quad x_2 = u^x - E_U x, \quad s_1 = L_L s - \ell^s, \quad \text{and} \quad s_2 = u^s - L_U s. \quad (2.2)$$

In some cases x_1 , x_2 , s_1 and s_2 are used to simplify the expressions appearing in certain equations, in others they are regarded as independent variables associated with the problem

$$\left. \begin{aligned}
 &\text{minimize} && f(x) \\
 &\text{subject to} && c(x) - s = 0, \quad Ax - b = 0, \\
 & && E_L x - x_1 = \ell^x, \quad L_L s - s_1 = \ell^s, \quad x_1 \geq 0, \quad s_1 \geq 0, \\
 & && E_U x + x_2 = u^x, \quad L_U s + s_2 = u^s, \quad x_2 \geq 0, \quad s_2 \geq 0, \\
 & && E_x x - b_x = 0, \quad L_x s - h_x = 0,
 \end{aligned} \right\} \quad (\text{NP})$$

which is equivalent to problem (NLP). In this case, the dual variables z_1^* , z_2^* , w_1^* , and w_2^* associated with the optimality conditions (2.1) are the Lagrange multipliers for the inequality constraints $x_1 \geq 0$, $x_2 \geq 0$, $s_1 \geq 0$, and $s_2 \geq 0$, respectively.

In the derivations that follow, the vectors z and w are defined as

$$z = E_x^\top z_x + E_L^\top z_1 - E_U^\top z_2, \quad \text{and} \quad w = L_x^\top w_x + L_L^\top w_1 - L_U^\top w_2. \quad (2.3)$$

3. The path-following equations

Penalty and barrier methods are closely related to path-following methods. These methods approximate a continuous path that passes through a solution of (NLP). In the simplest case, the path is parameterized by a positive scalar parameter that may be interpreted as a perturbation for the optimality conditions for the problem (NLP).

Let z_1^E and z_2^E , w_1^E and w_2^E denote nonnegative estimates of z_1^* and z_2^* , w_1^* and w_2^* . Similarly, let v^E , x^E and s^E denote estimates of v^* , x^* and s^* . Given small positive scalars μ^P , μ^A and μ^B , consider the perturbed optimality conditions

$$\left. \begin{aligned} \nabla f(x) - J(x)^T y - A^T v - E_x^T z_x - E_L^T z_1 + E_U^T z_2 &= 0, & z_1 &\geq 0, & z_2 &\geq 0, \\ y - L_x^T w_x - L_L^T w_1 + L_U^T w_2 &= 0, & w_1 &\geq 0, & w_2 &\geq 0, \\ c(x) - s &= \mu^P (y^E - y), & E_x x - b_x &= 0, & L_x s - h_x &= 0, \\ Ax - b &= \mu^A (v^E - v), \\ E_L x - \ell^X &\geq 0, & u^X - E_U x &\geq 0, \\ L_L s - \ell^S &\geq 0, & u^S - L_U s &\geq 0, \\ z_1 \cdot (E_L x - \ell^X) &= \mu^B (z_1^E - z_1) + \mu^B (E_L x^E - E_L x), \\ z_2 \cdot (u^X - E_U x) &= \mu^B (z_2^E - z_2) + \mu^B (E_U x^E - E_U x), \\ w_1 \cdot (L_L s - \ell^S) &= \mu^B (w_1^E - w_1) + \mu^B (L_L s^E - L_L s), \\ w_2 \cdot (u^S - L_U s) &= \mu^B (w_2^E - w_2) + \mu^B (L_U s^E - L_U s). \end{aligned} \right\} \quad (3.1)$$

Let v_P denote the vector of variables $v_P = (x, s, y, v, w_x, z_x, z_1, z_2, w_1, w_2)$. The primal-dual path-following equations are given by $F(v_P) = 0$, with

$$F(v_P) = \begin{pmatrix} \nabla f(x) - J(x)^T y - A^T v - E_x^T z_x - E_L^T z_1 + E_U^T z_2 \\ y - L_x^T w_x - L_L^T w_1 + L_U^T w_2 \\ c(x) - s + \mu^P (y - y^E) \\ Ax - b + \mu^A (v - v^E) \\ E_x x - b_x \\ L_x s - h_x \\ z_1 \cdot (E_L x - \ell^X) + \mu^B (z_1 - z_1^E) + \mu^B (E_L x - E_L x^E) \\ z_2 \cdot (u^X - E_U x) + \mu^B (z_2 - z_2^E) + \mu^B (E_U x^E - E_U x) \\ w_1 \cdot (L_L s - \ell^S) + \mu^B (w_1 - w_1^E) + \mu^B (L_L s - L_L s^E) \\ w_2 \cdot (u^S - L_U s) + \mu^B (w_2 - w_2^E) + \mu^B (L_U s^E - L_U s) \end{pmatrix} = \begin{pmatrix} \nabla f(x) - J(x)^T y - A^T v - z \\ y - w \\ c(x) - s + \mu^P (y - y^E) \\ Ax - b + \mu^A (v - v^E) \\ E_x x - b_x \\ L_x s - h_x \\ z_1 \cdot (E_L x - \ell^X) + \mu^B (z_1 - z_1^E) + \mu^B (E_L x - E_L x^E) \\ z_2 \cdot (u^X - E_U x) + \mu^B (z_2 - z_2^E) + \mu^B (E_U x^E - E_U x) \\ w_1 \cdot (L_L s - \ell^S) + \mu^B (w_1 - w_1^E) + \mu^B (L_L s - L_L s^E) \\ w_2 \cdot (u^S - L_U s) + \mu^B (w_2 - w_2^E) + \mu^B (L_U s^E - L_U s) \end{pmatrix}, \quad (3.2)$$

where the first $n+m$ equations are written in terms of z and w such that $z = E_x^T z_x + E_L^T z_1 - E_U^T z_2$ and $w = L_x^T w_x + L_L^T w_1 - L_U^T w_2$. (To simplify the notation, the dependence of F on the parameters μ^A , μ^P , μ^B , x^E , s^E , y^E , v^E , z_1^E , z_2^E , w_1^E , w_2^E is omitted.) Any

zero $(x, s, y, v, w_x, z_x, z_1, z_2, w_1, w_2)$ of F such that $\ell^x < E_L x, E_U x < u^x, \ell^s < L_L s, L_U < u^s, z_1 > 0, z_2 > 0, w_1 > 0,$ and $w_2 > 0$ approximates a point satisfying the optimality conditions (2.1), with the approximation becoming increasingly accurate as the terms $\mu^p(y - y^E), \mu^A(v - v^E), \mu^B(E_L x^E - E_L x), \mu^B(E_U x^E - E_U x), \mu^B(L_L s^E - L_L s), \mu^B(L_U s - L_L^E), \mu^B(z_1 - z_1^E), \mu^B(z_2 - z_2^E), \mu^B(w_1 - w_1^E)$ and $\mu^B(w_2 - w_2^E)$ approach zero. For any sequence of $x^E, s^E, z_1^E, z_2^E, w_1^E, w_2^E, v^E$ and y^E such that $x^E \rightarrow x^*, s^E \rightarrow s^*, z_1^E \rightarrow z_1^*, z_2^E \rightarrow z_2^*, w_1^E \rightarrow w_1^*, w_2^E \rightarrow w_2^*, v^E \rightarrow v^*$ and $y^E \rightarrow y^*$, it must hold that solutions $(x, s, y, v, z_1, z_2, w_1, w_2)$ of (3.1) must satisfy $z_1 \cdot (x - \ell^x) \rightarrow 0, z_2 \cdot (u^x - x) \rightarrow 0, w_1 \cdot (s - \ell^s) \rightarrow 0,$ and $w_2 \cdot (u^s - s) \rightarrow 0,$ This implies that any solution $(x, s, y, v, w_x, z_x, z_1, z_2, w_1, w_2)$ of (3.1) will approximate a solution of (2.1) independently of the values of μ^p, μ^A and μ^B (i.e., it is not necessary that $\mu^p \rightarrow 0, \mu^A \rightarrow 0$ and $\mu^B \rightarrow 0$).

If $v_P = (x, s, y, v, w_x, z_x, z_1, z_2, w_1, w_2)$ is a given approximate zero of $F(v_P)$ such that $\ell^x - \mu^B < E_L x, E_U x < u^x + \mu^B, \ell^s - \mu^B < L_L s, L_U s < u^s + \mu^B, z_1 > 0, z_2 > 0, w_1 > 0,$ and $w_2 > 0,$ the Newton equations for the change in variables $\Delta v_P = (\Delta x, \Delta s, \Delta y, \Delta v, \Delta w_x, \Delta z_x, \Delta z_1, \Delta z_2, \Delta w_1, \Delta w_2)$ are given by $F'(v_P)\Delta v_P = -F(v_P),$ with

$$F'(v_P) = \begin{pmatrix} H(x, y) & 0 & -J^T & -A^T & 0 & -E_X^T & -E_L^T & E_U^T & 0 & 0 \\ 0 & 0 & I_m & 0 & -L_X^T & 0 & 0 & 0 & -L_L^T & L_U^T \\ J & -I_m & D_Y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A & 0 & 0 & D_A & 0 & 0 & 0 & 0 & 0 & 0 \\ E_X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_X & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ Z_1^\mu E_L & 0 & 0 & 0 & 0 & 0 & X_1^\mu & 0 & 0 & 0 \\ -Z_2^\mu E_U & 0 & 0 & 0 & 0 & 0 & 0 & X_2^\mu & 0 & 0 \\ 0 & W_1^\mu L_L & 0 & 0 & 0 & 0 & 0 & 0 & S_1^\mu & 0 \\ 0 & -W_2^\mu L_U & 0 & 0 & 0 & 0 & 0 & 0 & 0 & S_2^\mu \end{pmatrix}, \quad (3.3)$$

where

$$\left. \begin{aligned} X_1^\mu &= \text{diag}(x_1 + \mu^B e), & X_2^\mu &= \text{diag}(x_2 + \mu^B e), & S_1^\mu &= \text{diag}(s_1 + \mu^B e), & S_2^\mu &= \text{diag}(s_2 + \mu^B e), \\ Z_1^\mu &= \text{diag}(z_1 + \mu^B e), & Z_2^\mu &= \text{diag}(z_2 + \mu^B e), & W_1^\mu &= \text{diag}(w_1 + \mu^B e), & W_2^\mu &= \text{diag}(w_2 + \mu^B e), \end{aligned} \right\} \quad (3.4)$$

with x_1, x_2, s_1 and s_2 given by (2.2). Any s may be written as $s = L_F^T s_F + L_X^T s_X,$ where L_F are the rows of I_m orthogonal to the rows of $L_X,$ i.e., $L_F^T L_X = 0.$ The vectors s_F and s_X are the components of s corresponding to the “free” and “fixed” components of $s,$ respectively. The variables $L_L s$ and $L_U s$ form a subset of $s_F.$ Throughout, we assume that s satisfies $L_X s - h_X = 0,$ in which case $\Delta s_X = 0$ and Δs satisfies

$$\Delta s = L_F^T \Delta s_F + L_X^T \Delta s_X = L_F^T \Delta s_F.$$

Similarly, any x may be written as $x = E_F^T x_F + E_X^T x_X,$ where x_F and x_X denote the components of x corresponding to the “free” and “fixed variables”, respectively. The variables $E_L x$ and $E_U x$ form a subset of $x_F.$ Throughout, we assume that x_X satisfies

$E_X x - b_X = 0$, in which case $\Delta x_X = 0$ and Δx satisfies

$$\Delta x = E_F^T \Delta x_F + E_X^T \Delta x_X = E_F^T \Delta x_F.$$

After premultiplying the first and fifth blocks of equations of (3.3) by E_F and L_F respectively, and substituting $\Delta x = E_F^T \Delta x_F$ and $\Delta s = L_F^T \Delta s_F$, the equations (3.3) can be written in the reduced form $\widehat{F}'(v_F) \Delta v_F = -\widehat{F}(v_F)$, where $\Delta v_F = (\Delta x_F, \Delta s_F, \Delta y, \Delta v, \Delta z_1, \Delta z_2, \Delta w_1, \Delta w_2)$,

$$\begin{pmatrix} H_F & 0 & -J_F^T & -A_F^T & -E_{LF}^T & E_{UF}^T & 0 & 0 \\ 0 & 0 & L_F & 0 & 0 & 0 & -L_{LF}^T & L_{UF}^T \\ J_F & -L_F^T & D_Y & 0 & 0 & 0 & 0 & 0 \\ A_F & 0 & 0 & D_A & 0 & 0 & 0 & 0 \\ Z_1^\mu E_{LF} & 0 & 0 & 0 & X_1^\mu & 0 & 0 & 0 \\ -Z_2^\mu E_{UF} & 0 & 0 & 0 & 0 & X_2^\mu & 0 & 0 \\ 0 & W_1^\mu L_{LF} & 0 & 0 & 0 & 0 & S_1^\mu & 0 \\ 0 & -W_2^\mu L_{UF} & 0 & 0 & 0 & 0 & 0 & S_2^\mu \end{pmatrix} \begin{pmatrix} \Delta x_F \\ \Delta s_F \\ \Delta y \\ \Delta v \\ \Delta z_1 \\ \Delta z_2 \\ \Delta w_1 \\ \Delta w_2 \end{pmatrix} = - \begin{pmatrix} g_F - J_F^T y - A_F^T v - E_{LF}^T z_1 + E_{UF}^T z_2 \\ y_F - L_{LF}^T w_1 + L_{UF}^T w_2 \\ c(x) - s + \mu^P (y - y^E) \\ Ax - b + \mu^A (v - v^E) \\ z_1 \cdot (E_L x - \ell^X) + \mu^B (z_1 - z_1^E) + \mu^B (E_L x - E_L x^E) \\ z_2 \cdot (u^X - E_U x) + \mu^B (z_2 - z_2^E) + \mu^B (E_U x^E - E_U x) \\ w_1 \cdot (L_L s - \ell^S) + \mu^B (w_1 - w_1^E) + \mu^B (L_L s - L_L s^E) \\ w_2 \cdot (u^S - L_U s) + \mu^B (w_2 - w_2^E) + \mu^B (L_U s^E - L_U s) \end{pmatrix},$$

where $H_F = E_F H E_F^T$, $J_F = J(x) E_F^T$, $A_F = A E_F^T$, $g_F = E_F \nabla f(x)$, $E_{LF} = E_L E_F^T$, $E_{UF} = E_U E_F^T$, $y_F = L_F y$, $L_{LF} = L_L L_F^T$ and $L_{UF} = L_U L_F^T$. The matrices J_F , A_F , E_{LF} and E_{UF} are the columns of $J(x)$, A , E_L and E_U associated with the “free” components of x . The matrices L_{LF} and L_{UF} are the columns of L_L and L_U associated with the “free” components of s . Then scaling the last four blocks of equations by (respectively) $(Z_1^\mu)^{-1}$, $(Z_2^\mu)^{-1}$, $(W_1^\mu)^{-1}$ and $(W_2^\mu)^{-1}$ gives

$$\begin{pmatrix} H_F & 0 & -J_F^T & -A_F^T & -E_{LF}^T & E_{UF}^T & 0 & 0 \\ 0 & 0 & L_F & 0 & 0 & 0 & -L_{LF}^T & L_{UF}^T \\ J_F & -L_F^T & D_Y & 0 & 0 & 0 & 0 & 0 \\ A_F & 0 & 0 & D_A & 0 & 0 & 0 & 0 \\ E_{LF} & 0 & 0 & 0 & D_1^Z & 0 & 0 & 0 \\ -E_{UF} & 0 & 0 & 0 & 0 & D_2^Z & 0 & 0 \\ 0 & L_{LF} & 0 & 0 & 0 & 0 & D_1^W & 0 \\ 0 & -L_{UF} & 0 & 0 & 0 & 0 & 0 & D_2^W \end{pmatrix} \begin{pmatrix} \Delta x_F \\ \Delta s_F \\ \Delta y \\ \Delta v \\ \Delta z_1 \\ \Delta z_2 \\ \Delta w_1 \\ \Delta w_2 \end{pmatrix} = - \begin{pmatrix} g_F - J_F^T y - A_F^T v - E_{LF}^T z_1 + E_{UF}^T z_2 \\ y_F - L_{LF}^T w_1 + L_{UF}^T w_2 \\ c(x) - s + \mu^P (y - y^E) \\ Ax - b + \mu^A (v - v^E) \\ D_1^Z (z_1 - \pi_1^Z) \\ D_2^Z (z_2 - \pi_2^Z) \\ D_1^W (w_1 - \pi_1^W) \\ D_2^W (w_2 - \pi_2^W) \end{pmatrix}, \quad (3.5)$$

where $A_F = A E_F^T$ are the columns of A associated with the “free” components of x , and

$$\begin{aligned} D_Y &= \mu^P I_m, & \pi^Y &= y^E - \frac{1}{\mu^P} (c - s), & D_A &= \mu^A I_A, & \pi^V &= v^E - \frac{1}{\mu^A} (Ax - b), \\ D_1^W &= S_1^\mu (W_1^\mu)^{-1}, & \pi_1^W &= \mu^B (S_1^\mu)^{-1} (w_1^E - s_1 + s_1^E), & D_1^Z &= X_1^\mu (Z_1^\mu)^{-1}, & \pi_1^Z &= \mu^B (X_1^\mu)^{-1} (z_1^E - x_1 + x_1^E), \\ D_2^W &= S_2^\mu (W_2^\mu)^{-1}, & \pi_2^W &= \mu^B (S_2^\mu)^{-1} (w_2^E - s_2 + s_2^E), & D_2^Z &= X_2^\mu (Z_2^\mu)^{-1}, & \pi_2^Z &= \mu^B (X_2^\mu)^{-1} (z_2^E - x_2 + x_2^E), \end{aligned}$$

with auxiliary quantities

$$x_1^E = E_L x^E - \ell^X, \quad x_2^E = u^X - E_U x^E, \quad s_1^E = L_L s^E - \ell^S, \quad \text{and} \quad s_2^E = u^S - L_U s^E.$$

Given the definitions (2.3), the vectors Δs and Δw_x are recovered as $\Delta s = L_F^T \Delta s_F$ and $\Delta w_x = [y + \Delta y - w]_x$. Similarly, Δx and Δz_x are recovered as $\Delta x = L_F^T \Delta x_F$ and $\Delta z_x = [g + H \Delta x - J^T(y + \Delta y) - z]_x$.

4. A shifted primal-dual penalty-barrier function

Consider the shifted primal-dual penalty-barrier problem applied to (NP):

$$\begin{aligned} & \underset{\substack{x, x_1, x_2, s, s_1, s_2, \\ y, v, z_1, z_2, w_1, w_2}}{\text{minimize}} && M(x, x_1, x_2, s, s_1, s_2, y, v, w_1, w_2; \mu^P, \mu^B, y^E, v^E, w_1^E, w_2^E) \\ & \text{subject to} && E_L x - x_1 = \ell^X, \quad L_L s - s_1 = \ell^S, \quad x_1 + \mu^B e > 0, \quad z_1 + \mu^B e > 0, \quad s_1 + \mu^B e > 0, \quad w_1 + \mu^B e > 0, \\ & && E_U x + x_2 = u^X, \quad L_U s + s_2 = u^S, \quad x_2 + \mu^B e > 0, \quad z_2 + \mu^B e > 0, \quad s_2 + \mu^B e > 0, \quad w_2 + \mu^B e > 0, \\ & && E_X x - b_X = 0, \quad L_X s - h_X = 0, \end{aligned}$$

where $M(x, x_1, x_2, s, s_1, s_2, y, v, z_1, z_2, w_1, w_2; \mu^P, \mu^B, y^E, v^E, z_1^E, z_2^E, w_1^E, w_2^E)$ is the shifted primal-dual penalty-barrier function

$$\begin{aligned} f(x) - (c(x) - s)^T y^E &+ \frac{1}{2\mu^P} \|c(x) - s\|^2 + \frac{1}{2\mu^P} \|c(x) - s + \mu^P(y - y^E)\|^2 \\ &- (Ax - b)^T v^E + \frac{1}{2\mu^A} \|Ax - b\|^2 + \frac{1}{2\mu^A} \|Ax - b + \mu^A(v - v^E)\|^2 \\ &- \sum_{j=1}^{n_L} \left\{ \mu^B ([z_1^E]_j + [x_1^E]_j + \mu^B) \ln ([z_1 + \mu^B e]_j [x_1 + \mu^B e]_j^2) - [z_1 \cdot (x_1 + \mu^B e)]_j - 2\mu^B [x_1]_j \right\} \\ &- \sum_{j=1}^{n_U} \left\{ \mu^B ([z_2^E]_j + [x_2^E]_j + \mu^B) \ln ([z_2 + \mu^B e]_j [x_2 + \mu^B e]_j^2) - [z_2 \cdot (x_2 + \mu^B e)]_j - 2\mu^B [x_2]_j \right\} \\ &- \sum_{i=1}^{m_L} \left\{ \mu^B ([w_1^E]_i + [s_1^E]_i + \mu^B) \ln ([w_1 + \mu^B e]_i [s_1 + \mu^B e]_i^2) - [w_1 \cdot (s_1 + \mu^B e)]_i - 2\mu^B [s_1]_i \right\} \\ &- \sum_{i=1}^{m_U} \left\{ \mu^B ([w_2^E]_i + [s_2^E]_i + \mu^B) \ln ([w_2 + \mu^B e]_i [s_2 + \mu^B e]_i^2) - [w_2 \cdot (s_2 + \mu^B e)]_i - 2\mu^B [s_2]_i \right\}. \quad (4.1) \end{aligned}$$

The gradient may be written as

$$\nabla M(x, x_1, x_2, s, s_1, s_2, y, v, z_1, z_2, w_1, w_2) = \begin{pmatrix} \nabla f(x) - A^T \left(2(v^E - \frac{1}{\mu^A}(Ax - b)) - v \right) - J(x)^T \left(2(y^E - \frac{1}{\mu^P}(c - s)) - y \right) \\ z_1 + 2\mu^B e - 2\mu^B (X_1^\mu)^{-1} (z_1^E + x_1^E + \mu^B e) \\ z_2 + 2\mu^B e - 2\mu^B (X_2^\mu)^{-1} (z_2^E + x_2^E + \mu^B e) \\ 2(y^E - \frac{1}{\mu^P}(c - s)) - y \\ w_1 + 2\mu^B e - 2\mu^B (S_1^\mu)^{-1} (w_1^E + s_1^E + \mu^B e) \\ w_2 + 2\mu^B e - 2\mu^B (S_2^\mu)^{-1} (w_2^E + s_2^E + \mu^B e) \\ c(x) - s + \mu^P (y - y^E) \\ Ax - b + \mu^A (v - v^E) \\ x_1 + \mu^B e - \mu^B (Z_1^\mu)^{-1} (z_1^E + x_1^E + \mu^B e) \\ x_2 + \mu^B e - \mu^B (Z_2^\mu)^{-1} (z_2^E + x_2^E + \mu^B e) \\ s_1 + \mu^B e - \mu^B (W_1^\mu)^{-1} (w_1^E + s_1^E + \mu^B e) \\ s_2 + \mu^B e - \mu^B (W_2^\mu)^{-1} (w_2^E + s_2^E + \mu^B e) \end{pmatrix},$$

where $X_1^\mu, X_2^\mu, S_1^\mu, S_2^\mu, Z_1^\mu, Z_2^\mu, W_1^\mu$ and W_2^μ are defined in (3.4). Equivalently,

$$\nabla M = \begin{pmatrix} \nabla f(x) - A^T (\pi^V + (\pi^V - v)) - J(x)^T (\pi^Y + (\pi^Y - y)) \\ z_1 - 2\pi_1^Z \\ z_2 - 2\pi_2^Z \\ \pi^Y + (\pi^Y - y) \\ w_1 - 2\pi_1^W \\ w_2 - 2\pi_2^W \\ -D_Y (\pi^Y - y) \\ -D_A (\pi^V - v) \\ -D_1^Z (\pi_1^Z - z_1) \\ -D_2^Z (\pi_2^Z - z_2) \\ -D_1^W (\pi_1^W - w_1) \\ -D_2^W (\pi_2^W - w_2) \end{pmatrix}.$$

The Hessian $\nabla^2 M(x, x_1, x_2, s, s_1, s_2, y, v, z_1, z_2, w_1, w_2)$ is given by

$$\begin{pmatrix} H_1 & 0 & 0 & -2J^T D_Y^{-1} & 0 & 0 & J^T & A^T & 0 & 0 & 0 & 0 \\ 0 & 2G_1^x & 0 & 0 & 0 & 0 & -I_m & 0 & I_L^x & 0 & 0 & 0 \\ 0 & 0 & 2G_2^x & 0 & 0 & 0 & 0 & 0 & 0 & I_U^x & 0 & 0 \\ -2D_Y^{-1} J & 0 & 0 & 2D_Y^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2G_1^s & 0 & 0 & 0 & 0 & 0 & I_L^s & 0 \\ 0 & 0 & 0 & 0 & 0 & 2G_2^s & 0 & 0 & 0 & 0 & 0 & I_U^s \\ J & 0 & 0 & -I_m & 0 & 0 & D_Y & 0 & 0 & 0 & 0 & 0 \\ A & 0 & 0 & 0 & 0 & 0 & 0 & D_A & 0 & 0 & 0 & 0 \\ 0 & I_L^x & 0 & 0 & 0 & 0 & 0 & 0 & G_1^z & 0 & 0 & 0 \\ 0 & 0 & I_U^x & 0 & 0 & 0 & 0 & 0 & 0 & G_2^z & 0 & 0 \\ 0 & 0 & 0 & 0 & I_L^s & 0 & 0 & 0 & 0 & 0 & G_1^w & 0 \\ 0 & 0 & 0 & 0 & 0 & I_U^s & 0 & 0 & 0 & 0 & 0 & G_2^w \end{pmatrix},$$

where $H_1 = H(x, 2\pi^y - y) + \frac{2}{\mu^A} A^T A + \frac{2}{\mu^B} J(x)^T J(x)$, and $I_L^x, I_U^x, I_L^s, I_U^s$ are identity matrices of size n_L, n_U, m_L, m_U respectively. In addition

$$\begin{aligned} G_1^x &= (X_1^\mu)^{-1} (\Pi_1^z + \mu^B I), & G_2^x &= (X_2^\mu)^{-1} (\Pi_2^z + \mu^B I), \\ G_1^s &= (S_1^\mu)^{-1} (\Pi_1^w + \mu^B I), & G_2^s &= (S_2^\mu)^{-1} (\Pi_2^w + \mu^B I), \\ G_1^z &= (Z_1^\mu)^{-1} (\Pi_1^x + \mu^B I), & G_2^z &= (Z_2^\mu)^{-1} (\Pi_2^x + \mu^B I), \\ G_1^w &= (W_1^\mu)^{-1} (\Pi_1^s + \mu^B I), & G_2^w &= (W_2^\mu)^{-1} (\Pi_2^s + \mu^B I), \end{aligned}$$

with $\Pi_1^z = \text{diag}(\pi_1^z)$, $\Pi_2^z = \text{diag}(\pi_2^z)$, $\Pi_1^w = \text{diag}(\pi_1^w)$, $\Pi_2^w = \text{diag}(\pi_2^w)$, $X_1^E = \text{diag}(x_1^E)$, $X_2^E = \text{diag}(x_2^E)$, $S_1^E = \text{diag}(s_1^E)$, $W_1^E = \text{diag}(w_1^E)$, $W_2^E = \text{diag}(w_2^E)$, $Z_1^E = \text{diag}(z_1^E)$ and $Z_2^E = \text{diag}(z_2^E)$.

5. Derivation of the primal-dual line-search direction

The primal-dual penalty-barrier problem may be written in the form

$$\text{minimize}_{p \in \mathcal{I}} M(p) \quad \text{subject to} \quad Cp = b_C,$$

where

$$\mathcal{I} = \{p : p = (x, x_1, x_2, s, s_1, s_2, y, v, z_1, z_2, w_1, w_2), \text{ with } x_i + \mu^B e > 0, s_i + \mu^B e > 0, z_i + \mu^B e > 0, w_i + \mu^B e > 0 \text{ for } i = 1, 2\},$$

and

$$C = \begin{pmatrix} E_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ E_L & -I_L^x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ E_U & 0 & I_U^x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_L & -I_L^s & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_U & 0 & I_U^s & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad b_C = \begin{pmatrix} b_x \\ \ell^x \\ u^x \\ h_x \\ \ell^s \\ u^s \end{pmatrix}. \quad (5.1)$$

Let p be any vector in \mathcal{I} such that $Cp = b_C$. The Newton direction Δp is given by the solution of the subproblem

$$\underset{\Delta p}{\text{minimize}} \quad \nabla M(p)^\top \Delta p + \frac{1}{2} \Delta p^\top \nabla^2 M(p) \Delta p \quad \text{subject to} \quad C \Delta p = b_C - Cp = 0. \quad (5.2)$$

Let N denote a matrix whose columns form a basis for $\text{null}(C)$, i.e., the columns of N are linearly independent and $CN = 0$. Every feasible direction Δp may be written in the form $\Delta p = Nd$. This implies that d satisfies the reduced equations $N^\top \nabla^2 M(p) Nd = -N^\top \nabla M(p)$. However, instead of solving (5.2), we formulate a linearly constrained *approximate* Newton method by approximating the Hessian $\nabla^2 M(p)$ by a matrix $B(p)$ such that $N^\top B(p)N$ is positive definite with $N^\top B(p)N \approx N^\top \nabla^2 M(p)N$. Consider the matrix B obtained by replacing π^Y by y , π_1^z by z_1 , π_2^z by z_2 , π_1^w by w_1 , π_2^w by w_2 , x_1^E by x_1 , x_2^E by x_2 , s_1^E by s_1 , s_2^E by s_2 , z_1^E by z_1 , z_2^E by z_2 , w_1^E by w_1 and w_2^E by w_2 in $\nabla^2 M(x, x_1, x_2, s, s_1, s_2, y, v, z_1, z_2, w_1, w_2)$. This gives an approximate Hessian $B(x, x_1, x_2, s, s_1, s_2, y, v, z_1, z_2, w_1, w_2)$ of the form

$$\begin{pmatrix} H^B + \frac{2}{\mu^A} A^\top A + \frac{2}{\mu^P} J^\top J & 0 & 0 & -2J^\top D_Y^{-1} & 0 & 0 & J^\top & A^\top & 0 & 0 & 0 & 0 \\ 0 & 2(D_1^z)^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & I_L^x & 0 & 0 & 0 \\ 0 & 0 & -2(D_2^z)^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & I_U^x & 0 & 0 \\ -2D_Y^{-1} J & 0 & 0 & 2D_Y^{-1} & 0 & 0 & -I_m & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(D_1^w)^{-1} & 0 & 0 & 0 & 0 & 0 & I_L^s & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(D_2^w)^{-1} & 0 & 0 & 0 & 0 & 0 & I_U^s \\ J & 0 & 0 & -I_m & 0 & 0 & D_Y & 0 & 0 & 0 & 0 & 0 \\ A & 0 & 0 & 0 & 0 & 0 & 0 & D_A & 0 & 0 & 0 & 0 \\ 0 & I_L^x & 0 & 0 & 0 & 0 & 0 & 0 & D_1^z & 0 & 0 & 0 \\ 0 & 0 & I_U^x & 0 & 0 & 0 & 0 & 0 & 0 & D_2^z & 0 & 0 \\ 0 & 0 & 0 & 0 & I_L^s & 0 & 0 & 0 & 0 & 0 & D_1^w & 0 \\ 0 & 0 & 0 & 0 & 0 & I_U^s & 0 & 0 & 0 & 0 & 0 & D_2^w \end{pmatrix},$$

where $H^B \approx H(x, y)$ is chosen so that the approximate reduced Hessian $N^\top B(p)N$ is positive definite (see Section 7). Given $B(p)$, an approximate Newton direction is given by the solution of the QP subproblem

$$\underset{\Delta p}{\text{minimize}} \quad \nabla M(p)^\top \Delta p + \frac{1}{2} \Delta p^\top B(p) \Delta p \quad \text{subject to} \quad C \Delta p = 0.$$

Let N denote a matrix whose columns form a basis for $\text{null}(C)$, i.e., the columns of N are linearly independent and $CN = 0$. Every feasible Δp may be written in the form $\Delta p = Nd$. This implies that d satisfies the reduced equations $N^T B(p)Nd = -N^T \nabla M(p)$. Consider the null-space basis defined from the columns of

$$N = \begin{pmatrix} E_F^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ E_{LF} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -E_{UF} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_F^T & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_{LF} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -L_{UF} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_A & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_L^x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_U^x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_L^s & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_U^s \end{pmatrix}, \quad (5.3)$$

where $E_{LF} = E_L E_F^T$, $E_{UF} = E_U E_F^T$, $L_{LF} = L_L L_F^T$ and $L_{UF} = L_U L_F^T$. The definition of N of (5.3) gives the reduced Hessian $N^T B(p)N$ such that

$$\begin{pmatrix} \hat{H}_F & -2J_F^T D_Y^{-1} L_F^T & J_F^T & A_F^T & E_{LF}^T & -E_{UF}^T & 0 & 0 \\ -2L_F D_Y^{-1} J_F & 2L_F (D_Y^{-1} + D_W^T) L_F^T & -L_F & 0 & 0 & 0 & L_{LF}^T & L_{UF}^T \\ J_F & -L_F^T & D_Y & 0 & 0 & 0 & 0 & 0 \\ A_F & 0 & 0 & D_A & 0 & 0 & 0 & 0 \\ E_{LF} & 0 & 0 & 0 & D_1^z & 0 & 0 & 0 \\ -E_{UF} & 0 & 0 & 0 & 0 & D_2^z & 0 & 0 \\ 0 & L_{LF} & 0 & 0 & 0 & 0 & D_1^w & 0 \\ 0 & -L_{UF} & 0 & 0 & 0 & 0 & 0 & D_2^w \end{pmatrix},$$

where $J_F = J(x)E_F^T$, $A_F = AE_F^T$, $\widehat{H}_F = E_F H^B E_F^T + \frac{2}{\mu^A} A_F^T A_F + \frac{2}{\mu^F} J_F^T J_F + 2(E_{LF}^T (D_1^Z)^{-1} E_{LF} + E_{UF}^T (D_2^Z)^{-1} E_{UF})$ and $D_W = ((L_L^T (D_1^W)^{-1} L_L + L_U^T (D_2^W)^{-1} L_U))^{\dagger}$. Similarly, the reduced gradient $N^T \nabla M(p)$ is given by

$$\begin{pmatrix} g_F - A_F^T (2\pi^V - v) - J_F^T (2\pi^Y - y) - E_{LF} (2\pi_1^Z - z_1) + E_{UF} (2\pi_2^Z - z_2) \\ 2\pi_F^Y - y_F - L_{LF} (2\pi_1^W - w_1) + L_{UF} (2\pi_2^W - w_2) \\ -D_Y (\pi^Y - y) \\ -D_A (\pi^V - v) \\ -D_1^Z (\pi_1^Z - z_1) \\ -D_2^Z (\pi_2^Z - z_2) \\ -D_1^W (\pi_1^W - w_1) \\ -D_2^W (\pi_2^W - w_2) \end{pmatrix},$$

where $g_F = E_F \nabla f(x)$, $\pi_F^Y = L_F \pi^Y$ and $y_F = L_F y$. The reduced approximate Newton equations $N^T B(p) N d = -N^T \nabla M(p)$ are then

$$\begin{pmatrix} \widehat{H}_F & -2J_F^T D_Y^{-1} L_F^T & J_F^T & A_F^T & E_{LF}^T & -E_{UF}^T & 0 & 0 \\ -2L_F D_Y^{-1} J_F & 2L_F (D_Y^{-1} + D_W^{\dagger}) L_F^T & -L_F & 0 & 0 & 0 & L_{LF}^T & L_{UF}^T \\ J_F & -L_F^T & D_Y & 0 & 0 & 0 & 0 & 0 \\ A_F & 0 & 0 & D_A & 0 & 0 & 0 & 0 \\ E_{LF} & 0 & 0 & 0 & D_1^Z & 0 & 0 & 0 \\ -E_{UF} & 0 & 0 & 0 & 0 & D_2^Z & 0 & 0 \\ 0 & L_{LF} & 0 & 0 & 0 & 0 & D_1^W & 0 \\ 0 & -L_{UF} & 0 & 0 & 0 & 0 & 0 & D_2^W \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8 \end{pmatrix} = - \begin{pmatrix} g_F - A_F^T (2\pi^V - v) - J_F^T (2\pi^Y - y) - E_{LF} (2\pi_1^Z - z_1) + E_{UF} (2\pi_2^Z - z_2) \\ 2\pi_F^Y - y_F - L_{LF} (2\pi_1^W - w_1) + L_{UF} (2\pi_2^W - w_2) \\ -D_Y (\pi^Y - y) \\ -D_A (\pi^V - v) \\ -D_1^Z (\pi_1^Z - z_1) \\ -D_2^Z (\pi_2^Z - z_2) \\ -D_1^W (\pi_1^W - w_1) \\ -D_2^W (\pi_2^W - w_2) \end{pmatrix}. \quad (5.4)$$

Given any nonsingular matrix R , the direction d satisfies $RN^T B(p)Nd = -RN^T \nabla M(p)$. In particular, consider the block upper-triangular matrix R such that

$$R = \begin{pmatrix} I_F^x & 0 & -2J_F^T D_Y^{-1} & -2A_F^T D_A^{-1} & -2E_{LF}^T (D_1^z)^{-1} & 2E_{UF}^T (D_2^z)^{-1} & 0 & 0 \\ & I_F^s & 2L_F D_Y^{-1} & 0 & 0 & 0 & -2L_{LF}^T (D_1^w)^{-1} & 2L_{UF}^T (D_2^w)^{-1} \\ & & I_m & 0 & 0 & 0 & 0 & 0 \\ & & & I_A & 0 & 0 & 0 & 0 \\ & & & & I_L^x & 0 & 0 & 0 \\ & & & & & I_U^x & 0 & 0 \\ & & & & & & I_L^s & 0 \\ & & & & & & & I_U^s \end{pmatrix},$$

where again, I_L^x , I_U^x , I_L^s , I_U^s are identity matrices of size n_L , n_U , m_L , and m_U respectively. Then R is nonsingular with

$$RN^T B(p)N = \begin{pmatrix} E_F H^B E_F^T & 0 & -J_F^T & -A_F^T & -E_{LF}^T & E_{UF}^T & 0 & 0 \\ 0 & 0 & L_F & 0 & 0 & 0 & -L_{LF}^T & L_{UF}^T \\ J_F & -L_F^T & D_Y & 0 & 0 & 0 & 0 & 0 \\ A_F & 0 & 0 & D_A & 0 & 0 & 0 & 0 \\ E_{LF} & 0 & 0 & 0 & D_1^z & 0 & 0 & 0 \\ -E_{UF} & 0 & 0 & 0 & 0 & D_2^z & 0 & 0 \\ 0 & L_{LF} & 0 & 0 & 0 & 0 & D_1^w & 0 \\ 0 & -L_{UF} & 0 & 0 & 0 & 0 & 0 & D_2^w \end{pmatrix}$$

Also,

$$RN^T \nabla M(p) = \begin{pmatrix} g_F - J_F^T y - A_F^T v - z_1 + z_2 \\ y_F - w_1 + w_2 \\ -D_Y(\pi^Y - y) \\ -D_A(\pi^V - v) \\ -D_1^z(\pi_1^z - z_1) \\ -D_2^z(\pi_2^z - z_2) \\ -D_1^w(\pi_1^w - w_1) \\ -D_2^w(\pi_2^w - w_2) \end{pmatrix}.$$

This gives the following (unsymmetric) reduced approximate Newton equations for d :

$$\begin{pmatrix} E_F H^B E_F^T & 0 & -J_F^T & -A_F^T & -E_{LF}^T & E_{UF}^T & 0 & 0 \\ 0 & 0 & L_F & 0 & 0 & 0 & -L_{LF}^T & L_{UF}^T \\ J_F & -L_F^T & D_Y & 0 & 0 & 0 & 0 & 0 \\ A_F & 0 & 0 & D_A & 0 & 0 & 0 & 0 \\ E_{LF} & 0 & 0 & 0 & D_1^Z & 0 & 0 & 0 \\ -E_{UF} & 0 & 0 & 0 & 0 & D_2^Z & 0 & 0 \\ 0 & L_{LF} & 0 & 0 & 0 & 0 & D_1^W & 0 \\ 0 & -L_{UF} & 0 & 0 & 0 & 0 & 0 & D_2^W \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8 \end{pmatrix} = - \begin{pmatrix} g_F - J_F^T y - A_F^T v - E_{LF}^T z_1 + E_{UF}^T z_2 \\ y_F - L_{LF}^T w_1 + L_{UF}^T w_2 \\ -D_Y(\pi^Y - y) \\ -D_A(\pi^V - v) \\ -D_1^Z(\pi_1^Z - z_1) \\ -D_2^Z(\pi_2^Z - z_2) \\ -D_1^W(\pi_1^W - w_1) \\ -D_2^W(\pi_2^W - w_2) \end{pmatrix}. \quad (5.5)$$

Then, the identity $\Delta p = Nd$ implies that

$$\Delta p = \begin{pmatrix} \Delta x \\ \Delta x_1 \\ \Delta x_2 \\ \Delta s \\ \Delta s_1 \\ \Delta s_2 \\ \Delta y \\ \Delta v \\ \Delta z_1 \\ \Delta z_2 \\ \Delta w_1 \\ \Delta w_2 \end{pmatrix} = Nd = \begin{pmatrix} E_F^T d_1 \\ d_1 \\ -d_1 \\ L_F^T d_2 \\ d_2 \\ -d_2 \\ d_3 \\ d_4 \\ d_5 \\ d_6 \\ d_7 \\ d_8 \end{pmatrix}. \quad (5.6)$$

These identities allow us to write equations (5.5) in the form

$$\begin{pmatrix} E_F H^B E_F^T & 0 & -J_F^T & -A_F^T & -E_{LF}^T & E_{UF}^T & 0 & 0 \\ 0 & 0 & L_F & 0 & 0 & 0 & -L_{LF}^T & L_{UF}^T \\ J_F & -L_F^T & D_Y & 0 & 0 & 0 & 0 & 0 \\ A_F & 0 & 0 & D_A & 0 & 0 & 0 & 0 \\ E_{LF} & 0 & 0 & 0 & D_1^Z & 0 & 0 & 0 \\ -E_{UF} & 0 & 0 & 0 & 0 & D_2^Z & 0 & 0 \\ 0 & L_{LF} & 0 & 0 & 0 & 0 & D_1^W & 0 \\ 0 & -L_{UF} & 0 & 0 & 0 & 0 & 0 & D_2^W \end{pmatrix} \begin{pmatrix} \Delta x_F \\ \Delta s_F \\ \Delta y \\ \Delta v \\ \Delta z_1 \\ \Delta z_2 \\ \Delta w_1 \\ \Delta w_2 \end{pmatrix} = - \begin{pmatrix} g_F - J_F^T y - A_F^T v - E_{LF}^T z_1 + E_{UF}^T z_2 \\ y_F - L_{LF}^T w_1 + L_{UF}^T w_2 \\ -D_Y(\pi^Y - y) \\ -D_A(\pi^V - v) \\ -D_1^Z(\pi_1^Z - z_1) \\ -D_2^Z(\pi_2^Z - z_2) \\ -D_1^W(\pi_1^W - w_1) \\ -D_2^W(\pi_2^W - w_2) \end{pmatrix}, \quad (5.7)$$

with $\Delta x = E_F^T \Delta x_F$, $\Delta s = L_F^T \Delta s_F$, $\Delta x_1 = \Delta x_F - (\ell^x - E_L x + x_1)$, $\Delta x_2 = -\Delta x_F + (u^x - E_U x - x_2)$, $\Delta s_1 = \Delta s_F - (\ell^s - L_L s + s_1)$ and $\Delta s_2 = -\Delta s_F + (u^s - L_U s - s_2)$.

The shifted penalty-barrier equations (5.7) are the same as the path-following equations (3.5) except for the (1, 1) block, where H_F is replaced by $E_F H^B E_F^T$.

6. The shifted primal-dual penalty-barrier direction

In this section we consider the solution of the shifted primal-dual penalty-barrier equations (5.7). Collecting terms and reordering the equations and unknowns, we obtain

$$\begin{pmatrix} D_A & 0 & 0 & 0 & 0 & 0 & A_F & 0 \\ 0 & D_1^Z & 0 & 0 & 0 & 0 & E_{LF} & 0 \\ 0 & 0 & D_2^Z & 0 & 0 & 0 & -E_{UF} & 0 \\ 0 & 0 & 0 & D_1^W & 0 & L_{LF} & 0 & 0 \\ 0 & 0 & 0 & 0 & D_2^W & -L_{UF} & 0 & 0 \\ 0 & 0 & 0 & -L_{LF}^T & L_{UF}^T & 0 & 0 & L_F \\ -A_F^T & -E_{LF}^T & E_{UF}^T & 0 & 0 & 0 & E_F H^B E_F^T & -J_F^T \\ 0 & 0 & 0 & 0 & 0 & -L_F^T & J_F & D_Y \end{pmatrix} \begin{pmatrix} \Delta v \\ \Delta z_1 \\ \Delta z_2 \\ \Delta w_1 \\ \Delta w_2 \\ \Delta s_F \\ \Delta x_F \\ \Delta y \end{pmatrix} = - \begin{pmatrix} D_A(v - \pi^V) \\ D_1^Z(z_1 - \pi_1^Z) \\ D_2^Z(z_2 - \pi_2^Z) \\ D_1^W(w_1 - \pi_1^W) \\ D_2^W(w_2 - \pi_2^W) \\ y_F - L_{LF}^T w_1 + L_{UF}^T w_2 \\ g_F - J_F^T y - A_F^T v - E_{LF}^T z_1 + E_{UF}^T z_2 \\ D_Y(y - \pi^Y) \end{pmatrix}. \quad (6.1)$$

Consider the diagonal matrices

$$D_W = (L_L^T (D_1^W)^{-1} L_L + L_U^T (D_2^W)^{-1} L_U)^\dagger \quad \text{and} \quad D_Z = (E_L^T (D_1^Z)^{-1} E_L + E_U^T (D_2^Z)^{-1} E_U)^\dagger,$$

There are several ways of computing Δw_1 and Δw_2 . Instead of using the block upper-triangular system above, we use the last two blocks of equations of (3.5) to give

$$\Delta w_1 = -(S_1^\mu)^{-1}(w_1 \cdot (L_L(s + \Delta s) - \ell^s + \mu^B e) - \mu^B w_1^E + \mu^B L_L(s - s^E + \Delta s)),$$

and

$$\Delta w_2 = -(S_2^\mu)^{-1}(w_2 \cdot (u^s - L_U(s + \Delta s) + \mu^B e) - \mu^B w_2^E + \mu^B L_U(s^E - s - \Delta s)).$$

Similarly, using (3.5) to solve for Δz_1 and Δz_2 yields

$$\Delta z_1 = -(X_1^\mu)^{-1}(z_1 \cdot (E_L(x + \Delta x) - \ell^x + \mu^B e) - \mu^B z_1^E + \mu^B E_L(x - x^E + \Delta x)),$$

and

$$\Delta z_2 = -(X_2^\mu)^{-1}(z_2 \cdot (u^x - E_U(x + \Delta x) + \mu^B e) - \mu^B z_2^E + \mu^B E_U(x^E - x - \Delta x)).$$

Similarly, using the first block of equations (6.1) to solve for Δv gives $\Delta v = -(v - \hat{\pi}^v)$, with $\hat{\pi}^v = v^E - \frac{1}{\mu^A}(A(x + \Delta x) - b)$. Finally, the vectors Δw_x and Δz_x are recovered as $\Delta w_x = [y + \Delta y - w]_x$ and $\Delta z_x = [g + H\Delta x - J^T(y + \Delta y) - z]_x$, where $w = L_x^T w_x + L_L^T w_1 - L_U^T w_2$ and $z = E_x^T z_x + E_L^T z_1 - E_U^T z_2$.

7. Summary: equations for the primal-dual line-search direction

The results of the preceding section imply that the solution of the path-following equations $F'(v_P)\Delta v_P = -F(v_P)$ with F and F' given by (3.2) and (3.3) may be computed as follows. Let x and s be given primal variables and slack variables such that $E_x x = b_x$, $L_x s = h_x$ with $\ell^x - \mu^B < E_L x$, $E_U x < u^x + \mu^B$, $\ell^s - \mu^B < L_L s$, $L_U s < u^s + \mu^B$. Similarly, let z_1 , z_2 , w_1 , w_2 and y denote dual variables such that $w_1 > 0$, $w_2 > 0$, $z_1 > 0$, and $z_2 > 0$. Consider the diagonal matrices $X_1^\mu = \text{diag}(E_L x - \ell^x + \mu^B e)$, $X_2^\mu = \text{diag}(u^x - E_U x + \mu^B e)$, $Z_1 = \text{diag}(z_1)$, $Z_2 = \text{diag}(z_2)$, $W_1 = \text{diag}(w_1)$, $W_2 = \text{diag}(w_2)$, $S_1^\mu = \text{diag}(L_L s - \ell^s + \mu^B e)$ and

$S_2^\mu = \text{diag}(u^s - L_v s + \mu^B e)$. Consider the quantities

$$\begin{aligned}
D_Y &= \mu^P I_m, & \pi^Y &= y^E - \frac{1}{\mu^P}(c - s), \\
D_A &= \mu^A I_A, & \pi^V &= v^E - \frac{1}{\mu^A}(Ax - b), \\
(D_1^Z)^{-1} &= (X_1^\mu)^{-1} Z_1^\mu, & (D_1^W)^{-1} &= (S_1^\mu)^{-1} W_1^\mu, \\
(D_2^Z)^{-1} &= (X_2^\mu)^{-1} Z_2^\mu, & (D_2^W)^{-1} &= (S_2^\mu)^{-1} W_2^\mu, \\
D_Z &= (E_L^T (D_1^Z)^{-1} E_L + E_U^T (D_2^Z)^{-1} E_U)^\dagger, & D_W &= (L_L^T (D_1^W)^{-1} L_L + L_U^T (D_2^W)^{-1} L_U)^\dagger, \\
\pi_1^Z &= \mu^B (X_1^\mu)^{-1} (z_1^E - x_1 + x_1^E), & \pi_1^W &= \mu^B (S_1^\mu)^{-1} (w_1^E - s_1 + s_1^E), \\
\pi_2^Z &= \mu^B (X_2^\mu)^{-1} (z_2^E - x_2 + x_2^E), & \pi_2^W &= \mu^B (S_2^\mu)^{-1} (w_2^E - s_2 + s_2^E), \\
\pi^Z &= E_L^T \pi_1^Z - E_U^T \pi_2^Z, & \pi^W &= L_L^T \pi_1^W - L_U^T \pi_2^W.
\end{aligned}$$

Choose H_F^B so that H_F^B approximates $E_F H(x, y) E_F^T$ and the KKT matrix

$$\begin{pmatrix} H_F^B + A_F^T D_A^{-1} A_F + E_F D_Z^\dagger E_F^T & J_F^T \\ J_F & -(D_Y + D_W) \end{pmatrix}$$

is nonsingular with m negative eigenvalues. (A common choice of H_F^B is the matrix $E_F (H(x, y) + \sigma I_n) E_F^T$ for some nonnegative scalar σ .) Solve the KKT system

$$\begin{pmatrix} H_F^B + A_F^T D_A^{-1} A_F + E_F D_Z^\dagger E_F^T & -J_F^T \\ J_F & D_Y + D_W \end{pmatrix} \begin{pmatrix} \Delta x_F \\ \Delta y \end{pmatrix} = - \begin{pmatrix} g_F - J_F^T y - A_F^T \pi^V - \pi_F^Z \\ D_W (y_F - \pi_F^W) + D_Y (y - \pi^Y) \end{pmatrix},$$

and set

$$\begin{aligned}
\Delta x &= E_F^T \Delta x_F, & \widehat{x} &= x + \Delta x, & \Delta z_1 &= -(X_1^\mu)^{-1} (z_1 \cdot (E_L \widehat{x} - \ell^X + \mu^B e) - \mu^B z_1^E + \mu^B E_L (x - x^E + \Delta x)), \\
& & \widehat{y} &= y + \Delta y, & \Delta z_2 &= -(X_2^\mu)^{-1} (z_2 \cdot (u^X - E_U \widehat{x} + \mu^B e) - \mu^B z_2^E + \mu^B E_U (x^E - x - \Delta x)), \\
& & \widehat{s} &= s + \Delta s, & \Delta s &= -D_W (\widehat{y} - \pi^W), \\
& & & & \Delta w_1 &= -(S_1^\mu)^{-1} (w_1 \cdot (L_L \widehat{s} - \ell^S + \mu^B e) - \mu^B w_1^E + \mu^B L_L (s - s^E + \Delta s)), \\
& & & & \Delta w_2 &= -(S_2^\mu)^{-1} (w_2 \cdot (u^S - L_U \widehat{s} + \mu^B e) - \mu^B w_2^E + \mu^B L_U (s^E - s - \Delta s)), \\
\widehat{\pi}^V &= v^E - \frac{1}{\mu^A} (A \widehat{x} - b), & \Delta v &= \widehat{\pi}^V - v, \\
w &= L_X^T w_x + L_L^T w_1 - L_U^T w_2, & z &= E_X^T z_x + E_L^T z_1 - E_U^T z_2, \\
\widehat{v} &= v + \Delta v, & \Delta w_x &= [\widehat{y} - w]_x, \\
& & \Delta z_x &= [\nabla f(x) + H(x) \Delta x - J(x)^T \widehat{y} - A^T \widehat{v} - z]_x.
\end{aligned}$$

The associated merit function (4.1) can be written as

$$\begin{aligned}
f(x) &- (c(x) - s)^T y^E + \frac{1}{2\mu^P} \|c(x) - s\|^2 + \frac{1}{2\mu^P} \|c(x) - s + \mu^P (y - y^E)\|^2 \\
&- (Ax - b)^T v^E + \frac{1}{2\mu^A} \|Ax - b\|^2 + \frac{1}{2\mu^A} \|Ax - b + \mu^A (v - v^E)\|^2 \\
&- \sum_{j=1}^{n_L} \left\{ \mu^B ([z_1^E]_j + [E_L x^E - \ell^X]_j + \mu^B) \ln ([z_1 + \mu^B e]_j [E_L x - \ell^X + \mu^B e]_j^2) - [z_1 \cdot (E_L x - \ell^X + \mu^B e)]_j - 2\mu^B [E_L x - \ell^X]_j \right\} \\
&- \sum_{j=1}^{n_U} \left\{ \mu^B ([z_2^E]_j + [u^X - E_U x^E]_j + \mu^B) \ln ([z_2 + \mu^B e]_j [u^X - E_U x + \mu^B e]_j^2) - [z_2 \cdot (u^X - E_U x + \mu^B e)]_j - 2\mu^B [u^X - E_U x]_j \right\} \\
&- \sum_{i=1}^{m_L} \left\{ \mu^B ([w_1^E]_i + [L_L s^E - \ell^S]_i + \mu^B) \ln ([w_1 + \mu^B e]_i [L_L s - \ell^S + \mu^B e]_i^2) - [w_1 \cdot (L_L s - \ell^S + \mu^B e)]_i - 2\mu^B [L_L s - \ell^S]_i \right\} \\
&- \sum_{i=1}^{m_U} \left\{ \mu^B ([w_2^E]_i + [u^S - L_U s^E]_i + \mu^B) \ln ([w_2 + \mu^B e]_i [u^S - L_U s + \mu^B e]_i^2) - [w_2 \cdot (u^S - L_U s + \mu^B e)]_i - 2\mu^B [u^S - L_U s]_i \right\}.
\end{aligned}$$

8. The primal-dual trust-region direction

Given a vector of primal-dual variables $p = (x, x_1, x_2, s, s_1, s_2, y, v, z_1, z_2, w_1, w_2)$, each iteration of a trust-region method for solving (NLP) involves finding a vector Δp of the form $\Delta p = Nd$, where N is a basis for the null-space of the matrix C of (5.1), and d is an approximate solution of the subproblem

$$\underset{d}{\text{minimize}} \quad g_N^T d + \frac{1}{2} d^T B_N(p) d \quad \text{subject to} \quad \|d\|_T \leq \delta, \quad (8.1)$$

where g_N and B_N are the reduced gradient and reduced Hessian $g_N = \nabla M$ and $B_N(p) = N^T B(p) N$, $\|d\|_T = (d^T T d)^{1/2}$, δ is the trust-region radius, and T is positive-definite. The subproblem (8.1) may be written as

$$\underset{\Delta v_M}{\text{minimize}} \quad g_N^T T^{-1/2} \Delta v_M + \frac{1}{2} \Delta v_M^T T^{-1/2} B_N(p) T^{-1/2} \Delta v_M \quad \text{subject to} \quad \|\Delta v_M\|_2 \leq \delta, \quad (8.2)$$

where $\Delta v_M = T^{1/2} d$. The application of the method of Moré and Sorensen [3] to solve the subproblem (8.2) requires the solution of the so-called *secular equations*, which have the form

$$(\bar{B}_N + \sigma I) \Delta v_M = -\bar{g}_N, \quad (8.3)$$

with σ a nonnegative scalar, $\bar{B}_N = T^{-1/2} B_N(p) T^{-1/2}$, and $\bar{g}_N = T^{-1/2} g_N$. In this note we consider the solution of the related equations

$$(B_N + \sigma T) d = -g_N, \quad (8.4)$$

and recover the solution of the secular equations (8.3) from the computed vector d .

The identity (5.6) allows the solution of the approximate Newton equations $B_N(p) d = -g_N$ (5.4) to be written in terms of

the change in the variables $(x, x_1, x_2, s, s_1, s_2, y, v, z_1, z_2, w_1, w_2)$. In particular, we have

$$\begin{pmatrix} \widehat{H}_F & -2J_F^T D_Y^{-1} L_F^T & J_F^T & A_F^T & E_{LF}^T & -E_{UF}^T & 0 & 0 \\ -2L_F D_Y^{-1} J_F & 2L_F (D_Y^{-1} + D_W^\dagger) L_F^T & -L_F & 0 & 0 & 0 & L_{LF}^T & L_{UF}^T \\ J_F & -L_F^T & D_Y & 0 & 0 & 0 & 0 & 0 \\ A_F & 0 & 0 & D_A & 0 & 0 & 0 & 0 \\ E_{LF} & 0 & 0 & 0 & D_1^Z & 0 & 0 & 0 \\ -E_{UF} & 0 & 0 & 0 & 0 & D_2^Z & 0 & 0 \\ 0 & L_{LF} & 0 & 0 & 0 & 0 & D_1^W & 0 \\ 0 & -L_{UF} & 0 & 0 & 0 & 0 & 0 & D_2^W \end{pmatrix} \begin{pmatrix} \Delta x_F \\ \Delta s_F \\ \Delta y \\ \Delta v \\ \Delta z_1 \\ \Delta z_2 \\ \Delta w_1 \\ \Delta w_2 \end{pmatrix} = - \begin{pmatrix} g_F - A_F^T (2\pi^v - v) - J_F^T (2\pi^y - y) - E_{LF} (2\pi_1^z - z_1) + E_{UF} (2\pi_2^z - z_2) \\ 2\pi_F^y - y_F - L_{LF} (2\pi_1^w - w_1) + L_{UF} (2\pi_2^w - w_2) \\ -D_Y (\pi^y - y) \\ -D_A (\pi^v - v) \\ -D_1^Z (\pi_1^z - z_1) \\ -D_2^Z (\pi_2^z - z_2) \\ -D_1^W (\pi_1^w - w_1) \\ -D_2^W (\pi_2^w - w_2) \end{pmatrix},$$

where

$$\widehat{H}_F = E_F H(x, y) E_F^T + \frac{2}{\mu^A} A_F^T A_F + \frac{2}{\mu^P} J_F^T J_F + 2(E_{LF}^T (D_1^Z)^{-1} E_{LF} + E_{UF}^T (D_2^Z)^{-1} E_{UF}),$$

with $H(x, y)$ the Hessian of the Lagrangian function, and

$$\begin{aligned} D_Y &= \mu^P I_m, & \pi^y &= y^E - \frac{1}{\mu^P} (c - s), & D_A &= \mu^A I_A, & \pi^v &= v^E - \frac{1}{\mu^A} (Ax - b), \\ D_1^W &= S_1^\mu (W_1^\mu)^{-1}, & \pi_1^w &= \mu^B (S_1^\mu)^{-1} (w_1^E - s_1 + s_1^E), & D_1^Z &= X_1^\mu (Z_1^\mu)^{-1}, & \pi_1^z &= \mu^B (X_1^\mu)^{-1} (z_1^E - x_1 + x_1^E), \\ D_2^W &= S_2^\mu (W_2^\mu)^{-1}, & \pi_2^w &= \mu^B (S_2^\mu)^{-1} (w_2^E - s_2 + s_2^E), & D_2^Z &= X_2^\mu (Z_2^\mu)^{-1}, & \pi_2^z &= \mu^B (X_2^\mu)^{-1} (z_2^E - x_2 + x_2^E). \end{aligned}$$

Note that in the trust-region case we make no assumption that B_N is positive definite.

The first step in the formulation of the trust-region equations (8.4) and their solution is to write the reduced gradient and Hessian of the merit function in terms of the vectors \vec{x} and \vec{y} that combine the primal variables (x, s) and dual variables

$(y, v, z_1, z_2, w_1, w_2)$. Let \vec{g} , \vec{H} , \vec{J} and \vec{D} denote the quantities

$$\vec{g} = \begin{pmatrix} g_F \\ 0 \end{pmatrix}, \quad \vec{H} = \begin{pmatrix} H_F & 0 \\ 0 & 0 \end{pmatrix}, \quad \vec{J} = \begin{pmatrix} J_F & -L_F^T \\ A_F & 0 \\ E_{LF} & 0 \\ -E_{UF} & 0 \\ 0 & L_{LF} \\ 0 & -L_{UF} \end{pmatrix} \quad \text{and} \quad \vec{D} = \begin{pmatrix} D_Y & 0 & 0 & 0 & 0 & 0 \\ 0 & D_A & 0 & 0 & 0 & 0 \\ 0 & 0 & D_1^z & 0 & 0 & 0 \\ 0 & 0 & 0 & D_2^z & 0 & 0 \\ 0 & 0 & 0 & 0 & D_1^w & 0 \\ 0 & 0 & 0 & 0 & 0 & D_2^w \end{pmatrix}.$$

Similarly, let $\vec{T}_x = \text{diag}(T^x, T^s)$ and $\vec{T}_y = \text{diag}(T^y, T^v, T_1^z, T_2^z, T_1^w, T_2^w)$. The equations $(B_N + \sigma T)\Delta p = -g_N$ may be written in the form

$$\begin{pmatrix} \vec{H} + 2\vec{J}^T \vec{D}^{-1} \vec{J} + \sigma \vec{T}_x & \vec{J}^T \\ \vec{J} & \vec{D} + \sigma \vec{T}_y \end{pmatrix} \begin{pmatrix} \Delta \vec{x} \\ \Delta \vec{y} \end{pmatrix} = - \begin{pmatrix} \vec{g} - \vec{J}^T \vec{\pi} - \vec{J}^T (\vec{\pi} - \vec{y}) \\ -\vec{D}(\vec{\pi} - \vec{y}) \end{pmatrix}, \quad (8.5)$$

where

$$\vec{y} = \begin{pmatrix} y \\ v \\ z_1 \\ z_2 \\ w_1 \\ w_2 \end{pmatrix}, \quad \vec{\pi} = \begin{pmatrix} \pi^y \\ \pi^v \\ \pi_1^z \\ \pi_2^z \\ \pi_1^w \\ \pi_2^w \end{pmatrix}, \quad \Delta \vec{x} = \begin{pmatrix} \Delta x_F \\ \Delta s_F \end{pmatrix}, \quad \text{and} \quad \Delta \vec{y} = \begin{pmatrix} \Delta y \\ \Delta v \\ \Delta z_1 \\ \Delta z_2 \\ \Delta w_1 \\ \Delta w_2 \end{pmatrix}.$$

Applying the nonsingular matrix $\begin{pmatrix} I & -2\vec{J}^T \vec{D}^{-1} \\ & I \end{pmatrix}$ to both sides of (8.5) gives the equivalent system

$$\begin{pmatrix} \vec{H} + \sigma \vec{T}_x & -\vec{J}^T (I + 2\sigma \vec{D}^{-1} \vec{T}_y) \\ \vec{J} & \vec{D} + \sigma \vec{T}_y \end{pmatrix} \begin{pmatrix} \Delta \vec{x} \\ \Delta \vec{y} \end{pmatrix} = - \begin{pmatrix} \vec{g} - \vec{J}^T \vec{y} \\ \vec{D}(\vec{y} - \vec{\pi}) \end{pmatrix}.$$

As in Gertz and Gill [1], we set $\vec{T}_x = I$ and $\vec{T}_y = \vec{D}$. With this choice, the associated vectors $\Delta \vec{x}$ and $\Delta \vec{y}$ satisfy the equations

$$\begin{pmatrix} \vec{H} + \sigma I & -\vec{J}^T \\ \vec{J} & \vec{D} \end{pmatrix} \begin{pmatrix} \Delta \vec{x} \\ (1 + 2\sigma)\Delta \vec{y} \end{pmatrix} = - \begin{pmatrix} \vec{g} - \vec{J}^T \vec{y} \\ \vec{D}(\vec{y} - \vec{\pi}) \end{pmatrix}, \quad (8.6)$$

where $\bar{\sigma} = (1 + \sigma)/(1 + 2\sigma)$. In terms of the original variables, the unsymmetric equations (8.6) are

$$\begin{pmatrix} H_F + \sigma I_F^x & 0 & -J_F^T & -A_F^T & -E_{LF}^T & E_{UF}^T & 0 & 0 \\ 0 & \sigma I_F^s & L_F & 0 & 0 & 0 & -L_{LF}^T & L_{UF}^T \\ J_F & -L_F^T & \bar{\sigma} D_Y & 0 & 0 & 0 & 0 & 0 \\ A_F & 0 & 0 & \bar{\sigma} D_A & 0 & 0 & 0 & 0 \\ E_{LF} & 0 & 0 & 0 & \bar{\sigma} D_1^z & 0 & 0 & 0 \\ -E_{UF} & 0 & 0 & 0 & 0 & \bar{\sigma} D_2^z & 0 & 0 \\ 0 & L_{LF} & 0 & 0 & 0 & 0 & \bar{\sigma} D_1^w & 0 \\ 0 & -L_{UF} & 0 & 0 & 0 & 0 & 0 & \bar{\sigma} D_2^w \end{pmatrix} \begin{pmatrix} \Delta x_F \\ \Delta s_F \\ (1 + 2\sigma)\Delta y \\ (1 + 2\sigma)\Delta v \\ (1 + 2\sigma)\Delta z_1 \\ (1 + 2\sigma)\Delta z_2 \\ (1 + 2\sigma)\Delta w_1 \\ (1 + 2\sigma)\Delta w_2 \end{pmatrix} = - \begin{pmatrix} E_F(g - J^T y - A^T v - z) \\ L_F(y - w) \\ c(x) - s + \mu^P(y - y^E) \\ Ax - b + \mu^A(v - v^E) \\ (Z_1^\mu)^{-1}(z_1 \cdot x_1 + \mu^B(z_1 - z_1^E + x_1 - x_1^E)) \\ (Z_2^\mu)^{-1}(z_2 \cdot x_2 + \mu^B(z_2 - z_2^E + x_2 - x_2^E)) \\ (W_1^\mu)^{-1}(w_1 \cdot s_1 + \mu^B(w_1 - w_1^E + s_1 - s_1^E)) \\ (W_2^\mu)^{-1}(w_2 \cdot s_2 + \mu^B(w_2 - w_2^E + s_2 - s_2^E)) \end{pmatrix}, \quad (8.7)$$

where $\bar{\sigma} = (1 + \sigma)/(1 + 2\sigma)$. Collecting terms and reordering the equations and unknowns, we obtain

$$\begin{pmatrix} \bar{\sigma} D_A & 0 & 0 & 0 & 0 & 0 & A_F & 0 \\ 0 & \bar{\sigma} D_1^z & 0 & 0 & 0 & 0 & E_{LF} & 0 \\ 0 & 0 & \bar{\sigma} D_2^z & 0 & 0 & 0 & -E_{UF} & 0 \\ 0 & 0 & 0 & \bar{\sigma} D_1^w & 0 & L_{LF} & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{\sigma} D_2^w & -L_{UF} & 0 & 0 \\ 0 & 0 & 0 & -L_{LF}^T & L_{UF}^T & \sigma I_F^s & 0 & L_F \\ -A_F^T & -E_{LF}^T & E_{UF}^T & 0 & 0 & 0 & H_F + \sigma I_F^x & -J_F^T \\ 0 & 0 & 0 & 0 & 0 & -L_F^T & J_F & \bar{\sigma} D_Y \end{pmatrix} \begin{pmatrix} \Delta \tilde{v} \\ \Delta \tilde{z}_1 \\ \Delta \tilde{z}_2 \\ \Delta \tilde{w}_1 \\ \Delta \tilde{w}_2 \\ \Delta \tilde{s}_F \\ \Delta \tilde{x}_F \\ \Delta \tilde{y} \end{pmatrix} = - \begin{pmatrix} D_A(v - \pi^V) \\ D_1^z(z_1 - \pi_1^z) \\ D_2^z(z_2 - \pi_2^z) \\ D_1^w(w_1 - \pi_1^w) \\ D_2^w(w_2 - \pi_2^w) \\ L_F(y - w) \\ E_F(g - J^T y - A^T v - z) \\ D_Y(y - \pi^Y) \end{pmatrix}, \quad (8.8)$$

where $\bar{D}_A = \bar{\sigma} D_A$, $\bar{D}_1^w = \bar{\sigma} D_1^w$, $\bar{D}_2^w = \bar{\sigma} D_2^w$, $\bar{D}_1^z = \bar{\sigma} D_1^z$, $\bar{D}_2^z = \bar{\sigma} D_2^z$, $\bar{D}_Y = \bar{\sigma} D_Y$, $\Delta \tilde{y} = (1 + 2\sigma)\Delta y$, $\Delta \tilde{v} = (1 + 2\sigma)\Delta v$, $\Delta \tilde{z}_1 = (1 + 2\sigma)\Delta z_1$, $\Delta \tilde{z}_2 = (1 + 2\sigma)\Delta z_2$, $\Delta \tilde{w}_1 = (1 + 2\sigma)\Delta w_1$, and $\Delta \tilde{w}_2 = (1 + 2\sigma)\Delta w_2$. We define

$$\bar{D}_W = (L_L^T(\bar{D}_1^w)^{-1}L_L + L_U^T(\bar{D}_2^w)^{-1}L_U)^\dagger = \bar{\sigma}(L_L^T(D_1^w)^{-1}L_L + L_U^T(D_2^w)^{-1}L_U)^\dagger = \bar{\sigma}D_W,$$

where

$$\tilde{H}_F = E_F \left(H(x, y) + \frac{1}{\bar{\sigma}} A^T D_A^{-1} A + \frac{1}{\bar{\sigma}} D_z^\dagger \right) E_F^T,$$

$w = L_x^T w_x + L_L^T w_1 - L_U^T w_2$, $z = E_x^T z_x + E_L^T z_1 - E_U^T z_2$, $\pi^w = L_L^T \pi_1^w - L_U^T \pi_2^w$ and $\pi^z = E_L^T \pi_1^z - E_U^T \pi_2^z$. Using block back-substitution, Δx_F and Δy may be computed by solving the equations

$$\begin{pmatrix} \tilde{H}_F + \sigma I_F^x & -J_F^T \\ J_F & \bar{\sigma}(D_Y + \check{D}_w) \end{pmatrix} \begin{pmatrix} \Delta x_F \\ \Delta \tilde{y} \end{pmatrix} = - \begin{pmatrix} E_F \left(g - J^T y - A^T v - z + \frac{1}{\bar{\sigma}} [A^T (v - \pi^v) + z - \pi^z] \right) \\ D_Y (y - \pi^y) + \check{D}_w (\bar{\sigma}(y - w) + w - \pi^w) \end{pmatrix}.$$

Once Δx_F and $\Delta \tilde{y}$ are known, the full vector Δx is computed as $\Delta x = E_F^T \Delta x_F$. Using the identity $\Delta s = L_F^T \Delta s_F$ in the sixth block of equations gives

$$\Delta s = -\bar{\sigma} \check{D}_w \left(y + (1 + 2\sigma) \Delta y - w + \frac{1}{\bar{\sigma}} [w - \pi^w] \right).$$

There are several ways of computing Δw_1 and Δw_2 . Instead of using the block upper-triangular system above, we use the last two blocks of equations of (8.7) to give

$$\begin{aligned} \Delta w_1 &= -\frac{1}{1 + \sigma} (S_1^\mu)^{-1} (w_1 \cdot (L_L(s + \Delta s) - \ell^s + \mu^B e) - \mu^B w_1^E + \mu^B L_L(s - s^E + \Delta s)) \quad \text{and} \\ \Delta w_2 &= -\frac{1}{1 + \sigma} (S_2^\mu)^{-1} (w_2 \cdot (u^s - L_U(s + \Delta s) + \mu^B e) - \mu^B w_2^E + \mu^B L_U(s^E - s - \Delta s)). \end{aligned}$$

Similarly, using (8.7) to solve for Δz_1 and Δz_2 yields

$$\begin{aligned} \Delta z_1 &= -\frac{1}{1 + \sigma} (X_1^\mu)^{-1} (z_1 \cdot (E_L(x + \Delta x) - \ell^x + \mu^B e) - \mu^B z_1^E + \mu^B E_L(x - x^E + \Delta x)) \quad \text{and} \\ \Delta z_2 &= -\frac{1}{1 + \sigma} (X_2^\mu)^{-1} (z_2 \cdot (u^x - E_U(x + \Delta x) + \mu^B e) - \mu^B z_2^E + \mu^B E_U(x^E - x - \Delta x)). \end{aligned}$$

Similarly, using the first block of equations (8.8) to solve for Δv gives $\Delta v = -(v - \hat{\pi}^v)/(1 + \sigma)$, with $\hat{\pi}^v = v^E - \frac{1}{\mu^A} (A(x + \Delta x) - b)$. Finally, the vectors Δw_x and Δz_x are recovered as $\Delta w_x = [y + \Delta y - w]_x$ and $\Delta z_x = [g + H \Delta x - J^T (y + \Delta y) - z]_x$.

9. Summary: equations for the trust-region direction

The results of the preceding section implies that the solution of the secular equations $(\bar{B}_N + \sigma I) \Delta v_M = -\bar{g}_N$, with σ a nonnegative scalar, $\bar{B}_N = T^{-1/2} B_N(p) T^{-1/2}$, and $\bar{g}_N = T^{-1/2} g_N$ may be computed as follows. Let x and s be given primal variables and slack variables such that $E_x x = b_x$, $L_x s = h_x$ with $\ell^x - \mu^B < E_L x$, $E_U x < u^x + \mu^B$, $\ell^s - \mu^B < L_L s$, $L_U s < u^s + \mu^B$. Similarly,

let z_1, z_2, w_1, w_2 and y denotes dual variables such that $w_1 > 0, w_2 > 0, z_1 > 0,$ and $z_2 > 0$. Consider the diagonal matrices $X_1^\mu = \text{diag}(E_L x - \ell^x + \mu^B e), X_2^\mu = \text{diag}(u^x - E_U x + \mu^B e), Z_1 = \text{diag}(z_1), Z_2 = \text{diag}(z_2), W_1 = \text{diag}(w_1), W_2 = \text{diag}(w_2), S_1^\mu = \text{diag}(L_L s - \ell^s + \mu^B e)$ and $S_2^\mu = \text{diag}(u^s - L_U s + \mu^B e)$. Given the quantities

$$\begin{aligned}
D_Y &= \mu^P I_m, & \pi^Y &= y^E - \frac{1}{\mu^P}(c - s), \\
D_A &= \mu^A I_A, & \pi^V &= v^E - \frac{1}{\mu^A}(Ax - b), \\
(D_1^Z)^{-1} &= (X_1^\mu)^{-1} Z_1^\mu, & (D_1^W)^{-1} &= (S_1^\mu)^{-1} W_1^\mu, \\
(D_2^Z)^{-1} &= (X_2^\mu)^{-1} Z_2^\mu, & (D_2^W)^{-1} &= (S_2^\mu)^{-1} W_2^\mu, \\
D_Z &= (E_L^T (D_1^Z)^{-1} E_L + E_U^T (D_2^Z)^{-1} E_U)^\dagger, & D_W &= (L_L^T (D_1^W)^{-1} L_L + L_U^T (D_2^W)^{-1} L_U)^\dagger, \\
\pi_1^Z &= \mu^B (X_1^\mu)^{-1} (z_1^E - x_1 + x_1^E), & \check{D}_W &= (D_W^\dagger + \sigma \bar{\sigma} I_F^s)^\dagger, \\
\pi_2^Z &= \mu^B (X_2^\mu)^{-1} (z_2^E - x_2 + x_2^E), & \pi_1^W &= \mu^B (S_1^\mu)^{-1} (w_1^E - s_1 + s_1^E), \\
\pi^Z &= E_L^T \pi_1^Z - E_U^T \pi_2^Z, & \pi_2^W &= \mu^B (S_2^\mu)^{-1} (w_2^E - s_2 + s_2^E), \\
& & \pi^W &= L_L^T \pi_1^W - L_U^T \pi_2^W,
\end{aligned}$$

solve the KKT system

$$\begin{aligned}
& \begin{pmatrix} E_F \left(H(x, y) + \sigma I_n + \frac{1}{\bar{\sigma}} A^T D_A^{-1} A + \frac{1}{\bar{\sigma}} D_Z^\dagger \right) E_F^T & -J_F^T \\ J_F & \bar{\sigma} (D_Y + \check{D}_W) \end{pmatrix} \begin{pmatrix} \Delta x_F \\ \Delta \tilde{y} \end{pmatrix} \\
& = - \begin{pmatrix} E_F \left(\nabla f(x) - J(x)^T y - A^T v - z + \frac{1}{\bar{\sigma}} [A^T (v - \pi^V) + z - \pi^Z] \right) \\ D_Y (y - \pi^Y) + \check{D}_W (\bar{\sigma} (y - w) + w - \pi^W) \end{pmatrix}.
\end{aligned}$$

Then

$$\begin{aligned}
\Delta x &= E_F^T \Delta x_F, & \hat{x} &= x + \Delta x, & \Delta z_1 &= -\frac{1}{1+\sigma} (X_1^\mu)^{-1} (z_1 \cdot (E_L \hat{x} - \ell^x + \mu^B e) - \mu^B z_1^E + \mu^B L_L (s - s^E + \Delta s)), \\
\Delta y &= \Delta \tilde{y} / (1 + 2\sigma), & \hat{y} &= y + \Delta y, & \Delta z_2 &= -\frac{1}{1+\sigma} (X_2^\mu)^{-1} (z_2 \cdot (u^x - E_U \hat{x} + \mu^B e) - \mu^B z_2^E + \mu^B L_U (s^E - s - \Delta s)), \\
& & \hat{s} &= s + \Delta s, & \Delta s &= -\bar{\sigma} \check{D}_w \left(y + (1 + 2\sigma) \Delta y - w + \frac{1}{\bar{\sigma}} [w - \pi^w] \right), \\
& & \hat{v}^V &= v^E - \frac{1}{\mu^A} (A \hat{x} - b), & \Delta w_1 &= -\frac{1}{1+\sigma} (S_1^\mu)^{-1} (w_1 \cdot (L_L \hat{s} - \ell^s + \mu^B e) - \mu^B w_1^E + \mu^B L_L (s - s^E + \Delta s)), \\
& & w &= L_X^T w_x + L_L^T w_1 - L_U^T w_2, & \Delta w_2 &= -\frac{1}{1+\sigma} (S_2^\mu)^{-1} (w_2 \cdot (u^s - L_U \hat{s} + \mu^B e) - \mu^B w_2^E + \mu^B L_U (s^E - s - \Delta s)), \\
& & \hat{v} &= v + \Delta v, & \Delta v &= -\frac{1}{1+\sigma} (v - \hat{\pi}^V), \\
& & z &= E_X^T z_x + E_L^T z_1 - E_U^T z_2, & \Delta w_x &= [\hat{y} - w]_x, \\
& & \Delta z_x &= [g + H \Delta x - J^T \hat{y} - z]_x.
\end{aligned}$$

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