DISCRETE HAMILTON–JACOBI THEORY

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Abstract. We develop a discrete analogue of the Hamilton–Jacobi theory in the framework of the discrete Hamiltonian mechanics. We first reinterpret the discrete Hamilton–Jacobi equation derived by Elnatanov and Schiff in the language of discrete mechanics. The resulting discrete Hamilton–Jacobi equation is discrete only in time, and is shown to recover the Hamilton–Jacobi equation in the continuous-time limit. The correspondence between discrete and continuous Hamiltonian mechanics naturally gives rise to a discrete analogue of Jacobi's solution to the Hamilton–Jacobi equation. We also prove a discrete analogue of the geometric Hamilton–Jacobi theorem of Abraham and Marsden. These results are readily applied to discrete optimal control setting, and some well-known results in discrete optimal control theory, such as the Bellman equation (discrete-time Hamilton–Jacobi–Bellman equation) of dynamic programming, follow immediately. We also apply the theory to discrete linear Hamiltonian systems, and show that the discrete Riccati equation follows as a special case of the discrete Hamilton–Jacobi equation.

1. Introduction

1.1. **Discrete Mechanics.** Discrete mechanics, a discrete-time version of Lagrangian and Hamiltonian mechanics, provides not only a systematic view of structure-preserving integrators but also a discrete-time counterpart to the theory of Lagrangian and Hamiltonian mechanics [see, e.g., 22; 25; 26]. The main feature of discrete mechanics is its use of a discrete version of variational principles. Namely discrete mechanics assumes that the dynamics is defined on discrete times from the outset, formulates a discrete variational principle on such dynamics, and then derives a discrete analogue of the Euler–Lagrange or Hamilton's equations from it. In other words, discrete mechanics is a reformulation of Lagrangian and Hamiltonian mechanics with discrete time, as opposed to a discretization of the equations in the continuous-time theory.

The advantage of this construction is that it naturally gives rise to discrete analogues of the concepts and ideas that share the same or similar properties with their continuous counterparts, such as symplectic forms, the Legendre transformation, momentum maps, and Noether's theorem [22]. This in turn provides us with the discrete ingredients that facilitate further theoretical developments, such as discrete analogues of the theories of complete integrability [see, e.g., 23; 25; 26] and also those of reduction and connections [13; 18; 20]. Whereas the main topic in discrete mechanics is the development of structure-preserving algorithms for Lagrangian and Hamiltonian systems [see, e.g., 22], the theoretical aspects of it are interesting in their own rights, and furthermore give insights into the numerical aspects as well.

Another notable feature of discrete mechanics, especially on the Hamiltonian side, is that it is a generalization of (nonsingular) discrete optimal control problems. In fact, as stated in Marsden and West [22], discrete mechanics is inspired by the formulations of discrete optimal control problems (see, for example, Jordan and Polak [14] and Cadzow [5]).

1.2. **Hamilton–Jacobi Theory.** In classical mechanics [see, e.g., 3; 10; 17; 21], the Hamilton–Jacobi equation is first introduced as a partial differential equation that the action integral satisfies. Specifically, let Q be a configuration space and T^*Q be its cotangent bundle, and suppose that $(\hat{q}(s), \hat{p}(s)) \in T^*Q$ is a solution of Hamilton's equations

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q}.$$

Then calculate the action integral along the solution starting from $s = 0$ and ending at $s = t$ with $t > 0$:

$$S(q, t) := \int_0^t \left[\hat{p}(s) \cdot \dot{\hat{q}}(s) - H(\hat{q}(s), \hat{p}(s)) \right] ds, \tag{1.1}$$

where $q := \hat{q}(t)$ and we regard the resulting integral as a function of the endpoint $(q, t) \in Q \times \mathbb{R}_+$, where $\mathbb{R}_+$ is the set of positive real numbers. Then by taking variation of the endpoint (q, t), one obtains a partial differential equation satisfied by $S(q, t)$:

$$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}\right) = 0.$$

This is the *Hamilton–Jacobi equation.*

Conversely, it is shown that if $S(q, t)$ is a solution of the Hamilton–Jacobi equation then $S(q, t)$ is a generating function of the canonical transformations (or symplectic flow) of the dynamics defined by Hamilton's equations. This result is the theory behind the powerful technique of exact integration called separation of variables.

The idea of the Hamilton–Jacobi theory is also useful in optimal control theory [see, e.g., 15]. Namely the Hamilton–Jacobi equation turns into the Hamilton–Jacobi–Bellman equation, which is a partial differential equation satisfied by the optimal cost function. It is also shown that the costate of the optimal solution is related to the solution of the Hamilton–Jacobi–Bellman equation.

1.3. **Discrete Hamilton–Jacobi Theory.** The main objective of this paper is to present a discrete analogue of the Hamilton–Jacobi theory using the framework of discrete Hamiltonian mechanics [16].

There are some previous works on discrete-time analogues of the Hamilton–Jacobi equation, such as Elnatanov and Schiff [8] and Lall and West [16]. Specifically, Elnatanov and Schiff [8] derived an equation for a generating function of a coordinate transformation that trivializes the dynamics. This derivation is a discrete analogue of the conventional derivation of the continuous-time Hamilton–Jacobi equation [see, e.g., 17, Chapter VIII]. Lall and West [16] formulated a discrete Lagrangian analogue of the Hamilton–Jacobi equation as a separable optimization problem.

1.4. **Main Results.** Our work was inspired by the result of Elnatanov and Schiff [8] and starts from a reinterpretation of their result in the language of discrete mechanics. This paper further extends the result by developing discrete analogues of results in the (continuous-time) Hamilton–Jacobi theory. Namely, we formulate a discrete analogue of Jacobi's solution, which relates the discrete action integral with a solution of the discrete Hamilton–Jacobi equation. This also provides a very simple derivation of the discrete Hamilton–Jacobi equation and exhibits a natural correspondence with the continuous-time theory. Another important result in this paper is a discrete analogue of the Hamilton–Jacobi theorem, which relates the solution of the discrete Hamilton–Jacobi equation with the solution of the discrete Hamilton's equations.

We also show that the discrete Hamilton–Jacobi equation is a generalization of the discrete Riccati equation and the Bellman equation (discrete Hamilton–Jacobi–Bellman equation). (See Fig. 1.) Specifically, we show that the discrete Hamilton–Jacobi equation applied to linear discrete Hamiltonian systems reduces to the discrete Riccati equation. This is again a discrete analogue of the well-known result that the Hamilton–Jacobi equation applied to linear Hamiltonian systems reduces to the Riccati equation [see, e.g., 15, p. 421]. Also we establish a link with discrete-time optimal control theory, and show that the Bellman equation of dynamic programming follows. This link makes it possible to interpret discrete analogues of Jacobi's solution and the Hamilton–Jacobi theorem in the optimal control setting. Namely we show that these results reduce to two well-known results in optimal control theory that relate the Bellman equation with the optimal solution.

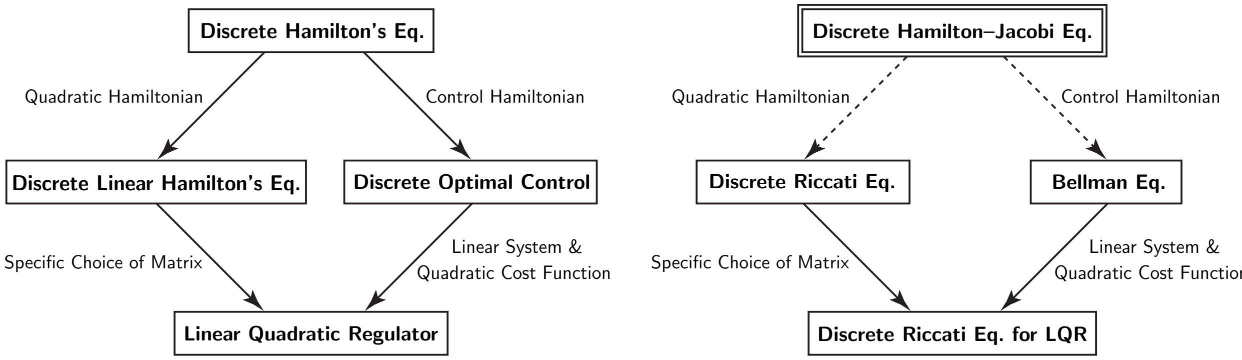


FIGURE 1. Discrete evolution equations (left) and corresponding discrete Hamilton–Jacobi-type equations (right). Dashed lines are the links established in the paper.

1.5. **Outline of the Paper.** We first present a brief review of discrete Lagrangian and Hamiltonian mechanics in Section 2. In Section 3 we describe a reinterpretation of the result of Elnatanov and Schiff [8] in the language of discrete mechanics and a discrete analogue of Jacobi's solution to the discrete Hamilton–Jacobi equation. The remainder of Section 3 is devoted to more detailed studies of the discrete Hamilton–Jacobi equation: its left and right variants, more explicit forms of them, and also a digression on the Lagrangian side. In Section 4 we prove a discrete version of the Hamilton–Jacobi theorem. Section 5 establishes the link with the discrete-time optimal control problem setting and show what the results in the preceding sections imply in this setting. In Section 6 we apply the theory to linear discrete Hamiltonian systems, and show that the discrete Riccati equation follows from the discrete Hamilton–Jacobi equation. We then take a harmonic oscillator as a simple physical example, and solve the discrete Hamilton–Jacobi equation explicitly. Finally, Section 7 discusses the continuous-time limit of the theory.

2. Discrete Mechanics

This section briefly reviews some key results of discrete mechanics following Marsden and West [22] and Lall and West [16].

2.1. **Discrete Lagrangian Mechanics.** A discrete Lagrangian flow $\{q_k\}$ for $k = 0, 1, \dots, N$ on a n-dimensional differentiable manifold Q can be described based on the following discrete variational principle. Let $S_{\rm d}^N$ be the following action sum of the discrete Lagrangian $L_{\rm d} : Q \times Q \to \mathbb{R}$:

$$S_{\rm d}^N(\{q_k\}_{k=0}^N) := \sum_{k=0}^{N-1} L_{\rm d}(q_k, q_{k+1}) \approx \int_0^{t_N} L(q(t), \dot{q}(t))\, dt, \tag{2.1}$$

which is an approximation of the action integral as shown above.

Consider discrete variations $q_k \mapsto q_k + \delta q_k$ for $k = 0, 1, \dots, N$ with $\delta q_0 = \delta q_N = 0$. Then the discrete variational principle $\delta S_{\rm d}^N = 0$ gives the discrete Euler–Lagrange equations:

$$D_2 L_{\rm d}(q_{k-1}, q_k) + D_1 L_{\rm d}(q_k, q_{k+1}) = 0. \tag{2.2}$$

This determines the discrete flow $F_{L_{\rm d}} : Q \times Q \to Q \times Q$:

$$F_{L_{\rm d}} : (q_{k-1}, q_k) \mapsto (q_k, q_{k+1}), \tag{2.3}$$

and this flow preserves the discrete Lagrangian symplectic one-forms $\Theta_{L_{\rm d}}^{\pm} : Q \times Q \to T^*(Q \times Q)$ defined by

$$\Theta_{L_{\rm d}}^{+} : (q_k, q_{k+1}) \mapsto D_2 L_{\rm d}(q_k, q_{k+1})\, dq_{k+1}, \tag{2.4a}$$

$$\Theta_{L_{\rm d}}^{-} : (q_k, q_{k+1}) \mapsto -D_1 L_{\rm d}(q_k, q_{k+1})\, dq_k. \tag{2.4b}$$

and hence it also preserves the discrete Lagrangian symplectic form

$$\Omega_{L_{\rm d}}(q_k, q_{k+1}) := d\Theta^{+}_{L_{\rm d}} = d\Theta^{-}_{L_{\rm d}} = D_1 D_2 L_{\rm d}(q_k, q_{k+1})\, dq_k \wedge dq_{k+1}. \tag{2.5}$$

Specifically, we have

$$(F_{L_{\rm d}})^{*}\Omega_{L_{\rm d}} = \Omega_{L_{\rm d}}. \tag{2.6}$$

2.2. **Discrete Hamiltonian Mechanics.** Introduce the *right and left discrete Legendre transforms* $\mathbb{F}L^{\pm}_{\rm d} : Q \times Q \to T^{*}Q$ by

$$\mathbb{F}L^{+}_{\rm d} : (q_k, q_{k+1}) \mapsto (q_{k+1}, D_2 L_{\rm d}(q_k, q_{k+1})), \tag{2.7a}$$

$$\mathbb{F}L^{-}_{\rm d} : (q_k, q_{k+1}) \mapsto (q_k, -D_1 L_{\rm d}(q_k, q_{k+1})). \tag{2.7b}$$

Then we find that the discrete Lagrangian symplectic forms Eq. (2.4) and (2.5) are pull-backs by these maps of the standard symplectic form on $T^{*}Q$:

$$\Theta^{\pm}_{L_{\rm d}} = (\mathbb{F}L^{\pm}_{\rm d})^{*}\Theta, \qquad \Omega^{\pm}_{L_{\rm d}} = (\mathbb{F}L^{\pm}_{\rm d})^{*}\Omega. \tag{2.8}$$

Let us define the momenta

$$p^{-}_{k,k+1} := -D_1 L_{\rm d}(q_k, q_{k+1}), \qquad p^{+}_{k,k+1} := D_2 L_{\rm d}(q_k, q_{k+1}). \tag{2.9}$$

Then the discrete Euler–Lagrange equations (2.2) becomes simply $p^{+}_{k-1,k} = p^{-}_{k,k+1}$. So defining

$$p_k := p^{+}_{k-1,k} = p^{-}_{k,k+1}, \tag{2.10}$$

one can rewrite the discrete Euler–Lagrange equations (2.2) as follows:

$$\begin{aligned} p_k &= -D_1 L_{\rm d}(q_k, q_{k+1}), \\ p_{k+1} &= D_2 L_{\rm d}(q_k, q_{k+1}). \end{aligned} \tag{2.11}$$

Furthermore, define the *discrete Hamiltonian map* $\tilde{F}_{L_{\rm d}} : T^{*}Q \to T^{*}Q$ by

$$\tilde{F}_{L_{\rm d}} : (q_k, p_k) \mapsto (q_{k+1}, p_{k+1}). \tag{2.12}$$

One may relate this map with the discrete Legendre transforms in Eq. (2.7) as follows:

$$\tilde{F}_{L_{\rm d}} = \mathbb{F}L^{+}_{\rm d} \circ (\mathbb{F}L^{-}_{\rm d})^{-1}. \tag{2.13}$$

Furthermore one can also show that this map is symplectic, i.e.,

$$(\tilde{F}_{L_{\rm d}})^{*}\Omega = \Omega. \tag{2.14}$$

This is the Hamiltonian description of the dynamics defined by the discrete Euler–Lagrange equation (2.2) introduced by Marsden and West [22]. However, notice that no discrete analogue of Hamilton's equations is introduced here, although the flow is now in the cotangent bundle $T^{*}Q$.

Lall and West [16] pushed this idea further to give discrete analogues of Hamilton's equations: From the point of view that a discrete Lagrangian is essentially a generating function of the first kind, we can apply Legendre transforms to the discrete Lagrangian to find the corresponding generating function of type two or three [10]. In fact, they turn out to be a natural Hamiltonian counterpart to the discrete Lagrangian mechanics described above. Specifically, with the discrete Legendre transform

$$p_{k+1} = \mathbb{F}L^{+}_{\rm d}(q_k, q_{k+1}) = D_2 L_{\rm d}(q_k, q_{k+1}), \tag{2.15}$$

we can define the following *right discrete Hamiltonian*:

$$H^{+}_{\rm d}(q_k, p_{k+1}) = p_{k+1} \cdot q_{k+1} - L_{\rm d}(q_k, q_{k+1}). \tag{2.16}$$

Then the discrete Hamiltonian map $\tilde{F}_{L_{\mathrm{d}}} : (q_k, p_k) \mapsto (q_{k+1}, p_{k+1})$ is defined implicitly by the *right discrete Hamilton's equations*

$$q_{k+1} = D_2 H_{\mathrm{d}}^{+}(q_k, p_{k+1}), \tag{2.17a}$$

$$p_k = D_1 H_{\mathrm{d}}^{+}(q_k, p_{k+1}). \tag{2.17b}$$

Similarly, with the discrete Legendre transform

$$p_k = \mathbb{F}L_{\mathrm{d}}^{-}(q_k, q_{k+1}) = -D_1 L_{\mathrm{d}}(q_k, q_{k+1}), \tag{2.18}$$

we can define the following *left discrete Hamiltonian*:

$$H_{\mathrm{d}}^{-}(p_k, q_{k+1}) = -p_k \cdot q_k - L_{\mathrm{d}}(q_k, q_{k+1}). \tag{2.19}$$

Then we have the *left discrete Hamilton's equations*

$$q_k = -D_1 H_{\mathrm{d}}^{-}(p_k, q_{k+1}), \tag{2.20a}$$

$$p_{k+1} = -D_2 H_{\mathrm{d}}^{-}(p_k, q_{k+1}). \tag{2.20b}$$

3. Discrete Hamilton–Jacobi Equation

3.1. **Derivation by Elnatanov and Schiff.**

Elnatanov and Schiff [8] derived a discrete Hamilton–Jacobi equation based on the idea that the Hamilton–Jacobi equation is an equation for a symplectic change of coordinates under which the dynamics becomes trivial. In this section we would like to reinterpret their derivation in the framework of discrete Hamiltonian mechanics reviewed above.

Theorem 3.1. *Suppose that the discrete dynamics $\{(q_k, p_k)\}_{k=0}^{N}$ is governed by the right discrete Hamilton's equations* (2.17)*. Consider the symplectic coordinate transformation $(q_k, p_k) \mapsto (\hat{q}_k, \hat{p}_k)$ that satisfies the following:*

(i) *The old and new coordinates are related by the type-1 generating function*[1] $S^k : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$*:*

$$\begin{aligned} \hat{p}_k &= -D_1 S^k(\hat{q}_k, q_k), \\ p_k &= D_2 S^k(\hat{q}_k, q_k); \end{aligned} \tag{3.1}$$

(ii) *the dynamics in the new coordinates $\{(\hat{q}_k, \hat{p}_k)\}_{k=0}^{N}$ is rendered trivial, i.e., $(\hat{q}_{k+1}, \hat{p}_{k+1}) = (\hat{q}_k, \hat{p}_k)$.*

Then the set of functions $\{S^k\}_{k=1}^{N}$ satisfies the discrete Hamilton–Jacobi equation*:*

$$S^{k+1}(\hat{q}_0, q_{k+1}) - S^k(\hat{q}_0, q_k) - D_2 S^{k+1}(\hat{q}_0, q_{k+1}) \cdot q_{k+1} + H_{\mathrm{d}}^{+}\left(q_k, D_2 S^{k+1}(\hat{q}_0, q_{k+1})\right) = 0, \tag{3.2}$$

or, with the shorthand notation $S_{\mathrm{d}}^k(q_k) := S^k(\hat{q}_0, q_k)$,

$$S_{\mathrm{d}}^{k+1}(q_{k+1}) - S_{\mathrm{d}}^k(q_k) - D S_{\mathrm{d}}^{k+1}(q_{k+1}) \cdot q_{k+1} + H_{\mathrm{d}}^{+}\left(q_k, D S_{\mathrm{d}}^{k+1}(q_{k+1})\right) = 0. \tag{3.3}$$

Proof. The key ingredient in the proof is the right discrete Hamiltonian in the new coordinates, i.e., a function $\hat{H}_{\mathrm{d}}^{+}(\hat{q}_k, \hat{p}_{k+1})$ that satisfies

$$\begin{aligned} \hat{q}_{k+1} &= D_2 \hat{H}_{\mathrm{d}}^{+}(\hat{q}_k, \hat{p}_{k+1}), \\ \hat{p}_k &= D_1 \hat{H}_{\mathrm{d}}^{+}(\hat{q}_k, \hat{p}_{k+1}), \end{aligned} \tag{3.4}$$

or equivalently,

$$\hat{p}_k \, d\hat{q}_k + \hat{q}_{k+1} \, d\hat{p}_{k+1} = d\hat{H}_{\mathrm{d}}^{+}(\hat{q}_k, \hat{p}_{k+1}). \tag{3.5}$$

[1] This is essentially the same as Eq. (2.11) in the sense that they are both transformations defined by generating functions of type one: Replace $(q_k, p_k, q_{k+1}, p_{k+1}, L_{\mathrm{d}})$ by $(\hat{q}_k, \hat{p}_k, q_k, p_k, S^k)$. However they have different interpretations: Eq. (2.11) describes the dynamics or time evolution whereas Eq. (3.1) is a change of coordinates.

Let us first write $\hat{H}_{\rm d}^{+}$ in terms of the original right discrete Hamiltonian $H_{\rm d}^{+}$ and the generating function S^k. For that purpose, first rewrite Eqs. (2.17) and (3.1) as follows:

$$p_k\,dq_k = -q_{k+1}\,dp_{k+1} + dH_{\rm d}^{+}(q_k, p_{k+1})$$

and

$$\hat{p}_k\,d\hat{q}_k = p_k\,dq_k - dS^k(\hat{q}_k, q_k),$$

respectively. Then, using the above relations, we have

$$\begin{aligned}
\hat{p}_k\,d\hat{q}_k + \hat{q}_{k+1}\,d\hat{p}_{k+1} &= \hat{p}_k\,d\hat{q}_k + d(\hat{p}_{k+1}\cdot\hat{q}_{k+1}) - \hat{p}_{k+1}\,d\hat{q}_{k+1} \\
&= p_k\,dq_k - dS^k(\hat{q}_k, q_k) + d(\hat{p}_{k+1}\cdot\hat{q}_{k+1}) - p_{k+1}\,dq_{k+1} + dS^{k+1}(\hat{q}_{k+1}, q_{k+1}) \\
&= -q_{k+1}\,dp_{k+1} + dH_{\rm d}^{+}(q_k, p_{k+1}) \\
&\quad - dS^k(\hat{q}_k, q_k) + d(\hat{p}_{k+1}\hat{q}_{k+1}) - p_{k+1}\,dq_{k+1} + dS^{k+1}(\hat{q}_{k+1}, q_{k+1}) \\
&= d\left(H_{\rm d}^{+}(q_k, p_{k+1}) + \hat{p}_{k+1}\cdot\hat{q}_{k+1} - p_{k+1}\cdot q_{k+1} + S^{k+1}(\hat{q}_{k+1}, q_{k+1}) - S^k(\hat{q}_k, q_k)\right).
\end{aligned}$$

Thus in view of Eq. (3.5), we obtain

$$\hat{H}_{\rm d}^{+}(\hat{q}_k, \hat{p}_{k+1}) = H_{\rm d}^{+}(q_k, p_{k+1}) + \hat{p}_{k+1}\cdot\hat{q}_{k+1} - p_{k+1}\cdot q_{k+1} + S^{k+1}(\hat{q}_{k+1}, q_{k+1}) - S^k(\hat{q}_k, q_k). \tag{3.6}$$

Now consider the choice of the new right discrete Hamiltonian $\hat{H}_{\rm d}^{+}$ that renders the dynamics trivial, i.e., $(\hat{q}_{k+1}, \hat{p}_{k+1}) = (\hat{q}_k, \hat{p}_k)$. It is clear from Eq. (3.4) that we can set

$$\hat{H}_{\rm d}^{+}(\hat{q}_k, \hat{p}_{k+1}) = \hat{p}_{k+1}\cdot\hat{q}_k. \tag{3.7}$$

Then Eq. (3.6) becomes

$$\hat{p}_{k+1}\cdot\hat{q}_k = H_{\rm d}^{+}(q_k, p_{k+1}) + \hat{p}_{k+1}\cdot\hat{q}_{k+1} - p_{k+1}\cdot q_{k+1} + S^{k+1}(\hat{q}_{k+1}, q_{k+1}) - S^k(\hat{q}_k, q_k),$$

and since $\hat{q}_{k+1} = \hat{q}_k = \cdots = \hat{q}_0$ we have

$$0 = H_{\rm d}^{+}(q_k, p_{k+1}) - p_{k+1}\cdot q_{k+1} + S^{k+1}(\hat{q}_0, q_{k+1}) - S^k(\hat{q}_0, q_k)$$

Eliminating p_{k+1} by using Eq. (3.1), we obtain Eq. (3.2). □

Remark 3.2. What Elnatanov and Schiff [8] refer to the *Hamilton–Jacobi difference equation* is the following:

$$S^{k+1}(\hat{q}_0, q_{k+1}) - S^k(\hat{q}_0, q_k) - D_2S^{k+1}(\hat{q}_0, q_{k+1})\cdot D_2H_{\rm d}^{+}(q_k, p_{k+1}) + H_{\rm d}^{+}(q_k, p_{k+1}) = 0. \tag{3.8}$$

It is clear that this is equivalent to Eq. (3.2) in view of Eq. (2.17)

3.2. **Discrete Analogue of Jacobi's Solution.** This section shows a discrete analogue of Jacobi's solution. This also gives an alternative derivation of the discrete Hamilton–Jacobi equation that is much simpler than the one shown above.

Theorem 3.3. *Consider the action sums Eq.* (2.1) *written in terms of the right discrete Hamiltonian, Eq.* (2.16)*:*

$$S_{\rm d}^k(q_k) := \sum_{l=0}^{k-1}\left[p_{l+1}\cdot q_{l+1} - H_{\rm d}^{+}(q_l, p_{l+1})\right] \tag{3.9}$$

evaluated along a solution of the right discrete Hamilton's equations (2.17)*; each* $S_{\rm d}^k(q_k)$ *is seen as a function of the end point coordinates* q_k *and the discrete end time* k*. Then these action sums satisfy the discrete Hamilton–Jacobi equation* (3.3)*.*

Proof. From Eq. (3.9), we have

$$S_{\rm d}^{k+1}(q_{k+1}) - S_{\rm d}^{k}(q_k) = p_{k+1} \cdot q_{k+1} - H_{\rm d}^{+}(q_k, p_{k+1}), \tag{3.10}$$

where p_{k+1} is considered to be a function of q_k and q_{k+1}, i.e., $p_{k+1} = p_{k+1}(q_k, q_{k+1})$. Taking the derivative of both sides with respect to q_{k+1}, we have

$$DS_{\rm d}^{k+1}(q_{k+1}) = p_{k+1} + \frac{\partial p_{k+1}}{\partial q_{k+1}} \cdot \left[q_{k+1} - D_2 H_{\rm d}^{+}(q_k, p_{k+1})\right].$$

However, the term in the brackets vanish because the right discrete Hamilton's equations (2.17) are assumed to be satisfied. Thus we have

$$p_{k+1} = DS_{\rm d}^{k+1}(q_{k+1}). \tag{3.11}$$

Substituting this into Eq. (3.10) gives Eq. (3.3). □

A couple of remarks are in order.

Remark 3.4. Recall that, in the derivation of the continuous Hamilton–Jacobi equation [see, e.g., 9, Section 23], we consider the variation of the action integral Eq. (1.1) with respect to the end point (q, t) and find

$$dS = p\,dq - H(q,p)\,dt. \tag{3.12}$$

This gives

$$\frac{\partial S}{\partial t} = -H(q,p), \qquad p = \frac{\partial S}{\partial q}, \tag{3.13}$$

and hence the Hamilton–Jacobi equation

$$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}\right) = 0. \tag{3.14}$$

In the above derivation of the discrete Hamilton–Jacobi equation (3.3), the difference in two action sums Eq. (3.10) is a natural discrete counterpart to the variation dS in Eq. (3.12). Notice also that Eq. (3.10) plays the same essential role as Eq. (3.12) does in deriving the Hamilton–Jacobi equation.

The table below summarizes the correspondence between the ingredients in the continuous and discrete theories (See also Remark 3.4):

3.3. **The Right and Left Discrete Hamilton–Jacobi Equations.** Recall that, in Eq. (3.9), we wrote the action sum Eq. (2.1) in terms of the right discrete Hamiltonian Eq. (2.16). We can also write it in terms of the left discrete Hamiltonian Eq. (2.19) as follows:

$$S_{\rm d}^{k}(q_k) = \sum_{l=0}^{k-1} \left[-p_l \cdot q_l - H_{\rm d}^{-}(p_l, q_{l+1})\right]. \tag{3.15}$$

Then we can proceed as in the proof of Theorem 3.3: First we have

$$S_{\rm d}^{k+1}(q_{k+1}) - S_{\rm d}^{k}(q_k) = -p_k \cdot q_k - H_{\rm d}^{-}(p_k, q_{k+1}). \tag{3.16}$$

where p_k is considered to be a function of q_k and q_{k+1}, i.e., $p_k = p_k(q_k, q_{k+1})$. Taking the derivative of both sides with respect to q_k, we have

$$-DS_{\rm d}^{k}(q_k) = -p_k - \frac{\partial p_k}{\partial q_k} \cdot \left[q_k + D_1 H_{\rm d}^{-}(p_k, q_{k+1})\right].$$

However, the term in the brackets vanish because the left discrete Hamilton's equations (2.20) are assumed to be satisfied. Thus we have

$$p_k = DS_{\rm d}^{k}(q_k). \tag{3.17}$$

Table 1. Correspondence between ingredients in continuous and discrete theories; $\mathbb{N}_0$ is the set of non-negative integers and $\mathbb{R}_{\geq 0}$ is the set of non-negative real numbers.

Continuous	**Discrete**
$(q,t) \in Q \times \mathbb{R}_{\geq 0}$	$(q_k, k) \in Q \times \mathbb{N}_0$
$\dot{q} = \partial H/\partial p$, $\dot{p} = -\partial H/\partial q$	$q_{k+1} = D_2 H_{\rm d}^{+}(q_k, p_{k+1})$, $p_k = D_1 H_{\rm d}^{+}(q_k, p_{k+1})$
$S(q,t) := \displaystyle\int_0^t \left[p(s)\cdot\dot{q}(s) - H(q(s),p(s))\right] ds$	$S_{\rm d}^k(q_k) := \displaystyle\sum_{l=0}^{k-1} \left[p_{l+1}\cdot q_{l+1} - H_{\rm d}^{+}(q_l, p_{l+1})\right]$
$dS = \dfrac{\partial S}{\partial q}\,dq + \dfrac{\partial S}{\partial t}\,dt$	$S_{\rm d}^{k+1}(q_{k+1}) - S_{\rm d}^k(q_k)$
$p\,dq - H(q,p)\,dt$	$p_{k+1}\cdot q_{k+1} - H_{\rm d}^{+}(q_k, p_{k+1})$
$\dfrac{\partial S}{\partial t} + H\left(q, \dfrac{\partial S}{\partial q}\right) = 0$	$S_{\rm d}^{k+1}(q_{k+1}) - S_{\rm d}^k(q_k) - DS_{\rm d}^{k+1}(q_{k+1})\cdot q_{k+1} + H_{\rm d}^{+}\left(q_k, D_2 S_{\rm d}^{k+1}(q_{k+1})\right) = 0$

Substituting this into Eq. (3.16) gives the discrete Hamilton–Jacobi equation with the left discrete Hamiltonian:

$$S_{\rm d}^{k+1}(q_{k+1}) - S_{\rm d}^k(q_k) + DS_{\rm d}^k(q_k)\cdot q_k + H_{\rm d}^{-}\left(DS_{\rm d}^k(q_k), q_{k+1}\right) = 0. \tag{3.18}$$

We refer to Eqs. (3.3) and (3.18) as the *right and left discrete Hamilton–Jacobi equations*, respectively.

As mentioned above, Eqs. (3.9) and (3.15) are the same action sum Eq.(2.1) expressed in different ways. Therefore we may summarize the above argument as follows:

Proposition 3.5. *The action sums, Eq.* (3.9) *or equivalently Eq.* (3.15)*, satisfy both the right and left discrete Hamilton–Jacobi equations* (3.3) *and* (3.18)*.*

3.4. **Explicit Forms of the Discrete Hamilton–Jacobi Equations.** The expressions of the right and left discrete Hamilton–Jacobi equations in Eqs. (3.3) and (3.18) are implicit in the sense that they contain two spatial variables q_k and q_{k+1}. However Theorem 3.3 suggests that q_k and q_{k+1} may be considered to be related by the dynamics defined by either Eq. (2.17) or (2.20), or equivalently, the discrete Hamiltonian map $\tilde{F}_{L_{\rm d}} : (q_k, p_k) \mapsto (q_{k+1}, p_{k+1})$ defined in Eq. (2.12). More specifically, we may write q_{k+1} in terms of q_k. This results in explicit forms of the discrete Hamilton–Jacobi equations, and we shall *define* the discrete Hamilton–Jacobi equations by the resulting explicit forms. We will see later in Section 5 that the explicit form is compatible with the formulation of the well-known Bellman equation.

For the right discrete Hamilton–Jacobi equation (3.3), we first define the map $f_k^{+} : Q \to Q$ as follows: Replace p_{k+1} in Eq. (2.17a) by $DS_{\rm d}^{k+1}(q_{k+1})$ as suggested by Eq. (3.11):

$$q_{k+1} = D_2 H_{\rm d}^{+}\left(q_k, DS_{\rm d}^{k+1}(q_{k+1})\right). \tag{3.19}$$

Assuming this equation is solvable for q_{k+1}, we define $f_k^{+} : Q \to Q$ by the resulting q_{k+1}, i.e., f_k^{+} is implicitly defined by

$$f_k^{+}(q_k) = D_2 H_{\rm d}^{+}\left(q_k, DS_{\rm d}^{k+1}(f_k^{+}(q_k))\right). \tag{3.20}$$

We may now identify q_{k+1} with $f_k^+(q_k)$ in the implicit form of the right Hamilton–Jacobi equation (3.3):

$$S_{\rm d}^{k+1}(f_k^+(q)) - S_{\rm d}^k(q) - DS_{\rm d}^{k+1}(f_k^+(q)) \cdot f_k^+(q) + H_{\rm d}^+\left(q, DS_{\rm d}^{k+1}(f_k^+(q))\right) = 0, \tag{3.21}$$

where we suppressed the subscript k of q_k since it is now clear that q_k is an independent variable as opposed to a function of the discrete time k. We *define* Eq. (3.21) to be the *right discrete Hamilton–Jacobi equation.* Notice that these are differential-difference equations defined on $Q \times \mathbb{N}$, with the spatial variable q and the discrete time k.

For the left discrete Hamilton–Jacobi equation (3.18), we define the map $f_k^- : Q \to Q$ as follows:

$$f_k^-(q_k) := \pi_Q \circ \tilde{F}_{L_{\rm d}}\left(dS_{\rm d}^k(q_k)\right), \tag{3.22}$$

where $\pi_Q : T^*Q \to Q$ is the cotangent bundle projection; equivalently, f_k^- is defined so that the diagram below commutes.

$$\begin{array}{ccc} T^*Q & \xrightarrow{\tilde{F}_{L_{\rm d}}} & T^*Q \\ {\scriptstyle dS_{\rm d}^k}\uparrow & & \downarrow{\scriptstyle \pi_Q} \\ Q & \underset{f_k^-}{\dashrightarrow} & Q \end{array} \qquad \begin{array}{ccc} dS_{\rm d}^k(q_k) & \longmapsto & \tilde{F}_{L_{\rm d}}\left(dS_{\rm d}^k(q_k)\right) \\ \uparrow & & \downarrow \\ q_k & \dashmapsto & f_k^-(q_k) \end{array} \tag{3.23}$$

Notice also that, since the map $\tilde{F}_{L_{\rm d}} : (q_k, p_k) \mapsto (q_{k+1}, p_{k+1})$ is defined by Eq. (2.20), f_k^- is defined implicitly by

$$q_k = -D_1 H_{\rm d}^-\left(DS_{\rm d}^k(q_k), f_k^-(q_k)\right). \tag{3.24}$$

In other words, replace p_k in Eq. (2.20a) by $DS_{\rm d}^k(q_k)$ as suggested by Eq. (3.17), and define $f_k^-(q_k)$ as the q_{k+1} in the resulting equation.

We may now identify q_{k+1} with $f_k^-(q_k)$ in Eq. (3.18):

$$S_{\rm d}^{k+1}(f_k^-(q)) - S_{\rm d}^k(q) + DS_{\rm d}^k(q) \cdot q + H_{\rm d}^-\left(DS_{\rm d}^k(q), f_k^-(q)\right) = 0, \tag{3.25}$$

where we again suppressed the subscript k of q_k. We *define* Eqs. (3.21) and (3.25) to be the *right and left discrete Hamilton–Jacobi equations*, respectively. Notice that these are differential-difference equations defined on $Q \times \mathbb{N}$, with the spatial variable q and the discrete time k.

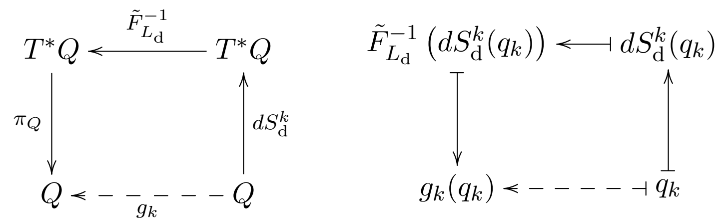
Remark 3.6. That the discrete Hamilton–Jacobi equation is a differential-difference equation defined on $Q \times \mathbb{N}$ corresponds to the fact that the continuous-time Hamilton–Jacobi equation (3.14) is a partial differential equation defined on $Q \times \mathbb{R}_+$.

Remark 3.7. Notice that the right discrete Hamilton–Jacobi equation (3.21) is more complicated than the left one (3.25), particularly because the map f_k^+ appears more often than f_k^- does in the latter; notice here that, as shown in Eq. (3.22), the maps $f_k^\pm$ in the discrete Hamilton–Jacobi equations (3.21) and (3.25) depend on the function $S_{\rm d}^k$, which is the unknown one has to solve for.

However, it is possible to define an equally simple variant of the right discrete Hamilton–Jacobi equation by writing q_{k-1} in terms of q_k: Let us first define $g_k : Q \to Q$ by

$$g_k(q_k) := \pi_Q \circ \tilde{F}_{L_{\rm d}}^{-1}\left(dS_{\rm d}^k(q_k)\right), \tag{3.26}$$

or so that the diagram below commutes.



(3.27)

Now, in Eq. (3.3), change the indices from $(k, k+1)$ to $(k-1, k)$ and identify q_{k-1} with $g_k(q_k)$ to obtain

$$S_{\rm d}^k(q) - S_{\rm d}^{k-1}(g_k(q)) - DS_{\rm d}^k(q) \cdot q + H_{\rm d}^+\left(g_k(q), DS_{\rm d}^k(q)\right) = 0, \tag{3.28}$$

where we again suppressed the subscript k of q_k. This is as simple as the left discrete Hamilton–Jacobi equation (3.25). However the map g_k is, being backward in time, rather unnatural compared to f_k. Furthermore, as we shall see in Section 5, in the discrete optimal control setting, the map f_k is defined by a given function and thus the formulation with f_k will turn out to be more convenient.

3.5. **The discrete Hamilton–Jacobi Equation on the Lagrangian Side.** First notice that Eq. (2.1) gives

$$S_{\rm d}^{k+1}(q_{k+1}) - S_{\rm d}^k(q_k) = L_{\rm d}(q_k, q_{k+1}). \tag{3.29}$$

This is essentially the Lagrangian equivalent of the discrete Hamilton–Jacobi equation (3.21) as Lall and West [16] suggest. Let us apply the same argument as above to obtain the explicit form for Eq. (3.29). Taking the derivative of the above equation with respect to q_k, we have

$$-D_1 L_{\rm d}(q_k, q_{k+1})\, dq_k = dS_{\rm d}^k(q_k),$$

and hence from the definition of the left discrete Legendre transform Eq. (2.7b),

$$\mathbb{F}L_{\rm d}^-(q_k, q_{k+1}) = dS_{\rm d}^k(q_k).$$

Assuming that $\mathbb{F}L_{\rm d}^-$ is invertible, we have

$$(q_k, q_{k+1}) = (\mathbb{F}L_{\rm d}^-)^{-1}\left(dS_{\rm d}^k(q_k)\right) =: (q_k, f_k^L(q_k)), \tag{3.30}$$

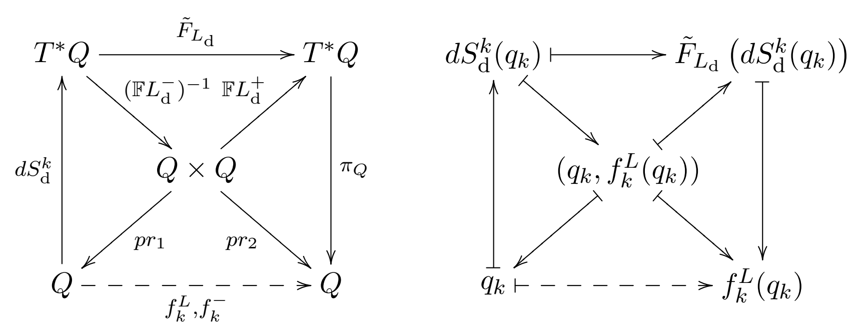
where we defined the map $f_k^L : Q \to Q$ as follows:

$$f_k^L(q_k) := pr_2 \circ (\mathbb{F}L_{\rm d}^-)^{-1}\left(dS_{\rm d}^k(q_k)\right), \tag{3.31}$$

where $pr_2 : Q \times Q \to Q$ is the projection to the second factor, i.e., $pr_2(q_1, q_2) = q_2$. Thus eliminating q_{k+1} from Eq. (3.29), and then replacing q_k by q, we obtain the discrete Hamilton–Jacobi equation on the Lagrangian side:

$$S_{\rm d}^{k+1}(f_k^L(q)) - S_{\rm d}^k(q) = L_{\rm d}\left(q, f_k^L(q)\right). \tag{3.32}$$

The map f_k^L defined in Eq. (3.31) is identical to f_k^- defined above in Eq. (3.22) as the commutative diagram below demonstrates:



(3.33)

The commutativity of the square in the diagram defines the f_k^- as we saw earlier, whereas that of the right-angled triangle on the lower left defines the f_k^L in Eq. (3.31); note the relation $\tilde{F}_{L_{\rm d}} = \mathbb{F}L_{\rm d}^+ \circ (\mathbb{F}L_{\rm d}^-)^{-1}$ from Eq. (2.13). Now f_k^L being identical to f_k^- implies that the discrete Hamilton–Jacobi equations on the Hamiltonian and Lagrangian sides, Eqs. (3.25) and (3.32), are equivalent.

4. Discrete Hamilton–Jacobi Theorem

The following gives a discrete analogue of the geometric Hamilton–Jacobi theorem (Theorem 5.2.4) by Abraham and Marsden [1]:

Theorem 4.1 (Discrete Hamilton–Jacobi). *Suppose that $S_{\rm d}^k$ satisfies the right discrete Hamilton–Jacobi equation* (3.21)*, and let $\{c_k\}_{k=0}^N \subset Q$ be a set of points such that*

$$c_{k+1} = f_k^+(c_k) \quad \textit{for} \quad k = 0, 1, \dots, N-1. \tag{4.1}$$

Then the set of points $\{(c_k, p_k)\}_{k=0}^N \subset T^*Q$ with

$$p_k := DS_{\rm d}^k(c_k) \tag{4.2}$$

is a solution of the right discrete Hamilton's equations (2.17)*.*

Similarly, suppose that $S_{\rm d}^k$ satisfies the left discrete Hamilton–Jacobi equation (3.25)*, and let $\{c_k\}_{k=0}^N \subset Q$ be a set of points that satisfy*

$$c_{k+1} = f_k^-(c_k) \quad \textit{for} \quad k = 0, 1, \dots, N-1. \tag{4.3}$$

Furthermore, assume that the Jacobian Df_k^- is invertible at each point c_k. Then the set of points $\{(c_k, p_k)\}_{k=0}^N \subset T^*Q$ with

$$p_k := DS_{\rm d}^k(c_k) \tag{4.4}$$

is a solution of the left discrete Hamilton's equations (2.20)*.*

Proof. To prove the first assertion, first recall the implicit definition of f_k^+ in Eq. (3.20):

$$f_k^+(q) = D_2 H_{\rm d}^+ \left(q, DS_{\rm d}^{k+1}(f_k^+(q)) \right). \tag{4.5}$$

In particular, for $q = c_k$, we have

$$c_{k+1} = D_2 H_{\rm d}^+ (c_k, p_k), \tag{4.6}$$

where we used Eq. (4.1) and (4.2). On the other hand, taking the derivative of Eq. (3.21) with respect to q,

$$\begin{aligned} DS_{\rm d}^{k+1}(f_k^+(q)) \cdot Df_k^+(q) - DS_{\rm d}^k(q) - Df_k^+(q) \cdot D^2 S_{\rm d}^{k+1}(f_k^+(q)) \cdot f_k^+(q) - DS_{\rm d}^{k+1}(f_k^+(q)) \cdot Df_k^+(q) \\ + D_1 H_{\rm d}^+ \left(q, DS_{\rm d}^{k+1}(f_k^+(q)) \right) + D_2 H_{\rm d}^+ \left(q, DS_{\rm d}^{k+1}(f_k^+(q)) \right) \cdot D^2 S_{\rm d}^{k+1}(f_k^+(q)) \cdot Df_k^+(q) = 0, \end{aligned}$$

which reduces to

$$-DS_{\rm d}^k(q) + D_1 H_{\rm d}^+ \left(q, DS_{\rm d}^{k+1}(f_k^+(q)) \right) = 0,$$

due to Eq. (4.5). Then substitution $q = c_k$ gives

$$-DS_{\rm d}^k(c_k) + D_1 H_{\rm d}^+ \left(c_k, DS_{\rm d}^{k+1}(f_k^+(c_k)) \right) = 0,$$

Using Eqs. (4.1) and (4.2), we obtain

$$p_k = D_1 H_{\rm d}^+ (c_k, p_{k+1}). \tag{4.7}$$

Eqs. (4.6) and (4.7) show that the sequence (c_k, p_k) satisfies the right discrete Hamilton's equations (2.17).

Now let us prove the latter assertion. First recall the implicit definition of f_k^- in Eq. (3.24):

$$q = -D_1 H_{\rm d}^- \left(DS_{\rm d}^k(q), f_k^-(q) \right) \tag{4.8}$$

In particular, for $q = c_k$, we have

$$c_k = -D_1 H_{\rm d}^- \left(p_k, c_{k+1}\right), \tag{4.9}$$

where we used Eq. (4.3) and (4.4). On the other hand, taking the derivative of Eq. (3.21) with respect to q,

$$\begin{aligned} DS_{\rm d}^{k+1}(f_k^-(q)) \cdot Df_k^-(q) - DS_{\rm d}^k(q) + D^2 S_{\rm d}^k(q) \cdot q + DS_{\rm d}^k(q) \\ + D_1 H_{\rm d}^- \left(DS_{\rm d}^k(q), f_k^-(q)\right) \cdot D^2 S_{\rm d}^k(q) + D_2 H_{\rm d}^- \left(DS_{\rm d}^k(q), f_k^-(q)\right) \cdot Df_k^-(q) = 0, \end{aligned}$$

which reduces to

$$\left[DS_{\rm d}^{k+1}(f_k^-(q)) + D_2 H_{\rm d}^- \left(DS_{\rm d}^k(q), f_k^-(q)\right)\right] \cdot Df_k^-(q) = 0.$$

due to Eq. (4.8). Then substitution $q = c_k$ gives

$$DS_{\rm d}^{k+1}(f_k^-(c_k)) = -D_2 H_{\rm d}^- \left(DS_{\rm d}^k(c_k), f_k^-(c_k)\right),$$

since $Df_k^-(c_k)$ is invertible by assumption. Then using Eqs. (4.3) and (4.4), we obtain

$$p_{k+1} = -D_2 H_{\rm d}^- \left(p_k, c_{k+1}\right). \tag{4.10}$$

Eqs. (4.9) and (4.10) show that the sequence (c_k, p_k) satisfies the left discrete Hamilton's equations (2.20). □

5. Relation to the Discrete-Time Hamilton–Jacobi–Bellman Equation

In this section we apply the above results to the optimal control setting. We will show that the (right) discrete Hamilton–Jacobi equation (3.21) gives the Bellman equation (discrete-time Hamilton–Jacobi–Bellman equation) as a special case.

5.1. Discrete Optimal Control Problem.

5.1. Discrete Optimal Control Problem. Let $\{q_k\}_{k=0}^N$ be the state variables in a vector space $V \cong \mathbb{R}^n$ with q_0 and q_N fixed and $u_{\rm d} := \{u_k\}_{k=0}^N$ be controls in the set $U \subset \mathbb{R}^m$. With a given function $C_{\rm d} : V \times U \to \mathbb{R}$, define the cost functional

$$J_{\rm d} := \sum_{k=0}^{N-1} C_{\rm d}(q_k, u_k). \tag{5.1}$$

Then a typical *discrete optimal control problem* is formulated as follows [see, e.g., 4; 5; 11; 14]:

Problem 5.1. Minimize the cost functional, i.e.,

$$\min_{u_{\rm d}} J_{\rm d} = \min_{u_{\rm d}} \sum_{k=0}^{N-1} C_{\rm d}(q_k, u_k) \tag{5.2}$$

subject to the constraint

$$q_{k+1} = f(q_k, u_k). \tag{5.3}$$

5.2. **Necessary Condition for Optimality and the Discrete-Time HJB Equation.** We would like to formulate the necessary condition for optimality. First introduce the augmented cost functional:

$$\begin{aligned}\hat{J}_{\rm d}^k(q_{\rm d},\,p_{\rm d},u_{\rm d}) &:= \sum_{l=0}^{k-1}\{C_{\rm d}(q_l,u_l) - p_{l+1}\cdot[q_{l+1} - f(q_l,u_l)]\}\\ &= -\sum_{l=0}^{k-1}\left[p_{l+1}\cdot q_{l+1} - \hat{H}_{\rm d}^+(q_l,p_{l+1},u_l)\right]\\ &= -\hat{S}_{\rm d}^k(q_{\rm d},\,p_{\rm d},u_{\rm d}),\end{aligned}$$

where we defined the Hamiltonian

$$\hat{H}_{\rm d}^+(q_l,p_{l+1},u_l) := p_{l+1}\cdot f(q_l,u_l) - C_{\rm d}(q_l,u_l), \tag{5.4}$$

and the action sum

$$\hat{S}_{\rm d}^k(q_{\rm d},p_{\rm d},u_{\rm d}) := \sum_{l=0}^{k-1}\left[p_{l+1}\cdot q_{l+1} - \hat{H}_{\rm d}^+(q_l,p_{l+1},u_l)\right], \tag{5.5}$$

with the shorthand notation $q_{\rm d} := \{q_l\}_{l=0}^k$, $p_{\rm d} := \{p_l\}_{l=1}^k$, and $u_{\rm d} := \{u_l\}_{l=0}^{k-1}$. Then the optimality condition Eq. (5.2) is restated as

$$\min_{q_{\rm d},\,p_{\rm d},u_{\rm d}} \hat{J}_{\rm d}^k(q_{\rm d},\,p_{\rm d},u_{\rm d}) = \min_{q_{\rm d},\,p_{\rm d},u_{\rm d}}\left\{-\sum_{l=0}^{k-1}\left[p_{l+1}\cdot q_{l+1} - \hat{H}_{\rm d}^+(q_l,p_{l+1},u_l)\right]\right\}, \tag{5.6}$$

which is equivalent to

$$\max_{q_{\rm d},\,p_{\rm d},\,u_{\rm d}} \hat{S}_{\rm d}^k(q_{\rm d},\,p_{\rm d},u_{\rm d}) = \max_{q_{\rm d},\,p_{\rm d},\,u_{\rm d}}\sum_{l=0}^{k-1}\left[p_{l+1}\cdot q_{l+1} - \hat{H}_{\rm d}^+(q_l,p_{l+1},u_l)\right]. \tag{5.7}$$

In particular, extremality with respect to the control $u_{\rm d}$ implies

$$D_3\hat{H}_{\rm d}^+(q_l,p_{l+1},u_l) = 0, \quad l = 0,1,\ldots,k-1. \tag{5.8}$$

Now we assume that $\hat{H}_{\rm d}^+$ is sufficiently regular so that the optimal control $u_{\rm d}^* := \{u_l^*\}_{l=0}^{k-1}$ is determined by

$$D_3\hat{H}_{\rm d}^+(q_l,p_{l+1},u_l^*) = 0, \quad l = 0,1,\ldots,k-1. \tag{5.9}$$

Therefore u_l^* is a function of q_l and p_{l+1}, i.e., $u_l^* = u_l^*(q_l,p_{l+1})$.

Then we can eliminate $u_{\rm d}$ in the maximization problem Eq. (5.7):

$$\max_{q_{\rm d},\,p_{\rm d}} S_{\rm d}(q_{\rm d},p_{\rm d}) = \max_{q_{\rm d},\,p_{\rm d}}\sum_{l=0}^{k-1}\left[p_{l+1}\cdot q_{l+1} - H_{\rm d}^+(q_l,p_{l+1})\right], \tag{5.10}$$

where we defined

$$H_{\rm d}^+(q_l,p_{l+1}) := \hat{H}_{\rm d}^+(q_l,p_{l+1},u_l^*) = p_{l+1}\cdot f(q_l,u_l^*) - C_{\rm d}(q_l,u_l^*), \tag{5.11}$$

and

$$S_{\rm d}^k(q_{\rm d},p_{\rm d}) := \hat{S}_{\rm d}^k(q_{\rm d},p_{\rm d},u_{\rm d}^*) = \sum_{l=0}^{k-1}\left[p_{l+1}\cdot q_{l+1} - H_{\rm d}^+(q_l,p_{l+1})\right]. \tag{5.12}$$

So now the problem is reduced to the maximization of the action sum Eq. (5.12) that has exactly the same form as the one in Eq. (3.9) formulated in the framework of discrete Hamiltonian mechanics.

The corresponding right discrete Hamilton's equations are, using the expression for the Hamiltonian in Eq. (5.11),

$$\begin{aligned} q_{k+1} &= f(q_k, u_k^*), \\ p_k &= p_{k+1} \cdot D_1 f(q_k, u_k^*) - D_1 C_{\rm d}(q_k, u_k^*). \end{aligned} \tag{5.13}$$

Therefore Eq. (3.20) gives the implicit definition of f_k^+ as follows:

$$f_k^+(q_k) = f\left(q_k, u_k^*\left(q_k, DS_{\rm d}^{k+1}(f_k^+(q_k))\right)\right). \tag{5.14}$$

Hence the (right) discrete Hamilton–Jacobi equation (3.21) applied to this case gives

$$S_{\rm d}^{k+1}(f(q_k, u_k^*)) - S_{\rm d}^k(q_k) - DS_{\rm d}^{k+1}(f(q_k, u_k^*)) \cdot f(q_k, u_k^*) + H_{\rm d}^+\left(q_k, DS_{\rm d}^{k+1}(f(q_k, u_k^*))\right) = 0, \tag{5.15}$$

and again using the expression for the Hamiltonian in Eq. (5.11), we obtain

$$S_{\rm d}^{k+1}(f(q_k, u_k^*)) - S_{\rm d}^k(q_k) - C_{\rm d}(q_k, u_k^*) = 0, \tag{5.16}$$

or equivalently

$$\max_{u_k}\left[S_{\rm d}^{k+1}(f(q_k, u_k)) - C_{\rm d}(q_k, u_k)\right] - S_{\rm d}^k(q_k) = 0, \tag{5.17}$$

which is the *discrete-time Hamilton–Jacobi–Bellman (HJB) equation* or, in short, the *Bellman equation* [see, e.g., 4].

Remark 5.2. Notice that the discrete HJB equation (5.17) is much simpler than the discrete Hamilton–Jacobi equations (3.21) and (3.25) because of the special form of the control Hamiltonian Eq. (5.11). Also notice that, as shown in Eq. (5.14), the term $f_k^+(q_k)$ is written in terms of the given function f. See Remark 3.7 for comparison.

5.3. **Relation between the Discrete HJ and HJB Equations and its Consequences.** Summarizing the observation made above, we have

Proposition 5.3. *The right discrete Hamilton–Jacobi equation* (3.21) *applied to the Hamiltonian formulation of the discrete optimal control problem 5.1 gives the discrete-time Hamilton–Jacobi–Bellman equation* (5.17)*.*

This observation leads to the following well-known facts:

Proposition 5.4. *The optimal cost function satisfies the discrete-time Hamilton–Jacobi–Bellman equation* (5.17)*.*

Proof. This follows from a reinterpretation of Theorem 3.3 through Proposition 5.3. □

Proposition 5.5. *Let $S_{\rm d}^k(q_k)$ be a solution to the discrete Hamilton–Jacobi–Bellman equation* (5.17)*. Then the costate p_k in the discrete maximum principle is given as follows:*

$$p_k = DS_{\rm d}^k(c_k), \tag{5.18}$$

where $c_{k+1} = f(c_k, u_k^*)$ with the optimal control u_k^*.

Proof. This follows from a reinterpretation of Theorem 4.1 through Proposition 5.3. □

6. Application To Discrete Linear Hamiltonian Systems

6.1. **Discrete Linear Hamiltonian Systems and Matrix Riccati Equation.**

Example 6.1 (Quadratic discrete Hamiltonian—discrete linear Hamiltonian systems)**.** Consider a discrete Hamiltonian system on $T^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$ (the configuration space is $Q = \mathbb{R}^n$) defined by the quadratic left discrete Hamiltonian

$$H_{\rm d}^-(p_k, q_{k+1}) = \frac{1}{2}p_k^T M^{-1} p_k + p_k^T L q_{k+1} + \frac{1}{2}q_{k+1}^T K q_{k+1}, \tag{6.1}$$

where M, K, and L are real $n \times n$ matrices; we assume that M and L are invertible and also that M and K are symmetric. The left discrete Hamilton's equations (2.20) are

$$\begin{aligned} q_k &= -(M^{-1}p_k + Lq_{k+1}), \\ p_{k+1} &= -(L^T p_k + K q_{k+1}), \end{aligned} \tag{6.2}$$

or

$$\begin{pmatrix} q_{k+1} \\ p_{k+1} \end{pmatrix} = \begin{pmatrix} -L^{-1} & -L^{-1}M^{-1} \\ KL^{-1} & KL^{-1}M^{-1} - L^T \end{pmatrix} \begin{pmatrix} q_k \\ p_k \end{pmatrix}. \tag{6.3}$$

and hence are a discrete linear Hamiltonian system (see Section A.1).

Now let us solve the left discrete Hamilton–Jacobi equation (3.25) for this system. For that purpose, we first generalize the problem to that with a set of initial points instead of a single initial point (q_0, p_0). More specifically, consider the set of initial points that is a Lagrangian affine space $\tilde{\mathcal{L}}(z_0)$ (see Definition A.2) which contains the point $z_0 := (q_0, p_0)$. Then the dynamics is formally written as, for any discrete time $k \in \mathbb{N}$,

$$\tilde{\mathcal{L}}_k := (\tilde{F}_{L_{\mathrm{d}}})^k \left(\tilde{\mathcal{L}}(z_0)\right) = \underbrace{\tilde{F}_{L_{\mathrm{d}}} \circ \cdots \circ \tilde{F}_{L_{\mathrm{d}}}}_{k} \left(\tilde{\mathcal{L}}(z_0)\right),$$

where $\tilde{F}_{L_{\mathrm{d}}} : T^*Q \to T^*Q$ is the discrete Hamiltonian map defined in Eq. (2.12). Since $\tilde{F}_{L_{\mathrm{d}}}$ is a symplectic map, Proposition A.4 implies that $\tilde{\mathcal{L}}_k$ is a Lagrangian affine space. Then, assuming that $\tilde{\mathcal{L}}_k$ is transversal to $\{0\} \oplus Q^*$, Corollary A.6 implies that there exists a set of functions S_{d}^k of the form

$$S_{\mathrm{d}}^k(q) = \frac{1}{2} q^T A_k q + b_k^T q + c_k \tag{6.4}$$

such that $\tilde{\mathcal{L}}_k = \operatorname{graph} dS_{\mathrm{d}}^k$; here A_k are symmetric $n \times n$ matrices, b_k are elements in $\mathbb{R}^n$, and c_k are in $\mathbb{R}$.

Now that we know the form of the solution, we substitute the above expression into the discrete Hamilton–Jacobi equation to find the equations for A_k, b_k, and c_k. Notice first that the map f_k^- is given by the first half of Eq. (6.3) with p_k replaced by $DS_{\mathrm{d}}^k(q)$:

$$\begin{aligned} f_k^-(q) &= -L^{-1}\left(q + M^{-1} DS_{\mathrm{d}}^k(q)\right) \\ &= -L^{-1}(I + M^{-1}A_k)q - L^{-1}M^{-1}b_k. \end{aligned} \tag{6.5}$$

Then substituting Eq. (6.4) into the left-hand side of the left discrete Hamilton–Jacobi equation (3.25) yields the following recurrence relations for A_k, b_k, and c_k:

$$A_{k+1} = L^T(I + A_k M^{-1})^{-1} A_k L - K, \tag{6.6a}$$

$$b_{k+1} = -L^T(I + A_k M^{-1})^{-1} b_k, \tag{6.6b}$$

$$c_{k+1} = c_k - \frac{1}{2} b_k^T (M + A_k)^{-1} b_k, \tag{6.6c}$$

where we assumed that $I + A_k M^{-1}$ is invertible.

Remark 6.2. We can rewrite Eq. (6.6a) as follows:

$$A_{k+1} = \left[KL^{-1} + (KL^{-1}M^{-1} - L^T)A_k\right](-L^{-1} - L^{-1}M^{-1}A_k)^{-1}. \tag{6.7}$$

Notice the exact correspondence between the coefficients in the above equation and the matrix entries in the discrete linear Hamiltonian equations (6.3). In fact, this is the discrete Riccati equation that corresponds to the iteration defined by Eq. (6.3). See Ammar and Martin [2] for details on this correspondence.

To summarize the above observation, we have

Proposition 6.3. *The discrete Hamilton–Jacobi equation* (3.25) *applied to the discrete linear Hamiltonian system* (6.3) *yields the discrete Riccati equation* (6.7).

In other words, the discrete Hamilton–Jacobi equation is a nonlinear generalization of the discrete Riccati equation.

A simple physical example that is described as a discrete linear Hamiltonian system is the following:

Example 6.4 (Harmonic oscillator)**.** Consider the one-dimensional harmonic oscillator with mass M and spring constant K. The configuration space is a real line, i.e., $Q = \mathbb{R}$, and the Lagrangian $L : T\mathbb{R} \to \mathbb{R}$ of the system is

$$L(q, \dot{q}) = \frac{M}{2}\dot{q}^2 + \frac{K}{2}q^2.$$

Introducing the angular frequency $\omega := \sqrt{K/M}$, we have

$$L(q, \dot{q}) = \frac{M}{2}\left(\dot{q}^2 + \omega^2 q^2\right).$$

It is easy to solve the (continuous) Euler–Lagrange equation and calculate Jacobi's solution explicitly:

$$S(q, t; q_0) := \int_0^t L(q(s), \dot{q}(s))\, ds = \frac{1}{2}M\omega\left[(q_0^2 + q^2)\cot(\omega t) - 2q_0 q\csc(\omega t)\right], \tag{6.8}$$

where q_0 is the initial position: $q(0) = q_0$. This gives the exact discrete Lagrangian [22] with step size h as follows:

$$L_{\mathrm{d}}^{\mathrm{ex}}(q_k, q_{k+1}) = S(q_{k+1}, h; q_k) = \frac{1}{2}M\omega\left[(q_k^2 + q_{k+1}^2)\cot(\omega h) - 2q_k q_{k+1}\csc(\omega h)\right] \tag{6.9}$$

The corresponding left discrete Hamiltonian (See Eq. (2.19)), which we shall call the *exact left discrete Hamiltonian*, is then

$$H_{\mathrm{d,ex}}^{-}(p_k, q_{k+1}) = \frac{1}{2}\left[\frac{p_k^2}{M\omega}\tan(\omega h) - 2p_k q_{k+1}\sec(\omega h) + M\omega\, q_{k+1}^2\tan(\omega h)\right]. \tag{6.10}$$

Comparing this with the general form of the quadratic Hamiltonian Eq. (6.1), we see that this is a special case with $n = 1$ and

$$M^{-1} = \frac{\tan(\omega h)}{M\omega}, \qquad L = -\sec(\omega h), \qquad K = M\omega\tan(\omega h).$$

Note that M, L, and K are also scalars now. Thus Eq. (6.5) gives

$$f_k^{-}(q) := \pi_{\mathbb{R}} \circ \tilde{F}_{L_{\mathrm{d}}}\left(dS_{\mathrm{d}}^k(q)\right) = \left(\cos(\omega h) + \frac{\sin(\omega h)}{M\omega}A_k\right)q + \frac{\sin(\omega h)}{M\omega}b_k. \tag{6.11}$$

Now the recurrence relations Eq. (6.6) reduce to

$$\begin{aligned} A_{k+1} &= \frac{M\omega\left[A_k\cos(\omega h) - M\omega\sin(\omega h)\right]}{M\omega\cos(\omega h) + A_k\sin(\omega h)}, \\ b_{k+1} &= \frac{M\omega}{M\omega\cos(\omega h) + A_k\sin(\omega h)}\, b_k, \\ c_{k+1} &= c_k - \frac{b_k^2}{A_k + M\omega\cot(\omega h)}. \end{aligned} \tag{6.12}$$

We impose the "initial condition" $S_{\mathrm{d}}^1(q_1) = L_{\mathrm{d}}^{\mathrm{ex}}(q_0, q_1)$, which follows from Eq. (3.9) or (3.15) for $k = 1$. This gives

$$A_1 = M\omega\cot(\omega h), \qquad b_1 = -M\omega q_0\csc(\omega h), \qquad c_1 = M\omega q_0^2\cot(\omega h). \tag{6.13}$$

Solving the above recurrence relations using *Mathematica*, we obtain

$$A_k = M\omega \cot(\omega kh), \qquad b_k = -M\omega q_0 \csc(\omega kh), \qquad c_k = M\omega q_0^2 \cot(\omega kh), \tag{6.14}$$

and hence the solution of the left discrete Hamilton–Jacobi equation

$$S_{\rm d}^k(q) = \frac{1}{2} M\omega \left[(q_0^2 + q^2) \cot(\omega kh) - 2q_0 q \csc(\omega kh) \right]. \tag{6.15}$$

Remark 6.5. Notice that, in the above example, we have $S_{\rm d}^k(q) = S(q, kh; q_0)$ from the explicit expression for Jacobi's solution Eq. (6.8) under the assumption that $q = q_k$. This is because we started with the exact discrete Lagrangian and hence the corresponding discrete dynamics is exact. Specifically, the exact discrete Lagrangian satisfies, by definition,

$$L_{\rm d}^{\rm ex}(q_l, q_{l+1}) = \int_{lh}^{(l+1)h} L(q(t), \dot{q}(t))\, dt, \qquad l \in \{0, 1, \dots, k-1\} \tag{6.16}$$

where $q(t)$ satisfies the continuous dynamics and the boundary conditions $q(lh) = q_l$ and $q((l+1)h) = q_{l+1}$. Hence

$$S_{\rm d}^k(q) := \sum_{l=0}^{k-1} L_{\rm d}^{\rm ex}(q_l, q_{l+1}) = \int_0^{kh} L(q(t), \dot{q}(t))\, dt =: S(q, kh; q_0), \tag{6.17}$$

which says that the discrete analogue of Jacobi's solution Eq. (3.9) is identical to Jacobi's solution Eq. (6.8) calculated using the continuous dynamics.

6.2. **Application of the Hamilton–Jacobi Theorem.** We illustrate how Theorem 4.1 works using the same example. Here we would like to see if we can "generate" the dynamics using the solution of the discrete Hamilton–Jacobi equations as in Theorem 4.1.

Example 6.6 (Harmonic oscillator)**.** Let us start from the solution obtained in Example 6.4:

$$S_{\rm d}^k(q) = \frac{1}{2} M\omega \left[(q_0^2 + q^2) \cot(\omega kh) - 2q_0 q \csc(\omega kh) \right]. \tag{6.18}$$

Notice that the expression for the right-hand side of Eq. (4.3) was already given in Eq. (6.11):

$$\pi_Q \circ \tilde{F}_{L_{\rm d}} \left(dS_{\rm d}^k(q_k) \right) = q_k \cos(\omega h) + \frac{1}{M\omega} DS_{\rm d}^k(q_k) \sin(\omega h)$$

Hence substituting Eq. (6.18) into Eq. (4.3) yields

$$q_{k+1} = \csc(\omega kh) \left\{ q_k \sin\left[\omega(k+1)h\right] - q_0 \sin(\omega h) \right\} \tag{6.19}$$

Then Eq. (4.4) gives

$$p_k = DS_{\rm d}^k(q_k) = M\omega \csc(\omega kh) \left[q_k \cos(\omega kh) - q_0 \right]. \tag{6.20}$$

It is easy to check these equations satisfy the left discrete Hamilton's equations (2.17) as Theorem 4.1 claims.

7. Continuous Limit

This section shows that the right and left discrete Hamilton–Jacobi equations (3.21) and (3.25) recover the original Hamilton–Jacobi equation (3.14) in the continuous-time limit. We reproduce the result of Elnatanov and Schiff [8] on the continuous limit of the right discrete Hamilton–Jacobi equation, applying the same argument simultaneously to the left discrete Hamilton–Jacobi equation. The main purpose of doing so here is to make it clear how the discrete ingredients are related to the corresponding continuous ones in our notation.

7.1. **Continuous Limit of Discrete Hamilton Equations.** Let us first look at the continuous-time limit of the right and left discrete Hamilton's equations (2.17) and (2.20). This makes it clear how the discrete and continuous Hamiltonians are related in the limit. First recall from Section 2.3 of Marsden and West [22] that the discrete Lagrangian $L_{\mathrm{d}}(q_k, q_{k+1})$ is consistent if it satisfies

$$\begin{aligned} L_{\mathrm{d}}(q_k, q_{k+1}) &= \int_{t_k}^{t_{k+1}} L(q(t), \dot{q}(t))\, dt + O(h^2) \\ &= \int_{t_k}^{t_{k+1}} [p(t)\cdot\dot{q}(t) - H(q(t), p(t))]\, dt + O(h^2). \end{aligned} \tag{7.1}$$

where $t_k = kh$, and the $(q(t), p(t))$ in the integrand is the flow defined by the continuous Lagrangian or Hamiltonian with $q(t_k) = q_k$ and $q(t_{k+1}) = q_k$. Consistency of a discrete Lagrangian implies that of the corresponding discrete flow, hence the terminology.

Lemma 7.1. *The right and left discrete Hamiltonians $H_{\mathrm{d}}^{+}(q_k, p_{k+1})$ and $H_{\mathrm{d}}^{-}(p_k, q_{k+1})$ defined as in Eq.* (2.16) *and* (2.19) *with a consistent discrete Lagrangian satisfies the following relations with the continuous Hamiltonian:*

$$H(q_k, p_k) = \lim_{h\to 0}\frac{1}{h}\left[H_{\mathrm{d}}^{+}(q_k, p_{k+1}) - p_{k+1}\cdot q_k\right] = \lim_{h\to 0}\frac{1}{h}\left[H_{\mathrm{d}}^{-}(p_k, q_{k+1}) + p_k\cdot q_{k+1}\right]. \tag{7.2}$$

Proof. Simple calculations with Eqs. (2.16) and (2.19) with Eq. (7.1) show

$$\frac{1}{h}\left[H_{\mathrm{d}}^{+}(q_k, p_{k+1}) - p_{k+1}\cdot q_k\right] = p_{k+1}\cdot\frac{q_{k+1}-q_k}{h} - \frac{1}{h}\int_{t_k}^{t_k+h}[p(t)\cdot\dot{q}(t) - H(q(t), p(t))]\, dt + O(h)$$

and

$$\frac{1}{h}\left[H_{\mathrm{d}}^{-}(p_k, q_{k+1}) + p_k\cdot q_{k+1}\right] = p_k\cdot\frac{q_{k+1}-q_k}{h} - \frac{1}{h}\int_{t_k}^{t_k+h}[p(t)\cdot\dot{q}(t) - H(q(t), p(t))]\, dt + O(h)$$

Taking the limit as $h \to 0$ on both sides in each of the above equations gives the result. □

Definition 7.2. We shall say that a right/left discrete Hamiltonian $H_{\mathrm{d}}^{\pm}$ is *consistent* if it satisfies Eq. (7.2).

Proposition 7.3. *With consistent discrete Hamiltonians, the right and left discrete Hamilton's equations* (2.17) *and* (2.20) *recover the continuous-time Hamilton's equations in the continuous limit.*

Proof. Simple calculations with Eqs. (2.17) and (2.20) show

$$\begin{aligned} \frac{q_{k+1}-q_k}{h} &= \frac{\partial}{\partial p_k}\left\{\frac{1}{h}\left[H_{\mathrm{d}}^{+}(q_k, p_{k+1}) - p_{k+1}\cdot q_k\right]\right\}, \\ \frac{p_{k+1}-p_k}{h} &= -\frac{\partial}{\partial q_{k+1}}\left\{\frac{1}{h}\left[H_{\mathrm{d}}^{+}(q_k, p_{k+1}) - p_{k+1}\cdot q_k\right]\right\} \end{aligned}$$

and

$$\begin{aligned} \frac{q_{k+1}-q_k}{h} &= \frac{\partial}{\partial p_{k+1}}\left\{\frac{1}{h}\left[H_{\mathrm{d}}^{-}(p_k, q_{k+1}) + p_k\cdot q_{k+1}\right]\right\}, \\ \frac{p_{k+1}-p_k}{h} &= -\frac{\partial}{\partial q_k}\left\{\frac{1}{h}\left[H_{\mathrm{d}}^{-}(p_k, q_{k+1}) + p_k\cdot q_{k+1}\right]\right\}. \end{aligned}$$

Taking the limit as $h \to 0$ on both sides in each of the above equations gives, with Eq. 7.2,

$$\dot{q}(t_k) = \frac{\partial H}{\partial p}(q(t_k), p(t_k)), \qquad \dot{p}(t_k) = -\frac{\partial H}{\partial q}(q(t_k), p(t_k)).$$

□

7.2. **Continuous Limit of Discrete Hamilton–Jacobi Equations.** Now we are ready to discuss the continuous limit of the right and left discrete Hamilton–Jacobi equations.

Proposition 7.4. *With consistent discrete Hamiltonians, the right and left discrete Hamilton–Jacobi equations* (3.3) *and* (3.18) *recover the continuous-time Hamilton–Jacobi equation.*

Proof. First define $S : Q \times \mathbb{R} \to \mathbb{R}$ that satisfies $S(q_k, t_k) = S_{\rm d}^k(q_k)$. Simple calculations with (3.3) and (3.18) yield

$$\frac{1}{h}\left[S(q_{k+1}, t_{k+1}) - S(q_k, t_k) - \frac{\partial S}{\partial q}(q_{k+1}, t_{k+1}) \cdot (q_{k+1} - q_k)\right] + \frac{1}{h}\left[H_{\rm d}^{+}\left(q_k, \frac{\partial S}{\partial q}(q_{k+1}, t_{k+1})\right) - \frac{\partial S}{\partial q}(q_{k+1}, t_{k+1}) \cdot q_k\right] = 0 \quad (7.3)$$

and

$$\frac{1}{h}\left[S(q_{k+1}, t_{k+1}) - S(q_k, t_k) - \frac{\partial S}{\partial q}(q_{k+1}, t_{k+1}) \cdot (q_{k+1} - q_k)\right] + \frac{1}{h}\left[H_{\rm d}^{-}\left(\frac{\partial S}{\partial q}(q_k, t_k), q_{k+1}\right) + \frac{\partial S}{\partial q}(q_k, t_k) \cdot q_{k+1}\right] = 0. \quad (7.4)$$

The first group of the terms in brackets is common to both of the above equations. Taylor expansion of the terms gives

$$\frac{1}{h}\left[S(q_{k+1}, t_{k+1}) - S(q_k, t_k) - \frac{\partial S}{\partial q}(q_{k+1}, t_{k+1}) \cdot (q_{k+1} - q_k)\right] = \frac{\partial S}{\partial t}(q_k, t_k) + \left[\frac{\partial S}{\partial q}(q_k, t_k) - \frac{\partial S}{\partial q}(q_{k+1}, t_{k+1})\right] \cdot \frac{q_{k+1} - q_k}{h} + O(h) \to \frac{\partial S}{\partial t}(q_k, t_k)$$

as $h \to 0$. On the other hand, by Lemma 7.1, the limit as $h \to 0$ of the second group of the terms in each of Eqs. (7.3) and (7.4) is

$$\lim_{h \to 0} \frac{1}{h}\left[H_{\rm d}^{+}\left(q_k, \frac{\partial S}{\partial q}(q_{k+1}, t_{k+1})\right) - \frac{\partial S}{\partial q}(q_{k+1}, t_{k+1}) \cdot q_k\right] = H\left(q_k, \frac{\partial S}{\partial q}(q_k, t_k)\right),$$

and

$$\lim_{h \to 0} \frac{1}{h}\left[H_{\rm d}^{-}\left(\frac{\partial S}{\partial q}(q_k, t_k), q_{k+1}\right) + \frac{\partial S}{\partial q}(q_k, t_k) \cdot q_{k+1}\right] = H\left(q_k, \frac{\partial S}{\partial q}(q_k, t_k)\right).$$

As a result, both the right and left discrete Hamilton–Jacobi equations give, in the limit as $h \to 0$,

$$\frac{\partial S}{\partial t}(q_k, t_k) + H\left(q_k, \frac{\partial S}{\partial q}(q_k, t_k)\right) = 0,$$

which is the continuous-time Hamilton–Jacobi equation. □

8. Conclusion and Future Work

We developed a discrete-time analogue of the Hamilton–Jacobi theory starting from the discrete variational Hamilton equations formulated by Lall and West [16]. We reinterpreted and extended the discrete Hamilton–Jacobi equation given by Elnatanov and Schiff [8] to show that it possesses theoretical significance in discrete mechanics that is equivalent to that of the (continuous-time) Hamilton–Jacobi equation in Hamiltonian mechanics. Furthermore, we showed that the discrete Hamilton–Jacobi equation reduces to the discrete Riccati equation with a quadratic Hamiltonian, and also that it specializes to the Bellman equation of dynamic programming if applied to discrete optimal control problems. This again gives discrete analogues of the corresponding known results in the continuous-time theory. Application to discrete optimal control also revealed that Theorems 3.3 and 4.1 specialize to two well-known results in discrete optimal control theory.

We are interested in the following topics for future work:

- *Application to integrable discrete systems.* Theorem 4.1 gives a discrete analogue of the theory behind the technique of solution by separation of variables in the sense that the theorem relates a solution of the discrete Hamilton–Jacobi equations with that of the discrete Hamilton's equations. An interesting question then is whether or not separation of variables applies to integrable discrete systems, e.g., discrete rigid bodies of Moser and Veselov [23] and various others discussed by Suris [25, 26].
- *Development of numerical methods based on the discrete Hamilton–Jacobi equation.* Hamilton–Jacobi-based numerical methods made seminal contributions to the development of structured integrators for Hamiltonian systems [see, e,g., 6, and also references therein]. The present theory, being intrinsically discrete in time, potentially provides a variant of such numerical methods.
- *Extension to discrete nonholonomic and Dirac mechanics.* The present work is concerned only with unconstrained systems. Extensions to nonholonomic and Dirac mechanics, more specifically discrete-time versions of the nonholonomic Hamilton–Jacobi theory [7; 12; 24] and Dirac Hamilton–Jacobi theory [19], are another direction of future research.
- *Relation to the power method and iterations on the Grassmannian manifold.* Ammar and Martin [2] established links between the power method, iterations on the Grassmannian manifold, and the Riccati equation. The discussion on iterations of Lagrangian subspaces and its relation to the Riccati equation in Sections 6.1 and A.2 is a special case of such links. On the other hand, Proposition 6.3 suggests that the discrete Hamilton–Jacobi equation is a generalization of the Riccati equation. We are interested in seeing possible further links implied by the generalization.

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Appendix A. Discrete Linear Hamiltonian Systems

A.1. **Discrete Linear Hamiltonian Systems.** Suppose that the configuration space Q is an n-dimensional vector space, and that the discrete Hamiltonian $H_{\rm d}^{+}$ or $H_{\rm d}^{-}$ is quadratic as in Eq. (6.1). Also assume that the corresponding discrete Hamiltonian map $\tilde{F}_{L_{\rm d}} : (q_k, p_k) \mapsto (q_{k+1}, p_{k+1})$ is invertible. Then the discrete Hamilton's equations (2.17) or (2.20) reduce to the discrete linear Hamiltonian system

$$z_{k+1} = A_{L_{\rm d}} z_k, \tag{A.1}$$

where $z_k \in \mathbb{R}^{2n}$ is a coordinate expression for $(q_k, p_k) \in Q \times Q^*$ and $A_{L_{\rm d}} : Q \times Q^* \to Q \times Q^*$ is the matrix representation of the map $\tilde{F}_{L_{\rm d}}$ under the same basis. Since $\tilde{F}_{L_{\rm d}}$ is symplectic, $A_{L_{\rm d}}$ is an $2n \times 2n$ symplectic matrix, i.e.,

$$A_{L_{\rm d}}^T \mathbb{J} A_{L_{\rm d}} = \mathbb{J}, \tag{A.2}$$

where the matrix $\mathbb{J}$ is defined by

$$\mathbb{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

with I the $n \times n$ identity matrix.

A.2. **Lagrangian Subspaces and Lagrangian Affine Spaces.** First recall the definition of a Lagrangian subspace:

Definition A.1. Let V be a symplectic vector space with the symplectic form Ω. A subspace $\mathcal{L}$ of V is said to be *Lagrangian* if $\Omega(v, w) = 0$ for any $v, w \in \mathcal{L}$ and $\dim \mathcal{L} = \dim V/2$.

We introduce the following definition for later convenience:

Definition A.2. A subset $\tilde{\mathcal{L}}(b)$ of a symplectic vector space V is called a *Lagrangian affine space* if $\tilde{\mathcal{L}}(b) = b + \mathcal{L}$ for some element $b \in V$ and a Lagrangian subspace $\mathcal{L} \subset V$.

The following fact is well-known [see, e.g., 15, Theorem 6 on p. 417]:

Proposition A.3. *Let $\mathcal{L}$ be a Lagrangian subspace of V and $A : V \to V$ be a symplectic transformation. Then $A^k(\mathcal{L})$ is also a Lagrangian subspace of V for any $k \in \mathbb{N}$.*

A similar result holds for Lagrangian affine spaces:

Proposition A.4. *Let $\tilde{\mathcal{L}}(b) = b + \mathcal{L}$ be a Lagrangian affine space of V and $A : V \to V$ be a symplectic transformation. Then $A^k\left(\tilde{\mathcal{L}}(b)\right)$ is also a Lagrangian affine space of V for any $k \in \mathbb{N}$. More explicitly, we have*

$$A^k\left(\tilde{\mathcal{L}}(b)\right) = A^k b + A^k(\mathcal{L}).$$

Proof. Follows from a straightforward calculation. □

A.3. **Generating Functions.** Now consider the case where $V = Q \oplus Q^*$. This is a symplectic vector space with the symplectic form $\Omega : (Q \oplus Q^*) \times (Q \oplus Q^*) \to \mathbb{R}$ defined by

$$\Omega : (v, w) \mapsto v^T \mathbb{J} w.$$

The key result here regarding Lagrangian subspaces on $Q \oplus Q^*$ is the following:

Proposition A.5. *A Lagrangian subspace of $Q \oplus Q^*$ that is transversal to $\{0\} \oplus Q^*$ is the graph of an exact one-form, i.e., $\mathcal{L} = \operatorname{graph} dS$ for some function $S : Q \to \mathbb{R}$ which has the form*

$$S(q) = \frac{1}{2} \langle Aq, q\rangle + C \tag{A.3}$$

with some symmetric linear map $A : Q \to Q^*$ and an arbitrary real scalar constant C. Moreover, the correspondence between the Lagrangian subspaces and such functions (modulo the constant term) is one-to-one.

Proof. First recall that a Lagrangian submanifold of T^*Q that projects diffeomorphically onto Q is the graph of a closed one-forms on Q [See 1, Proposition 5.3.15 and the subsequent paragraph on p. 410]. In our case, Q is a vector space, and so the cotangent bundle T^*Q is identified with the direct sum $Q \oplus Q^*$. Now a Lagrangian subspace of $Q \oplus Q^*$ that is transversal to $\{0\} \oplus Q^*$ projects diffeomorphically onto Q, and so is the graph of a closed one-form. Then by the Poincaré lemma, it follows that any such Lagrangian subspace $\mathcal{L}$ is identified with the graph of an exact one-form dS with some function S on Q, i.e., $\mathcal{L} = \operatorname{graph} dS$.

However, as shown in, e.g., Jurdjevic [15, Theorem 3 on p. 233], the space of Lagrangian subspaces that are transversal to $\{0\} \oplus Q^*$ is in one-to-one correspondence with the space of all symmetric maps $A : Q \to Q^*$, with the correspondence given by $\mathcal{L} = \operatorname{graph} A$. Hence $\operatorname{graph} dS = \operatorname{graph} A$, or more specifically,

$$dS(q) = A_{ij} q^j \, dq^i.$$

This implies that S has the form

$$S(q) = \frac{1}{2} A_{ij} q^i q^j + C,$$

with an arbitrary real scalar constant C. □

Corollary A.6. *Let $\tilde{\mathcal{L}}(z_0) = z_0 + \mathcal{L}$ be a Lagrangian affine space, where $z_0 = (q_0, p_0)$ is an element in $Q \oplus Q^*$ and $\mathcal{L}$ is a Lagrangian subspace of $Q \oplus Q^*$ that is transversal to $\{0\} \oplus Q^*$. Then $\tilde{\mathcal{L}}(z_0)$ is the graph of an exact one-form $d\tilde{S}$ with a function $\tilde{S} : Q \to \mathbb{R}$ of the form*

$$\tilde{S}(q) = \frac{1}{2} \langle Aq, q \rangle + \langle p_0 - Aq_0, q \rangle + C,$$

with an arbitrary real scalar constant C.

Proof. From the above proposition, there exists a function $S : Q \to \mathbb{R}$ of the form Eq. (A.3) such that $\mathcal{L} = \operatorname{graph} dS$. Let $\tilde{S} : Q \to R$ be defined by $\tilde{S}(q) := S(q - q_0) + \langle p_0, q \rangle$. Then

$$d\tilde{S}(q) = A(q - q_0) + p_0. \tag{A.4}$$

and thus

$$\begin{aligned} \operatorname{graph} d\tilde{S} &= \{(q, d\tilde{S}(q)) \mid q \in Q\} \\ &= \{(q, A(q - q_0) + p_0) \mid q \in Q\} \\ &= (q_0, p_0) + \{(q - q_0, A(q - q_0)) \mid q \in Q\} \\ &= z_0 + \mathcal{L} \\ &= \tilde{\mathcal{L}}(z_0). \end{aligned}$$

The form Eq. (A.4) follows from a direct calculation. □

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