DISCRETE DIRAC STRUCTURES AND VARIATIONAL DISCRETE DIRAC MECHANICS

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Abstract. We construct discrete analogues of Dirac structures by considering the geometry of symplectic maps and their associated generating functions, in a manner analogous to the construction of continuous Dirac structures in terms of the geometry of symplectic vector fields and their associated Hamiltonians. We demonstrate that this framework provides a means of deriving implicit discrete Lagrangian and Hamiltonian systems, while incorporating discrete Dirac constraints. In particular, this yields implicit nonholonomic Lagrangian and Hamiltonian integrators. We also introduce a discrete Hamilton–Pontryagin variational principle on the discrete Pontryagin bundle, which provides an alternative derivation of the same set of integration algorithms. In so doing, we explicitly characterize the discrete Dirac structures that are preserved by Hamilton–Pontryagin integrators. In addition to providing a unified treatment of discrete Lagrangian and Hamiltonian mechanics in the more general setting of Dirac mechanics, it provides a generalization of symplectic and Poisson integrators to the broader category of Dirac integrators. Since discrete Lagrangians and discrete Hamiltonians are essentially generating functions of different types, the theoretical framework described in this paper is sufficiently general to encompass all possible Dirac integrators through an appropriate choice of generating functions.

Contents

1. Introduction

Dirac structures, which can be viewed as simultaneous generalizations of symplectic and Poisson structures, were introduced in Courant [9, 10]. In the context of geometric mechanics [1; 2; 24], Dirac structures are of interest as they can directly incorporate Dirac constraints that arise in degenerate Lagrangian systems [19], interconnected systems [8], and nonholonomic systems [4], and thereby provide a unified geometric framework for studying such problems.

From the Hamiltonian perspective, these systems are described by implicit Hamiltonian systems, and a comprehensive review of Dirac structures in this setting can be found in Dalsmo and van der Schaft [11]. This approach is motivated by earlier work on almost-Poisson structures that describe nonholonomic systems using brackets that fail to satisfy the Jacobi identity [6; 27]. In the context of systems with symmetry, Dirac analogues of symplectic [3] and Poisson [17] reduction have been developed.

On the Lagrangian side, degenerate, interconnected, and nonholonomic systems can be described by implicit Lagrangian systems, which were introduced in the context of Dirac structures in Yoshimura and Marsden [28]. The corresponding variational description of implicit Lagrangian systems was developed in [29], with the introduction of the Hamilton–Pontryagin principle on the Pontryagin bundle $TQ \oplus T^*Q$, which yields the Legendre transformation, as well as Hamilton's principle for Lagrangian systems and Hamilton's phase space principle for Hamiltonian systems. Implicit Lagrangian systems with constraints and external forces are described by the Lagrange–d'Alembert–Pontryagin principle, and in the presence of symmetries, the corresponding Lagrangian reduction theory was developed in [30], and the cotangent bundle reduction theory in [31]. Generalizations of Dirac manifolds to Dirac anchored vector bundles provide a category that is closed under Dirac reduction, which provides the necessary setting for Dirac reduction by stages [7].

In the context of geometric numerical integration [14; 21], which is concerned with the development of numerical methods that preserve geometric properties of the corresponding continuous flow, variational integrators that preserve the symplectic structure can be systematically derived from a discrete Hamilton's principle [25], and can be extended to asynchronous variational integrators [23] that preserve the multisymplectic structure of Hamiltonian partial differential equations. The discrete variational formulation of Hamiltonian mechanics was developed in [20] as the dual, in the sense of optimization, to discrete Lagrangian mechanics. Discrete analogues of the Hamilton–Pontryagin principle were introduced in [5; 18] for particular choices of discrete Lagrangians.

Contributions of this paper. In this paper, we introduce discrete analogues of Dirac structures, and show how they describe discrete implicit Lagrangian and Hamiltonian systems. The construction relies on the observation that the continuous Dirac structures used to define implicit Lagrangian systems arise from geometric properties of infinitesimally symplectic vector fields. By analogy, we construct discrete Dirac structures that are derived from properties of symplectic maps, and demonstrate that they yield implicit discrete Lagrangian and Hamiltonian systems, and recover nonholonomic integrators that are typically derived from a discrete Lagrange–d'Alembert principle.

We also introduce a discrete Hamilton–Pontryagin principle on the discrete Pontryagin bundle $(Q \times Q) \oplus T^*Q$, that provides a variational characterization of implicit discrete Lagrangian and Hamiltonian systems that we previously described in terms of discrete Dirac structures, and which reduce to the standard variational Lagrangian integrators [25] and variational Hamiltonian integrators [20]. Furthermore, we introduce a discrete Lagrange–d'Alembert–Pontryagin principle to allow the incorporation of Dirac constraints, and which recover nonholonomic integrators. We also describe a discrete Hamilton's phase space principle, which provides a variational formulation of discrete Hamiltonian mechanics, and a discrete Hamilton–d'Alembert principle in phase space, which addresses the issue of constraints.

In addition to providing a characterization of the discrete geometric structure that is preserved by Hamilton–Pontryagin integrators, we also characterize the corresponding discrete variational principles in an intrinsic manner that provides a theory of discrete Dirac mechanics that is valid semi-globally on the discrete Pontryagin bundle, i.e., on the preimage of a neighborhood of the diagonal of $Q \times Q$. We also provide a correspondence between discrete Lagrangian and Hamiltonian mechanics by introducing a discrete generalized Legendre transformation that is valid even without the assumption of hyperregularity.

Outline of this paper. The paper is organized as follows. The first part of the paper is concerned with the geometry of discrete Dirac mechanics. In Section 2, we review the theory of generating functions of symplectic maps, and derive discrete maps that we will use in the construction of discrete Dirac structures. In Section 3, we describe the corresponding theory of infinitesimally symplectic vector fields, and derive continuous maps that describe continuous Dirac structures. In Section 4, we review the continuous theory of Dirac structures and implicit Lagrangian and Hamiltonian systems, and construct a corresponding discrete theory.

The second part of the paper addresses the discrete variational structure of discrete Dirac mechanics. In Section 5, we review the Hamilton–Pontryagin principle and implicit Lagrangian systems. In Section 6, we introduce the discrete generalized Legendre transformation. In Section 7, we introduce both the local and intrinsic discrete Hamilton–Pontryagin principle, and show that they yield implicit discrete Lagrangian and Hamiltonian systems. In Section 8, we introduce the local and intrinsic discrete Lagrange–d'Alembert–Pontryagin principle, which incorporates discrete constraints that can model both interconnections and nonholonomic constraints. In Sections 9 and 10, we consider the corresponding local and intrinsic discrete variational principles on the Hamiltonian side, which are respectively the discrete Hamilton's principle on phase space, and the discrete Hamilton–d'Alembert principle on phase space. In Section 11, we provide some concluding remarks and future directions.

2. The Geometry of Generating Functions

2.1. **Generating Functions.** Let us first review the theory of generating functions following [1] and [24]. The key fact is the following (Proposition 5.2.1 of [1]):

Proposition 2.1. *Let (P_0, Ω_0) and (P_1, Ω_1) be symplectic manifolds, and $\pi_i : P_0 \times P_1 \to P_i$ be the projections onto P_i for $i = 0, 1$, and let us define $\Omega_{P_0\times P_1} \in \bigwedge^2(P_0 \times P_1)$ as follows:*

$$\Omega_{P_0\times P_1} := \pi_1^*\Omega_1 - \pi_0^*\Omega_0. \tag{2.1}$$

Then

(i) *$\Omega_{P_0\times P_1}$ is a symplectic form on $P_0 \times P_1$.*
(ii) *A map $F : P_0 \to P_1$ is symplectic if and only if $i_F^*\Omega_{P_0\times P_1} = 0$, where $i_F : \Gamma_F \to P_0 \times P_1$ is an inclusion and Γ_F is the graph of F.*

In particular we are interested in the case $(P_i, \Omega_i) = (T^*Q_i, \Omega_i)$ with the canonical symplectic form $\Omega_i = -d\Theta_i$ on T^*Q_i for $i = 0, 1$. Then

$$\Omega_{T^*Q_0\times T^*Q_1} := \pi_1^*\Omega_1 - \pi_0^*\Omega_0 = \pi_1^*(-d\Theta_1) - \pi_0^*(-d\Theta_0) = -d(\pi_1^*\Theta_1 - \pi_0^*\Theta_0) = -d\Theta_{T^*Q_0\times T^*Q_1}. \tag{2.2}$$

where we defined $\Theta_{T^*Q_0\times T^*Q_1} \in \bigwedge^1(T^*Q_0 \times T^*Q_1)$ by

$$\Theta_{T^*Q_0\times T^*Q_1} := \pi_1^*\Theta_1 - \pi_0^*\Theta_0. \tag{2.3}$$

So

$$i_F^*\Omega_{T^*Q_0\times T^*Q_1} = i_F^*(-d\Theta_{T^*Q_0\times T^*Q_1}) = -d(i_F^*\Theta_{T^*Q_0\times T^*Q_1}). \tag{2.4}$$

Therefore Part (ii) of the above proposition now reads

$$F : T^*Q_0 \to T^*Q_1 \text{ is symplectic} \iff d(i_F^*\Theta_{T^*Q_0\times T^*Q_1}) = 0. \tag{2.5}$$

However, by the Poincaré lemma, locally $d(i_F^*\Theta_{T^*Q_0\times T^*Q_1}) = 0$ if and only if $i_F^*\Theta_{T^*Q_0\times T^*Q_1} = dS$ for some function $S : \Gamma_F \to \mathbb{R}$. Such a function S is called a *generating function*. To summarize,

$$F : T^*Q_0 \to T^*Q_1 \text{ is symplectic} \overset{\text{locally}}{\iff} i_F^*\Theta_{T^*Q_0\times T^*Q_1} = dS \text{ for some function } S : \Gamma_F \to \mathbb{R}. \tag{2.6}$$

2.2. **Parametrization of Γ_F.** Suppose there exists a manifold M and a map $\varphi_M : M \to \Gamma_F$ that gives a (local) parametrization of the graph Γ_F. Now define $\tilde{S} : M \to \mathbb{R}$ to be $\tilde{S} := \varphi_M^* S = S \circ \varphi_M$. Then we can restate Eq. (2.6) in the following way:

$$F : T^*Q_0 \to T^*Q_1 \text{ is symplectic} \overset{\text{locally}}{\iff} (i_F^M)^*\Theta_{T^*Q_0\times T^*Q_1} = d\tilde{S} \text{ for some function } \tilde{S} : M \to \mathbb{R}, \tag{2.7}$$

where $i_F^M : M \to T^*Q_0 \times T^*Q_1$ is defined by $i_F^M := i_F \circ \varphi_M$.

For the remainder of this section, we consider the special case with $Q_0 = Q_1 = Q$. This is a natural setting for doing mechanics since in this case $F : T^*Q \to T^*Q$ describes a flow on the cotangent bundle T^*Q. We choose three different parametrization based on the classification given by Goldstein et al. [13].

2.3. **Generating Function of Type 1 and the Map $\gamma_Q^d : T^*Q \times T^*Q \to T^*(Q \times Q)$.** Let M be $Q \times Q$. Then the flow F on T^*Q is symplectic if and only if there exists $S_1 : Q \times Q \to \mathbb{R}$ such that

$$(i_F^{Q\times Q})^*\Theta_{T^*Q\times T^*Q} = dS_1. \tag{2.8}$$

Suppose $F : (q_0, p_0) \mapsto (q_1, p_1)$, or equivalently,

$$i_F^{Q\times Q} : Q \times Q \to T^*Q \times T^*Q;\ (q_0, q_1) \mapsto ((q_0, p_0), (q_1, p_1)), \tag{2.9}$$

where p_0 and p_1 are considered to be functions of q_0 and q_1. Then we can write Eq. (2.8) as follows:

$$p_1 dq_1 - p_0 dq_0 = D_1S_1 dq_0 + D_2S_1 dq_1, \tag{2.10}$$

which gives

$$p_0 = -D_1S_1, \qquad p_1 = D_2S_1. \tag{2.11}$$

This gives rise to a map $\kappa_Q^d : T^*Q \times T^*Q \to T^*(Q \times Q)$ so that the diagram

$$\begin{tikzcd}
T^*Q \times T^*Q \arrow[rr, "\kappa_Q^d"] & & T^*(Q \times Q) \\
& Q \times Q \arrow[ul, "i_F^{Q \times Q}"] \arrow[ur, "dS_1"'] &
\end{tikzcd} \tag{2.12a}$$

commutes. To be more specific, one has

$$\begin{tikzcd}
((q_0, p_0), (q_1, p_1)) \arrow[rr] & & (q_0, q_1, D_1 S_1, D_2 S_1) \\
& (q_0, q_1) \arrow[ul] \arrow[ur] &
\end{tikzcd} \tag{2.12b}$$

In view of Eq. (2.11), we obtain

$$\kappa_Q^d : ((q_0, p_0), (q_1, p_1)) \mapsto (q_0, q_1, -p_0, p_1). \tag{2.13}$$

2.4. **Generating Function of Type 2 and the Map $\Omega_{d+}^\flat : T^*Q \times T^*Q \to T^*\mathcal{H}_+$.** Let M be $\mathcal{H}_+$, whose local coordinates are (q_0, p_1)[1]. Then the flow F on T^*Q is symplectic if and only if there exists $S_2 : \mathcal{H}_+ \to \mathbb{R}$ such that

$$(i_F^{\mathcal{H}_+})^* \Theta^{(2)}_{T^*Q \times T^*Q} = dS_2, \tag{2.14}$$

where

$$\Theta^{(2)}_{T^*Q \times T^*Q} := d(q_1 p_1) - \Theta_{T^*Q \times T^*Q} = p_0 dq_0 + q_1 dp_1. \tag{2.15}$$

Note that using $\Theta^{(2)}_{T^*Q \times T^*Q}$ in place of $\Theta_{T^*Q \times T^*Q}$ does not affect the argument outlined in Section 2.2, since $d\Theta^{(2)}_{T^*Q \times T^*Q} = -d\Theta_{T^*Q \times T^*Q} = \Omega_{T^*Q \times T^*Q}$.

Suppose $F : (q_0, p_0) \mapsto (q_1, p_1)$, or equivalently,

$$i_F^{\mathcal{H}_+} : \mathcal{H}_+ \to T^*Q \times T^*Q;\ (q_0, p_1) \mapsto ((q_0, p_0), (q_1, p_1)), \tag{2.16}$$

where p_0 and q_1 are considered to be functions of q_0 and p_1. Then we can write Eq. (2.14) as follows:

$$p_0 dq_0 + q_1 dp_1 = D_1 S_2 dq_0 + D_2 S_2 dp_1, \tag{2.17}$$

which gives

$$p_0 = D_1 S_2, \qquad q_1 = D_2 S_2. \tag{2.18}$$

This gives rise to a map $\Omega_{d+}^\flat : T^*Q \times T^*Q \to T^*\mathcal{H}_+$ so that the diagram

$$\begin{tikzcd}
T^*Q \times T^*Q \arrow[rr, "\Omega_{d+}^\flat"] & & T^*\mathcal{H}_+ \\
& \mathcal{H}_+ \arrow[ul, "i_F^{\mathcal{H}_+}"] \arrow[ur, "dS_2"'] &
\end{tikzcd} \tag{2.19a}$$

commutes. To be more specific, one has

$$\begin{tikzcd}
((q_0, p_0), (q_1, p_1)) \arrow[rr] & & (q_0, p_1, D_1 S_2, D_2 S_2) \\
& (q_0, p_1) \arrow[ul] \arrow[ur] &
\end{tikzcd} \tag{2.19b}$$

In view of Eq. (2.18), we obtain

$$\Omega_{d+}^\flat : ((q_0, p_0), (q_1, p_1)) \mapsto (q_0, p_1, p_0, q_1). \tag{2.20}$$

[1]We can think of $\mathcal{H}_+$ as a submanifold of $Q \times T^*Q$ with local coordinates $(q_0, (q_1, p_1))$ where q_1 is dependent on q_0 and p_1.

2.5. **Generating Function of Type 3 and the Map $\Omega_{d-}^\flat : T^*Q \times T^*Q \to T^*\mathcal{H}_-$.** Let M be $\mathcal{H}_-$, whose local coordinates are (p_1, q_0)[2]. Then the flow F on T^*Q is symplectic if and only if there exists $S_3 : \mathcal{H}_- \to \mathbb{R}$ such that

$$(i_F^{\mathcal{H}_-})^* \Theta^{(3)}_{T^*Q\times T^*Q} = dS_3, \tag{2.21}$$

where

$$\Theta^{(3)}_{T^*Q\times T^*Q} := -d(q_0 p_0) - \Theta_{T^*Q\times T^*Q} = -q_0 dp_0 - p_1 dq_1. \tag{2.22}$$

Again, we can use $\Theta^{(3)}_{T^*Q\times T^*Q}$ in place of $\Theta_{T^*Q\times T^*Q}$ since $d\Theta^{(3)}_{T^*Q\times T^*Q} = -d\Theta_{T^*Q\times T^*Q} = \Omega_{T^*Q\times T^*Q}$.

Suppose $F : (q_0, p_0) \mapsto (q_1, p_1)$, or equivalently,

$$i_F^{\mathcal{H}_-} : \mathcal{H}_- \to T^*Q \times T^*Q;\ (p_0, q_1) \mapsto ((q_0, p_0), (q_1, p_1)), \tag{2.23}$$

where q_0 and p_1 are considered to be functions of p_0 and q_1. Then we can write Eq. (2.21) as follows:

$$-q_0 dp_0 - p_1 dq_1 = D_1 S_3 dp_0 + D_2 S_3 dq_1, \tag{2.24}$$

which gives

$$q_0 = -D_1 S_3, \qquad p_1 = -D_2 S_3. \tag{2.25}$$

This gives rise to a map $\Omega_{d-}^\flat : T^*Q \times T^*Q \to T^*\mathcal{H}_-$ so that the diagram

$$\begin{tikzcd} T^*Q \times T^*Q \arrow[rr, "\Omega_{d-}^\flat"] & & T^*\mathcal{H}_- \\ & \mathcal{H}_- \arrow[ul, "i_F^{\mathcal{H}_-}"] \arrow[ur, "dS_3"'] & \end{tikzcd} \tag{2.26a}$$

commutes. To be more specific, one has

$$\begin{tikzcd} ((q_0, p_0), (q_1, p_1)) \arrow[rr] & & (p_0, q_1, D_1 S_3, D_2 S_3) \\ & (p_0, q_1) \arrow[ul] \arrow[ur] & \end{tikzcd} \tag{2.26b}$$

In view of Eq. (2.25), we obtain

$$\Omega_{d-}^\flat : ((q_0, p_0), (q_1, p_1)) \mapsto (p_0, q_1, -q_0, -p_1). \tag{2.27}$$

3. Symplectic Flows as the Infinitesimal Limit of Symplectic Maps

We can "infinitesimalize" the above discussions to recover the familiar notions of symplectic flows, Hamiltonian and Lagrangian systems, and also the maps $\kappa_Q : TT^*Q \to T^*TQ$ and $\Omega^\flat : TT^*Q \to T^*T^*Q$ that are found in Yoshimura and Marsden [28].

3.1. **Hamiltonian Flows and the Map $\Omega^\flat : TT^*Q \to T^*T^*Q$.** The key idea is to regard the flow $F_X : T^*Q \to T^*Q$ of a vector field $X \in \mathfrak{X}(T^*Q)$ as the infinitesimal limit of the above discussions of symplectic maps. In fact the following definition of symplecticity of the flow F_X is analogous to Eq. (2.5):

Definition 3.1. The flow $F_X : T^*Q \to T^*Q$ of a vector field $X \in \mathfrak{X}(T^*Q)$ is called *symplectic* if $\pounds_X \Omega = 0$.

By Cartan's magic formula, and since Ω is a closed two-form, i.e., $d\Omega = 0$, a symplectic vector field X can be equivalently characterized by the property,

$$0 = \pounds_X \Omega = i_X(d\Omega) + d(i_X\Omega) = d(i_X\Omega), \tag{3.1}$$

which is to say that a vector field $X \in \mathfrak{X}(T^*Q)$ is symplectic if $i_X\Omega$ is closed, i.e., $d(i_X\Omega) = 0$.

Now again by the Poincaré lemma, locally we can restate this as follows:

$$F_X : T^*Q \to T^*Q \text{ is symplectic} \overset{\text{locally}}{\iff} i_X\Omega = dH \text{ for some function } H : T^*Q \to \mathbb{R}, \tag{3.2}$$

[2]We can think of $\mathcal{H}_-$ as a submanifold of $T^*Q \times Q$ with local coordinates $((q_0, p_0), q_1)$ where q_0 is dependent on p_0 and q_1.

which is again analogous to Eq. (2.6). So the Hamiltonian $H : T^*Q \to \mathbb{R}$ is an infinitesimal analogue of generating functions. Furthermore, the above local statement leads to the well-known *global* definition of the Hamiltonian flows:

Definition 3.2. The flow $F_X : T^*Q \to T^*Q$ of a vector field $X \in \mathfrak{X}(T^*Q)$ is called *Hamiltonian* if $i_X\Omega$ is exact, i.e., $i_X\Omega = dH$ for some function $H : T^*Q \to \mathbb{R}$.

Now analogously to Section 2, we can *define* $\Omega^\flat$ so that

$$\begin{tikzcd}
TT^*Q \arrow[rr, "\Omega^\flat"] & & T^*T^*Q \\
& T^*Q \arrow[ul, "X"] \arrow[ur, "dH"'] &
\end{tikzcd} \tag{3.3a}$$

commutes. To be more specific, one has

$$\begin{tikzcd}
(q,p,\dot{q},\dot{p}) \arrow[rr] & & (q,p,\partial H/\partial q, \partial H/\partial p) \\
& (q,p) \arrow[ul] \arrow[ur] &
\end{tikzcd} \tag{3.3b}$$

Note that, in coordinates, we can write $i_X\Omega = dH$ as follows:

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q}. \tag{3.4}$$

So in view of this set of equations, we obtain

$$\Omega^\flat : (q, p, \delta q, \delta p) \mapsto (q, p, -\delta p, \delta q). \tag{3.5}$$

3.2. **Lagrangian Flows and the Map $\kappa_Q : TT^*Q \to T^*TQ$.** We consider the Lagrangian analogue of the above construction. Given the Legendre transformation $\mathbb{F}L : TQ \to T^*Q$, we construct $\Omega_L = \mathbb{F}L^*\Omega$, and consider a second-order vector field $X_L \in \mathfrak{X}(TQ)$ that preserves the Lagrangian symplectic form, i.e.,

$$\pounds_{X_L}\Omega_L = 0. \tag{3.6}$$

We introduce the Lagrange one-form $\Theta_L = \mathbb{F}L^*\Theta = \frac{\partial L}{\partial v}dq$. Then, $\Omega_L = -d\Theta_L$, and

$$0 = \pounds_{X_L}\Omega_L = \pounds_{X_L}(-d\Theta_L) = -d\pounds_{X_L}\Theta_L, \tag{3.7}$$

as the Lie derivative commutes with the exterior derivative. Since $\pounds_{X_L}\Theta_L$ is closed, by the Poincaré lemma, there exists a local function $L : TQ \to \mathbb{R}$, such that,

$$\pounds_{X_L}\Theta_L = dL. \tag{3.8}$$

This is the intrinsic Euler–Lagrange equation expressed in terms of the Lagrangian (Section 3.4.2 of [16]), and is equivalent to the intrinsic Euler–Lagrange equation written in terms of the energy function (Equation (7.3.5) of [24]),

$$i_{X_L}\Omega_L = dE, \tag{3.9}$$

as the following discussion demonstrates. By applying Cartan's magic formula, we obtain

$$dL = \pounds_{X_L}\Theta_L = i_{X_L}(d\Theta_L) + d(i_{X_L}\Theta_L) = -i_{X_L}\Omega_L + d\left(i_{X_L}(\mathbb{F}L^*\Theta)\right), \tag{3.10}$$

which implies

$$i_{X_L}\Omega_L = d\left(i_{X_L}(\mathbb{F}L^*\Theta) - L\right). \tag{3.11}$$

The expression in the parentheses on the right hand side is precisely the intrinsic expression for the energy function $E : TQ \to \mathbb{R}$, as the following coordinate computation shows:

$$E(q,v) = \frac{\partial L}{\partial v}v - L(q,v) = \left\langle \frac{\partial L}{\partial v}dq, (\dot{q}, \dot{v}) \right\rangle - L(q,v) = \left(i_{X_L}(\mathbb{F}L^*\Theta) - L\right)(q,v), \tag{3.12}$$

where we used the fact that X_L is second-order, i.e., $\dot{q} = v$. As such, (3.8) is equivalent to (3.9). In coordinates, we can write $\pounds_{X_L}\Theta_L = dL$, expressed in terms of the $\mathbb{F}L$-related vector field $X \in \mathfrak{X}(T^*Q)$, i.e., $X \circ \mathbb{F}L = T\mathbb{F}L \circ X_L$, as follows:

$$p = \frac{\partial L}{\partial v}, \qquad \dot{q} = v, \qquad \dot{p} = \frac{\partial L}{\partial q}. \tag{3.13}$$

Now we can define κ_Q so that

$$\begin{tikzcd}
TT^*Q \arrow[rr, bend left, "\kappa_Q"] & TTQ \arrow[l, "T\mathbb{F}L"'] & T^*TQ \\
T^*Q \arrow[u, "X"] & TQ \arrow[l, "\mathbb{F}L"] \arrow[u, "X_L"'] \arrow[ur, "dL"']
\end{tikzcd} \tag{3.14a}$$

commutes. To be more specific, one has

$$\begin{tikzcd}
(q,p,\dot{q},\dot{p}) \arrow[rr, bend left] & (q,v,\dot{q},\dot{v}) \arrow[l] & (q,v,\partial L/\partial q, \partial L/\partial v) \\
(q,p) \arrow[u] & (q,v) \arrow[l] \arrow[u] \arrow[ur]
\end{tikzcd} \tag{3.14b}$$

In view of Eq. (3.13), we obtain

$$\kappa_Q : (q, p, \delta q, \delta p) \mapsto (q, \delta q, \delta p, p). \tag{3.15}$$

4. Discrete Dirac Structures and Implicit Discrete Lagrangian Systems

The maps κ_Q^d and $\Omega_{d\pm}^\flat$ defined in Eqs. (2.13), (2.20), and (2.27) provide a discrete counterpart of the framework for (continuous) Dirac mechanics developed by Yoshimura and Marsden [28; 29]. Considering the fact that the discrete Lagrangian and Hamiltonians are generating functions of Types 1, 2, and 3 [20], this is a natural setting for the implicit discrete Lagrangian and Hamiltonian systems, as we shall see.

For the purpose of comparison, let us first briefly review continuous Dirac mechanics following [28; 29].

4.1. **Continuous Dirac Mechanics.**

4.1.1. *The Big Diagram.* The maps κ_Q and $\Omega^\flat$ defined in Eqs. (3.15) and (3.5) give rise to the following diagram.

$$\begin{tikzcd}
T^*T^*Q \arrow[dr, "\pi_{T^*Q}"'] \arrow[drrr, "\pi^2"] & & TT^*Q \arrow[ll, "\Omega^\flat"'] \arrow[rr, "\kappa_Q"] \arrow[dl, "\tau_{T^*Q}"'] \arrow[d, "\rho_{TT^*Q}"] \arrow[dr, "T\pi_Q"] & & T^*TQ \arrow[llll, bend right, "\gamma_Q"'] \arrow[dl, "\pi_{TQ}"] \arrow[dlll, "\pi^1"'] \\
 & T^*Q & TQ \oplus T^*Q \arrow[l, "pr_2"] \arrow[r, "pr_1"'] & TQ &
\end{tikzcd} \tag{4.1a}$$

$$
\begin{tikzcd}
(q,p,-\delta p,\delta q) \arrow[dr] \arrow[drrr] & & (q,p,\delta q,\delta p) \arrow[ll] \arrow[rr] \arrow[dl] \arrow[d] \arrow[dr] & & (q,\delta q,\delta p,p) \arrow[llll, bend right] \arrow[dlll] \arrow[dl] \\
 & (q,p) & (q,\delta q)\oplus(q,p) \arrow[l] \arrow[r] & (q,\delta q) &
\end{tikzcd}
\tag{4.1b}
$$

4.1.2. *Symplectic Forms.* The induced symplectic one-forms on TT^*Q are

$$
\chi := (\Omega^\flat)^*\Theta_{T^*T^*Q} = -\delta p\, dq + \delta q\, dp, \qquad \lambda := (\kappa_Q)^*\Theta_{T^*TQ} = \delta p\, dq + p\, d(\delta q). \tag{4.2}
$$

and

$$
\Omega_{TT^*Q} := -d\lambda = d\chi = dq \wedge d(\delta p) + d(\delta q) \wedge dp. \tag{4.3}
$$

4.1.3. *Dirac Structure and Implicit Lagrangian System.* A distribution $\Delta_Q \subset TQ$ induces a Dirac structure on T^*Q as follows:

$$
D_{\Delta_Q}(z) := \{(v_z, \alpha_z) \in T_zT^*Q \times T_z^*T^*Q \mid v_z \in \Delta_{T^*Q}(z),\, \alpha_z(w_z) = \Omega(v_z, w_z) \text{ for } w_z \in \Delta_{T^*Q}(z)\}, \tag{4.4}
$$

or equivalently,

$$
D_{\Delta_Q}(z) := \left\{(v_z, \alpha_z) \in T_zT^*Q \times T_z^*T^*Q \mid v_z \in \Delta_{T^*Q}(z),\, \alpha_z - \Omega^\flat(v_z) \in \Delta^\circ_{T^*Q}(z)\right\} \tag{4.5}
$$

where $\Delta_{T^*Q} := (T\pi_Q)^{-1}(\Delta_Q)$.

Let $\gamma_Q := \Omega^\flat \circ (\kappa_Q)^{-1} : T^*TQ \to T^*T^*Q$. For a given Lagrangian $L : TQ \to \mathbb{R}$, define $\mathfrak{D}L := \gamma_Q \circ dL$. Let $X \in \mathfrak{X}(T^*Q)$, i.e., $X : T^*Q \to TT^*Q$. Then an *implicit Lagrangian system* (L, Δ_Q, X) is defined as follows:

$$
(X, \mathfrak{D}L) \in D_{\Delta_Q}. \tag{4.6}
$$

This gives the *implicit Lagrangian system*:

$$
\dot{q} = v \in \Delta_Q(q), \qquad p = \frac{\partial L}{\partial v}, \qquad \dot{p} - \frac{\partial L}{\partial q} \in \Delta^\circ_Q(q). \tag{4.7}
$$

Consider the special case $\Delta_Q = TQ$. Then

$$
D_{\Delta_Q}(z) := \left\{(v_z, \alpha_z) \in T_zT^*Q \times T_z^*T^*Q \mid \alpha_z = \Omega^\flat(v_z)\right\} \tag{4.8}
$$

Then the implicit Lagrangian system is written, in coordinates, as follows:

$$
\dot{q} = v, \quad \dot{p} = \frac{\partial L}{\partial q}, \quad p = \frac{\partial L}{\partial v}. \tag{4.9}
$$

4.1.4. *Dirac Structure and Implicit Hamiltonian System.* Given a Hamiltonian $H : T^*Q \to \mathbb{R}$, an *implicit Hamiltonian system* (H, Δ_Q, X) is defined as follows:

$$
(X, dH) \in D_{\Delta_Q}, \tag{4.10}
$$

which gives the *implicit Hamiltonian system*:

$$
\dot{q} = \frac{\partial H}{\partial p} \in \Delta_Q(q), \qquad \dot{p} + \frac{\partial H}{\partial q} \in \Delta^\circ_Q(q). \tag{4.11}
$$

Again, for the special case $\Delta_Q = TQ$, the implicit Hamiltonian system becomes

$$
\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}, \tag{4.12}
$$

which is the standard Hamiltonian system.

4.2. **(+)-Discrete Dirac Mechanics.**

4.2.1. *The Big Diagram.* The maps κ_Q^d and $\Omega_{d+}^\flat$ defined in Eqs. (2.13) and (2.20) give rise to the following diagram.

$$
\begin{tikzcd}
T^*\mathcal{H}_+ \arrow[dr, "\pi_{\mathcal{H}_+}"'] \arrow[drrr, "\pi^2_{d+}" near start] & & T^*Q \times T^*Q \arrow[ll, "\Omega^\flat_{d+}"'] \arrow[rr, "\kappa^d_Q"] \arrow[dl, "\tau_{\mathcal{H}_+}"'] \arrow[d, "\rho^{d+}_{(T^*Q)^2}"'] \arrow[dr, "\pi_Q \times \pi_Q"] & & T^*(Q \times Q) \arrow[llll, bend right, "\gamma^{d+}_Q"'] \arrow[dlll, "\pi^1_{d+}"' near start] \arrow[dl, "\pi_{Q\times Q}"] \\
& \mathcal{H}_+ & (Q \times Q) \oplus \mathcal{H}_+ \arrow[l, "pr_2^{d+}"] \arrow[r, "pr_1^{d+}"'] & Q \times Q &
\end{tikzcd}
\tag{4.13a}
$$

$$
\begin{tikzcd}
(q_0, p_1, p_0, q_1) \arrow[dr] \arrow[drrr] & & ((q_0, p_0), (q_1, p_1)) \arrow[ll] \arrow[rr] \arrow[dl] \arrow[d] \arrow[dr] & & (q_0, q_1, -p_0, p_1) \arrow[llll, bend right] \arrow[dlll] \arrow[dl] \\
& (q_0, p_1) & (q_0, q_1) \oplus (q_0, p_1) \arrow[l] \arrow[r] & (q_0, q_1) &
\end{tikzcd}
\tag{4.13b}
$$

4.2.2. *Symplectic Forms.* We can reinterpret the maps κ_Q^d and $\Omega_{d+}^\flat$ in connection to the discussions by Yoshimura and Marsden [28] in the following way. The maps κ_Q^d and $\Omega_{d+}^\flat$ induce two symplectic one-forms on $T^*Q \times T^*Q$. One is

$$\chi_{d+} := (\Omega_{d+}^\flat)^* \Theta_{T^*\mathcal{H}_+} = p_0\, dq_0 + q_1\, dp_1 = \Theta^{(2)}_{T^*Q\times T^*Q}, \tag{4.14}$$

and the other is

$$\lambda_{d+} := (\kappa_Q^d)^* \Theta_{T^*(Q\times Q)} = -p_0\, dq_0 + p_1\, dq_1 = \Theta_{T^*Q\times T^*Q}. \tag{4.15}$$

As shown above, they are related as follows:

$$\Omega_{T^*Q\times T^*Q} = -d\lambda_{d+} = d\chi_{d+} = dq_1 \wedge dp_1 - dq_0 \wedge dp_0. \tag{4.16}$$

This nicely parallels with the corresponding discussions of the maps κ_Q and $\Omega^\flat$ in [28] and Section 4.1.2.

4.2.3. *Discrete constraint distributions .* We first introduce the discrete constraint distribution[3] which we denote $\Delta_Q^d \subset Q \times Q$, which is a submanifold of $Q \times Q$ with the property that it contains the diagonal of $Q \times Q$, i.e.,

$$\Delta = \{(q, q) \mid q \in Q\} \subset \Delta_Q^d. \tag{4.17}$$

This discrete constraint distribution induces a continuous constraint distribution $\Delta_Q \subset TQ$ as follows. Consider smooth curves on Q, endowed with the equivalence relation $\varphi \sim \psi \iff \varphi(0) = \psi(0), D\varphi(0) = D\psi(0)$. Given a curve $\varphi \in C^\infty((-\epsilon, \epsilon), Q)$ with the properties $\varphi(0) = q$, $D\varphi(0) = v$, we identify the equivalence class of curves $[\varphi]$ with the tangent vector $v_q \in T_qQ$. Now, we introduce the class of curves on Q that are compatible with the discrete constraint distribution,

$$\mathcal{C}_{\Delta_Q^d} := \left\{\varphi \in C^\infty((-\epsilon, \epsilon), Q) \mid \forall \tau \in (0, \epsilon), (\varphi(-\tau), \varphi(0)), (\varphi(0), \varphi(\tau)) \in \Delta_Q^d \right\}. \tag{4.18}$$

Then, the continuous constraint distribution Δ_Q is defined by the property,

$$\varphi \in \mathcal{C}_{\Delta_Q^d} \implies [\varphi] \in \Delta_Q. \tag{4.19}$$

[3] We will adopt the notational convention that a continuous constraint distribution on M is denoted by $\Delta_M \subset TM$, and a discrete constraint distribution on M is denoted by $\Delta_M^d \subset M \times M$.

The annihilator of Δ_Q is denoted by $\Delta_Q^\circ \subset T^*Q$, and is defined, for each $q \in Q$, as

$$\Delta_Q^\circ(q) := \left\{ \alpha_q \in T_q^*Q \mid \forall v_q \in \Delta_Q, \langle \alpha_q, v_q \rangle = 0 \right\}. \tag{4.20}$$

Finally, these induce the discrete constraint distribution, $\Delta_{T^*Q}^d := (\pi_Q \times \pi_Q)^{-1}(\Delta_Q^d) \subset T^*Q \times T^*Q$, and the annihilator distribution on the discrete Pontryagin bundle, $\Delta_{\mathcal{H}_\pm}^\circ := \left(\Omega_{d\pm}^\flat\right)\left(\Delta_Q^\circ \times \Delta_Q^\circ\right) \subset T^*\mathcal{H}_\pm$, which are explicitly given by

$$\Delta_{T^*Q}^d = \left\{ ((q_0, p_0), (q_1, p_1)) \in T^*Q \times T^*Q \mid (q_0, q_1) \in \Delta_Q^d \right\}, \tag{4.21}$$

$$\Delta_{\mathcal{H}_+} = \left\{ (q_0, p_1, p_0, q_1) \in T^*\mathcal{H}_+ \mid p_0 \in \Delta_Q^\circ(q_0), p_1 \in \Delta_Q^\circ(q_1) \right\}, \tag{4.22}$$

$$\Delta_{\mathcal{H}_-} = \left\{ (p_0, q_1, -q_0, -p_1) \in T^*\mathcal{H}_+ \mid p_0 \in \Delta_Q^\circ(q_0), p_1 \in \Delta_Q^\circ(q_1) \right\}. \tag{4.23}$$

4.2.4. $(+)$-*Discrete Dirac Structure and Implicit Discrete Lagrangian System.* Let us define the $(+)$-*discrete Dirac structure* by

$$D_{\Delta_Q}^{d+}(z) := \left\{ ((z, z^1), \alpha_{z_+}) \in (\{z\} \times T^*Q) \times T_{z_+}^*\mathcal{H}_+ \mid \left(z, z^1\right) \in \Delta_{T^*Q}^d, \alpha_{z_+} - \Omega_{d+}^\flat\left((z, z^1)\right) \in \Delta_{\mathcal{H}_+}^\circ \right\}, \tag{4.24}$$

where if $z = (q, p)$ and $z^1 = (q^1, p^1)$ then $z_+ := (q, p^1)$.

Let $\gamma_Q^{d+} := \Omega_{d+}^\flat \circ (\kappa_Q^d)^{-1} : T^*(Q \times Q) \to T^*\mathcal{H}_+$, and for a given *discrete Lagrangian* $L_d : Q \times Q \to \mathbb{R}$, define $\mathfrak{D}^+ L_d := \gamma_Q^{d+} \circ dL$. Now let us write $X_d^k = ((q_k^0, p_k^0), (q_{k+1}^0, p_{k+1}^0)) \in T^*Q \times T^*Q$, and define an *implicit discrete Lagrangian system* (L_d, Δ_Q^d, X_d) as follows:

$$\left(X_d^k, \mathfrak{D}^+ L_d(q_k^0, q_k^1)\right) \in D_{\Delta_Q}^{d+} \iff (q_k^0, q_{k+1}^0) \in \Delta_Q^d \text{ and } \mathfrak{D}^+ L_d - \Omega_{d+}^\flat(X_d^k) \in \Delta_{\mathcal{H}_+}^\circ. \tag{4.25}$$

Now

$$\mathfrak{D}^+ L_d(q_k^0, q_k^1) = \gamma_Q^{d+}(q_k^0, q_k^1, D_1 L_d, D_2 L_d) = (q_k^0, D_2 L_d, -D_1 L_d, q_k^1), \tag{4.26}$$

and

$$\Omega_{d+}^\flat(X_d^k) = (q_k^0, p_{k+1}^0, p_k^0, q_{k+1}^0). \tag{4.27}$$

So we obtain the set of equations,

$$p_{k+1}^0 = D_2 L_d(q_k^0, q_k^1) \in \Delta_Q^\circ(q_k^1), \qquad p_k^0 + D_1 L_d(q_k^0, q_k^1) \in \Delta_Q^\circ(q_k^0), \qquad q_k^1 = q_{k+1}^0, \qquad (q_k^0, q_{k+1}^0) \in \Delta_Q^d, \tag{4.28}$$

which we shall call the *implicit discrete Euler–Lagrange equations.*

Consider the special case $\Delta_Q^d = Q \times Q$, which implies $\Delta_Q = TQ$. Then

$$D_{\Delta_Q}^{d+}(z) := \left\{ ((z, z^1), \alpha_{z_+}) \in (\{z\} \times T^*Q) \times T_{z_+}^*\mathcal{H}_+ \mid \alpha_{z_+} = \Omega_{d+}^\flat\left((z, z^1)\right) \right\}, \tag{4.29}$$

Then the implicit discrete Euler–Lagrange equations have the form

$$p_{k+1}^0 = D_2 L_d(q_k^0, q_k^1), \qquad p_k^0 = -D_1 L_d(q_k^0, q_k^1), \qquad q_k^1 = q_{k+1}^0. \tag{4.30}$$

4.2.5. $(+)$-*Discrete Dirac Structure and Implicit* $(+)$-*Discrete Hamiltonian System.* Let $H_{d+} : \mathcal{H}_+ \to \mathbb{R}$ be given. Then

$$dH_{d+}(q_k^0, p_k^1) = D_1 H_{d+}(q_k^0, p_k^1)\, dq_k^0 + D_2 H_{d+}(q_k^0, p_k^1)\, dp_k^1 \in T^*\mathcal{H}_+ \tag{4.31}$$

An *implicit* $(+)$-*discrete Hamiltonian system* $(H_{d+}, \Delta_Q^d, X_d)$ is defined as

$$\left(X_d^k, dH_{d+}(q_k^0, p_k^1)\right) \in D_{\Delta_Q}^{d+} \iff (q_k^0, q_{k+1}^0) \in \Delta_Q \text{ and } dH_{d+} - \Omega_{d+}^\flat(X_d^k) \in \Delta_{\mathcal{H}_+}^\circ. \tag{4.32}$$

Now

$$dH_{d+}(q_k^0, q_k^1) = (q_k^0, p_k^1, D_1 H_{d+}, D_2 H_{d+}), \tag{4.33}$$

and

$$\Omega_{d+}^\flat(X_d^k) = (q_k^0, p_{k+1}^0, p_k^0, q_{k+1}^0). \tag{4.34}$$

So we obtain the set of equations

$$p_k^0 - D_1 H_{d+}(q_k^0, p_k^1) \in \Delta_Q^\circ(q_k^0), \qquad q_{k+1}^0 = D_2 H_{d+}(q_k^0, p_k^1), \qquad p_k^1 - p_{k+1}^0 \in \Delta_Q^\circ(q_k^1), \qquad (q_k^0, q_{k+1}^0) \in \Delta_Q^d, \tag{4.35}$$

which we shall call the *implicit* $(+)$-*discrete Hamilton's equations.*

If $\Delta_Q^d = Q \times Q$, the discrete Dirac structure is given by (4.29), and the implicit (+)-discrete Hamilton's equations have the form

$$p_k^0 = D_1 H_{d+}(q_k^0, p_k^1), \qquad q_{k+1}^0 = D_2 H_{d+}(q_k^0, p_k^1), \qquad p_k^1 = p_{k+1}^0. \tag{4.36}$$

Note that this is essentially the (+)-discrete Hamilton's equations in Lall and West [20].

4.3. $(-)$-Discrete Dirac Mechanics.

4.3.1. *The Big Diagram.* The maps κ_Q^d and $\Omega_{d-}^\flat$ defined in Eqs. (2.13) and (2.27) give rise to the following diagram.

$$\begin{tikzcd}
T^*\mathcal{H}_- \arrow[dr, "\pi_{\mathcal{H}_-}"'] \arrow[drrr, "\pi^2_{d-}" near start] & & T^*Q \times T^*Q \arrow[ll, "\Omega^\flat_{d-}"'] \arrow[rr, "\kappa^d_Q"] \arrow[dl, "\tau_{\mathcal{H}_-}"'] \arrow[d, "\rho^{d-}_{(T^*Q)^2}"] \arrow[dr, "\pi_Q \times \pi_Q"] & & T^*(Q \times Q) \arrow[llll, bend right, "\gamma_Q^{d-}"'] \arrow[dlll, "\pi^1_{d-}"' near start] \arrow[dl, "\pi_{Q \times Q}"] \\
& \mathcal{H}_- & (Q \times Q) \oplus \mathcal{H}_- \arrow[l, "pr_2^{d-}"] \arrow[r, "pr_1^{d-}"'] & Q \times Q &
\end{tikzcd} \tag{4.37a}$$

$$\begin{tikzcd}
(p_0, q_1, -q_0, -p_1) \arrow[dr] \arrow[drrr] & & ((q_0, p_0), (q_1, p_1)) \arrow[ll] \arrow[rr] \arrow[dl] \arrow[d] \arrow[dr] & & (q_0, q_1, -p_0, p_1) \arrow[llll, bend right] \arrow[dlll] \arrow[dl] \\
& (p_0, q_1) & (q_0, q_1) \oplus (p_0, q_1) \arrow[l] \arrow[r] & (q_0, q_1) &
\end{tikzcd} \tag{4.37b}$$

4.3.2. *Symplectic Forms.* As in the (+)-discrete case, we can reinterpret the maps κ_Q^d and $\Omega_{d-}^\flat$ as follows. We have

$$\chi_{d-} := (\Omega_{d-}^\flat)^* \Theta_{T^*T^*Q} = -p_1\, dq_1 - q_0\, dp_0 = \Theta^{(3)}_{T^*Q \times T^*Q}, \tag{4.38}$$

and

$$\lambda_{d-} := (\kappa_Q^d)^* \Theta_{T^*(Q \times Q)} = -p_0\, dq_0 + p_1\, dq_1 = \Theta_{T^*Q \times T^*Q}, \tag{4.39}$$

and then

$$\Omega_{T^*Q \times T^*Q} := -d\lambda_{d-} = d\chi_{d-} = dq_1 \wedge dp_1 - dq_0 \wedge dp_0. \tag{4.40}$$

4.3.3. $(-)$-*Discrete Dirac Structure and Implicit Discrete Lagrangian System.* Let us define the $(-)$-*discrete Dirac structure* by

$$D^{d-}_{\Delta_Q}(z) := \left\{ ((z, z^1), \alpha_{z_-}) \in (\{z\} \times T^*Q) \times T^*_{z_-}\mathcal{H}_- \;\middle|\; \left(z, z^1\right) \in \Delta^d_{T^*Q}, \alpha_{z_-} - \Omega^\flat_{d-}\left((z, z^1)\right) \in \Delta^\circ_{\mathcal{H}_-} \right\}, \tag{4.41}$$

where if $z = (q, p)$ and $z^1 = (q^1, p^1)$ then $z_- := (q^1, p)$.

Let $\gamma_Q^{d-} := \Omega_{d-}^\flat \circ (\kappa_Q^d)^{-1} : T^*(Q \times Q) \to T^*\mathcal{H}_-$, and for a given discrete Lagrangian $L_d : Q \times Q \to \mathbb{R}$, define $\mathfrak{D}^- L_d := \gamma_Q^{d-} \circ dL$. Now define an *implicit discrete Lagrangian system* (L_d, Δ_Q^d, X_d) as follows:

$$\left(X_d^k, \mathfrak{D}^- L_d(q_k^0, q_k^1)\right) \in D^{d-}_{\Delta_Q} \iff (q_k^0, q_{k+1}^0) \in \Delta_Q^d \text{ and } \mathfrak{D}^- L_d - \Omega^\flat_{d-}(X_d^k) \in \Delta^\circ_{\mathcal{H}_-}. \tag{4.42}$$

Now

$$\mathfrak{D}^- L_d(q_k^0, q_k^1) = \gamma_Q^{d-}(q_k^0, q_k^1, D_1 L_d, D_2 L_d) = (-D_2 L_d, q_{k+1}^0, -q_k^0, -D_1 L_d), \tag{4.43}$$

and

$$\Omega^\flat_{d-}(X_d^k) = (p_k^0, q_{k+1}^0, -q_k^0, -p_{k+1}^0). \tag{4.44}$$

So we recover the *implicit discrete Euler–Lagrange equations*

$$p^0_{k+1} = D_2L_d(q^0_k, q^1_k) \in \Delta^\circ_Q(q^1_k), \qquad p^0_k + D_1L_d(q^0_k, q^1_k) \in \Delta^\circ_Q(q^0_k), \qquad q^1_k = q^0_{k+1}, \qquad (q^0_k, q^0_{k+1}) \in \Delta^d_Q, \tag{4.45}$$

that we previously obtained in (4.28).

If $\Delta^d_Q = Q \times Q$, then

$$D^{d-}_{\Delta_Q}(z) := \left\{((z, z^1), \alpha_{z_-}) \in (\{z\} \times T^*Q) \times T^*_{z_-}\mathcal{H}_- \mid \alpha_{z_-} = \Omega^\flat_{d-}\left((z, z^1)\right)\right\}, \tag{4.46}$$

which yields,

$$p^0_{k+1} = D_2L_d(q^0_k, q^1_k), \qquad p^0_k = -D_1L_d(q^0_k, q^1_k), \qquad q^1_k = q^0_{k+1}. \tag{4.47}$$

4.3.4. $(-)$*-Discrete Dirac Structure and Implicit* $(-)$*-Discrete Hamiltonian System.* Let $H_{d-} : \mathcal{H}_- \to \mathbb{R}$ be given. Then

$$dH_{d-}(p^0_k, q^1_k) = D_2H_{d-}(p^1_k, q^0_k)\, dp^0_k + D_1H_{d-}(p^1_k, q^0_k)\, dq^1_k \in T^*\mathcal{H}_- \tag{4.48}$$

An *implicit* $(-)$*-discrete Hamiltonian system* $(H_{d-}, \Delta^d_Q, X_d)$ is defined as

$$(X^k_d, dH_{d-}) \in D^{d-}_{\Delta_Q} \iff (q^0_k, q^0_{k+1}) \in \Delta^d_Q \text{ and } dH_{d-} - \Omega^\flat_{d-}(X^k_d) \in \Delta^\circ_{\mathcal{H}_-}. \tag{4.49}$$

Now

$$dH_{d-}(q^1_k, p^0_k) = (p^0_k, q^1_k, D_1H_{d-}, D_2H_{d-}), \tag{4.50}$$

and

$$\Omega^\flat_{d-}(X^k_d) = (p^0_k, q^0_{k+1}, -q^0_k, -p^0_{k+1}). \tag{4.51}$$

So we obtain the set of equations

$$q^0_k = -D_1H_{d-}(p^0_k, q^1_k), \qquad p^0_{k+1} + D_2H_{d-}(p^0_k, q^1_k) \in \Delta^\circ_Q(q^1_k), \qquad q^1_k = q^0_{k+1}, \qquad (q^0_k, q^0_{k+1}) \in \Delta^d_Q, \tag{4.52}$$

which we shall call the *implicit* $(-)$*-discrete Hamilton's equations.*

If $\Delta^d_Q = Q \times Q$, the discrete Dirac structure is given by (4.46), and the implicit $(-)$-discrete Hamilton's equations have the form

$$q^0_k = -D_1H_{d-}(p^0_k, q^1_k), \qquad p^0_{k+1} = -D_2H_{d-}(p^0_k, q^1_k), \qquad q^1_k = q^0_{k+1}. \tag{4.53}$$

Note that this is essentially the $(-)$-discrete Hamilton's equations in Lall and West [20].

5. Hamilton–Pontryagin Principle and Implicit Lagrangian Systems

We will first review the continuous Hamilton–Pontryagin principle in coordinates and without constraints, to motivate the construction of the discrete Hamilton–Pontryagin principle. A variational derivation of the implicit Euler–Lagrange equations can be obtained by considering the augmented variational principle given by,

$$\delta \int [L(q, v) - p(v - \dot{q})] = 0, \tag{5.1}$$

where we impose the second-order curve condition, $v = \dot{q}$ using Lagrange multipliers p. This variational principle is also referred to as the *Hamilton–Pontryagin Principle*, and is a variational principle on the *Pontryagin bundle* $TQ \oplus T^*Q$.

In a local trivialization Q is represented by an open set U in a linear space E, so the Pontryagin bundle is represented by $(U \times E) \oplus (U \times E^*) \cong U \times E \times E^*$, with local coordinates (q, v, p). If we consider q, v, and p as independent variables, we have that,

$$\begin{aligned}\delta \int [L(q, v) - p(v - \dot{q})]dt &= \int \left[\frac{\partial L}{\partial q}\delta q + \left(\frac{\partial L}{\partial v} - p\right)\delta v - (v - \dot{q})\delta p + p\delta\dot{q}\right] dt \\ &= \int \left[\left(\frac{\partial L}{\partial q} - \dot{p}\right)\delta q + \left(\frac{\partial L}{\partial v} - p\right)\delta v - (v - \dot{q})\delta p\right] dt\end{aligned} \tag{5.2}$$

where we used integration by parts, and the fact that the variation δq vanishes at the endpoints. This yields the *implicit Euler–Lagrange equations*,

$$\dot{p} = \frac{\partial L}{\partial q}, \qquad p = \frac{\partial L}{\partial v}, \qquad v = \dot{q}. \tag{5.3}$$

5.1. **Variational Principle in Phase Space.** By starting with the Hamilton–Pontryagin Principle, and considering the necessary stationarity conditions in different orders, we will obtain the variational principle on phase space, and the usual variational principle on TQ.

By taking variations with respect to v, we obtain the relation,

$$\frac{\partial L}{\partial v}(q, v) - p = 0. \tag{5.4}$$

We introduce the *Hamiltonian*, $H : T^*Q \to \mathbb{R}$, defined to be,

$$H(q, p) = pv - L(q, v)|_{p=\partial L/\partial v(q,v)} \,. \tag{5.5}$$

From the definition of the Hamiltonian, we can express the augmented variational principle as,

$$\delta \int [p\dot{q} - H(q, p)] = 0, \tag{5.6}$$

which is the *variational principle in phase space.*

By first taking variations of the augmented variational principle with respect to p, we obtain the usual variational principle on TQ,

$$\delta \int L(q, \dot{q}) = 0. \tag{5.7}$$

5.2. **Hamilton's Equations.** By considering Hamilton's principle in phase space, we obtain,

$$\begin{aligned} 0 &= \delta \int [p\dot{q} - H(q, p)]dt \\ &= \int \left[\dot{q}\delta p + p\delta\dot{q} - \frac{\partial H}{\partial q}\delta q - \frac{\partial H}{\partial p}\delta p \right] dt \\ &= \int \left[\left(-\dot{p} - \frac{\partial H}{\partial q} \right) \delta q + \left(\dot{q} - \frac{\partial H}{\partial p} \right) \delta p \right] dt. \end{aligned} \tag{5.8}$$

By the fundamental theorem of the calculus of variations, this is equivalent to *Hamilton's equations*,

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q}. \tag{5.9}$$

6. Discrete Generalized Legendre Transform

In this section, we introduce the discrete Lagrangian and Hamiltonian constraints, and develop the discrete generalized Legendre transform that relates the two discrete constraint submanifolds, by following the approach of §2 of [29] and using the symplectic structures $\Omega_{T^*Q\times T^*Q} = -d\lambda_{d\pm} = d\chi_{d\pm}$ on $T^*Q \times T^*Q$, and the maps κ_Q^d, and $\Omega_{d\pm}^\flat$.

6.1. **Discrete Lagrangian constraints.** Let L_d be a discrete Lagrangian on $\Delta_Q^d \subset Q \times Q$. The symplectic manifold $(T^*Q \times T^*Q, \Omega_{T^*Q\times T^*Q} = -d\lambda_{d\pm})$ is defined by the quadruple $(T^*Q \times T^*Q, Q \times Q, \pi_Q \times \pi_Q, \lambda_{d\pm})$ and the set

$$\begin{aligned} N_d := \big\{ &(z, z^1) \in T^*Q \times T^*Q \,|\, (\pi_Q \times \pi_Q)\left((z, z^1)\right) \in \Delta_Q^d, \\ &\lambda_{d+}\left((w, w^1)\right) = \left\langle dL_d\left((\pi_Q \times \pi_Q)\left((z, z^1)\right)\right), T(\pi_Q \times \pi_Q)\left((w, w^1)\right)\right\rangle, \\ &\forall (w, w^1) \in T_{(z,z^1)}(T^*Q \times T^*Q) \text{ s.t. } T_{(z,z^1)}(\pi_Q \times \pi_Q)\left((w, w^1)\right) \in T_{(\pi_Q\times\pi_Q)((z,z^1))}\Delta_Q^d \big\} \end{aligned} \tag{6.1}$$

is a Lagrangian submanifold of $(T^*Q \times T^*Q, \Omega_{T^*Q\times T^*Q} = -d\lambda_{d\pm})$ of dimension $\frac{1}{2}\dim(T^*Q \times T^*Q)$, where the submanifold

$$\Delta_Q^d = (\pi_Q \times \pi_Q)(N_d) \subset Q \times Q \tag{6.2}$$

is the discrete constraint distribution on $Q \times Q$ called the *discrete Lagrangian constraint.* Then, the discrete Lagrangian L_d is a generating function of N_d, since $N_d \subset T^*Q \times T^*Q$ is the graph of $(\kappa_Q^d)^{-1}(dL_d)$.

6.2. **Discrete Hamiltonian constraints.** Let $H_{d\pm}$ be a $(\pm)$-discrete Hamiltonian on $P_{d\pm} \subset \mathcal{H}_{\pm}$. The symplectic manifold $(T^*Q\times T^*Q, \Omega_{T^*Q\times T^*Q} = d\chi_{d\pm})$ is defined by the quadruple $(T^*Q\times T^*Q, \mathcal{H}_{\pm}, \tau_{\mathcal{H}_{\pm}}, \chi_{d\pm})$ and the set

$$\begin{aligned} N_{d\pm} := \big\{ (z, z^1) \in T^*Q \times T^*Q \,|\, & \tau_{\mathcal{H}_{\pm}}\left((z, z^1)\right) \in P_{d\pm}, \\ & \chi_{d\pm}\left((w, w^1)\right) = \left\langle dH_{d\pm}\left(\tau_{\mathcal{H}_{\pm}}\left((z, z^1)\right)\right), T\tau_{\mathcal{H}_{\pm}}\left((w, w^1)\right)\right\rangle, \\ & \forall (w, w^1) \in T_{(z,z^1)}(T^*Q \times T^*Q) \text{ s.t. } T_{(z,z^1)}\tau_{\mathcal{H}_{\pm}}\left((w, w^1)\right) \in T_{\tau_{\mathcal{H}_{\pm}}((z,z^1))}P_{d\pm} \big\} \end{aligned} \tag{6.3}$$

is a Lagrangian submanifold of $(T^*Q \times T^*Q, \Omega_{T^*Q\times T^*Q} = d\chi_{d\pm})$ of dimension $\frac{1}{2}\dim(T^*Q \times T^*Q)$, where the submanifold

$$P_{d\pm} = (\pi_Q \times \pi_Q)(N_{d\pm}) \subset \mathcal{H}_{\pm} \tag{6.4}$$

is the *discrete constraint momentum space* called a *discrete Hamiltonian constraint*. Then, the discrete Hamiltonian $H_{d\pm}$ is a generating function of $N_{d\pm}$, since $N_{d\pm} \subset T^*Q \times T^*Q$ is the graph of $(\Omega^{\flat}_{d\pm})^{-1}(dH_{d\pm})$.

6.3. **Symplectomorphism and the discrete momentum function.** Consider the identity map, $\varphi : (P_1 = T^*Q \times T^*Q, \Omega_1 = -d\lambda_{d\pm}) \to (P_2 = T^*Q \times T^*Q, \Omega_2 = d\chi_{d\pm})$. Since $P_1 = P_2$ and $\Omega_2 = \Omega_1$, it follows that $\varphi^*\Omega_2 = \Omega_1$, and φ is a symplectomorphism.

The graph of the symplectomorphism φ is a submanifold of $P_1 \times P_2$, which is denoted by

$$\Gamma(\varphi) \subset P_1 \times P_2. \tag{6.5}$$

Let $I_\varphi : \Gamma(\varphi) \to P_1 \times P_2$ be the inclusion and $\pi_i : P_1 \times P_2 \to P_i$ be the canonical projection. As in (2.1), we define

$$\begin{aligned} \Omega_{P_1\times P_2} &= \pi_1^*\Omega_1 - \pi_2^*\Omega_2 \\ &= \pi_1^*(-d\lambda_{d\pm}) - \pi_2^*(d\chi_{d\pm}). \end{aligned} \tag{6.6}$$

Since φ is symplectic, by Proposition 2.1, we have that $i_\varphi^*\Omega_{P_1\times P_2} = 0$. We can write $\Omega_{P_1\times P_2} = -d\Theta_{d\pm}$, where $\Theta_{d\pm} = \lambda_{d\pm} \oplus \chi_{d\pm} = \pi_1^*\lambda_{d\pm} + \pi_2^*\chi_{d\pm}$. Also, $\Gamma(\varphi)$ is a maximally isotropic submanifold with half the dimension of $P_1 \times P_2 = (T^*Q \times T^*Q) \times (T^*Q \times T^*Q)$.

Given the diagonal map $\Psi : (T^*Q\times T^*Q) \to (T^*Q\times T^*Q)\times(T^*Q\times T^*Q)$, we have the one-forms $\Psi^*\Theta_{d\pm}$ on $(T^*Q \times T^*Q)$, given by

$$\begin{aligned} \Psi^*\Theta_{d+} &= \Psi^*(\lambda_{d+} \oplus \chi_{d+}) = \lambda_{d+} + \varphi^*\chi_{d+} = (-p_0 dq_0 + p_1 dq_1) + (p_0 dq_0 + q_1 dp_1) \\ &= p_1 dq_1 + q_1 dp_1 = d(p_1 q_1) = d(G_{d+} \circ \rho^{d+}_{(T^*Q)^2}), \end{aligned} \tag{6.7}$$

and

$$\begin{aligned} \Psi^*\Theta_{d-} &= \Psi^*(\lambda_{d-} \oplus \chi_{d-}) = \lambda_{d-} + \varphi^*\chi_{d-} = (-p_0 dq_0 + p_1 dq_1) + (-p_1 dq_1 - q_0 dp_0) \\ &= -p_0 dq_0 - q_0 dp_0 = d(-p_0 q_0) = d(G_{d-} \circ \rho^{d-}_{(T^*Q)^2}), \end{aligned} \tag{6.8}$$

where

$$G_{d+}\left((q_0, q_1) \oplus (q_0, p_1)\right) = p_1 q_1, \tag{6.9}$$

$$G_{d-}\left((q_0, q_1) \oplus (p_0, q_1)\right) = -p_0 q_0, \tag{6.10}$$

which we refer to as the $(\pm)$-*discrete momentum functions*.

6.4. **Discrete Generalized Legendre Transforms.** As we saw in (6.1) and (6.3), the discrete constraint manifold on $T^*Q \times T^*Q$ can be realized as graphs of one-forms on $Q \times Q$ and $\mathcal{H}_{\pm}$. These various representations are related by the *discrete generalized Legendre transform*, which is a procedure for constructing the submanifold $\mathcal{K}_{d\pm}$ of the discrete Pontryagin bundle $(Q \times Q) \oplus \mathcal{H}_{\pm}$ from a submanifold N_d of $T^*Q \times T^*Q$ associated with $(T^*Q\times T^*Q, Q\times Q, \pi_Q \times \pi_Q, \lambda_{d\pm})$ and with a discrete Lagrangian L_d on $\Delta_Q^d \subset Q\times Q$, as in (6.1). We define the submanifold $\mathcal{K}_{d\pm}$ to be the image of N_d under the map $\left((\pi_Q \times \pi_Q) \times \tau_{\mathcal{H}_{\pm}}\right) \circ \Psi$, which is given by

$$\mathcal{K}_{d\pm} = \left((\pi_Q \times \pi_Q) \times \tau_{\mathcal{H}_{\pm}}\right) \circ \Psi(N_d) \subset (Q \times Q) \times \mathcal{H}_{\pm}, \tag{6.11}$$

which is the graph of the discrete Legendre transforms $\mathbb{F}L_{d\pm} : Q \times Q \to \mathcal{H}_\pm$ with respect to the discrete constraint distribution $\Delta_Q^d \subset Q \times Q$. Define the *discrete generalized energy* $E_{d\pm}$ on $(Q\times Q)\oplus\mathcal{H}_\pm$ in terms of the $(\pm)$-discrete momentum functions, and $pr_1^{d\pm}$, by

$$E_{d\pm} = G_{d\pm} - L_d \circ pr_1^{d\pm}, \tag{6.12}$$

or more explicitly,

$$E_{d+}\left((q_0,q_1)\oplus(q_0,p_1)\right) = p_1 q_1 - L_d(q_0,q_1), \tag{6.13}$$

$$E_{d-}\left((q_0,q_1)\oplus(p_0,q_1)\right) = -p_0 q_0 - L_d(q_0,q_1). \tag{6.14}$$

These expressions agree with the $(\pm)$-discrete Hamiltonians $H_{d\pm} : \mathcal{H}_\pm \to \mathbb{R}$, once we appropriately impose the discrete Legendre transforms $\mathbb{F}L_{d\pm}$. More precisely, the following diagram commutes:

$$\begin{tikzcd} Q\times Q \arrow[r, "\mathbf{1}_{Q\times Q}\oplus\mathbb{F}L_{d\pm}"] \arrow[d, "\mathbb{F}L_{d\pm}"'] & (Q\times Q)\oplus\mathcal{H}_\pm \arrow[d, "E_{d\pm}"] \\ \mathcal{H}_\pm \arrow[r, "H_{d\pm}"] & \mathbb{R} \end{tikzcd} \tag{6.15}$$

The maps $\mathbf{1}_{Q\times Q}\oplus\mathbb{F}L_{d+}$ and $\mathbf{1}_{Q\times Q}\oplus\mathbb{F}L_{d-}$ are given by $(q_0,q_1)\mapsto(q_0,q_1,D_2L_d(q_0,q_1))$ and $(q_0,q_1)\mapsto(q_0,q_1,-D_1L_d(q_0,q_1))$, respectively. Then, the submanifold $\mathcal{K}_{d\pm}$ may be given by

$$\begin{aligned}\mathcal{K}_{d+} &= \left\{(q_0,q_1,p_1)\in(Q\times Q)\oplus\mathcal{H}_+ \mid \forall(q_0,p_1)\in\mathcal{H}_+,(q_0,q_1)\in\Delta_Q^d \text{ is a stationary point of } E_{d+}\right\} &(6.16)\\ &= \left\{(q_0,q_1,p_1)\in(Q\times Q)\oplus\mathcal{H}_+ \mid (q_0,q_1)\in\Delta_Q^d, p_1 = D_2L_d(q_0,q_1)\right\}, &(6.17)\end{aligned}$$

and

$$\begin{aligned}\mathcal{K}_{d-} &= \left\{(q_0,q_1,p_0)\in(Q\times Q)\oplus\mathcal{H}_- \mid \forall(p_0,q_1)\in\mathcal{H}_-,(q_0,q_1)\in\Delta_Q^d \text{ is a stationary point of } E_{d-}\right\} &(6.18)\\ &= \left\{(q_0,q_1,p_0)\in(Q\times Q)\oplus\mathcal{H}_- \mid (q_0,q_1)\in\Delta_Q^d, p_0 = -D_1L_d(q_0,q_1)\right\}. &(6.19)\end{aligned}$$

7. Discrete Hamilton–Pontryagin Principle and Implicit Discrete Lagrangian Systems

We first relax the discrete second-order curve condition, and consider the augmented discrete variational principle given by,

$$\delta\sum\left[L_d(q_k^0,q_k^1) - p_{k+1}(q_k^1 - q_{k+1}^0)\right] = 0, \tag{7.1}$$

where we impose the second-order curve condition, $q_k^1 = q_{k+1}^0$ using Lagrange multipliers p_{k+1}, and keep the endpoints q_0^0 and q_N^0 fixed. We will refer to this discrete variational principle as the *Discrete Hamilton–Pontryagin Principle.*

If we consider q_k^0, q_k^1 and p_k as independent variables, with the condition that $\delta q_0^0 = 0$ and $\delta q_{N-1}^1 = 0$, we obtain,

$$\begin{aligned}\delta\sum\left[L_d(q_k^0,q_k^1) - p_{k+1}(q_k^1 - q_{k+1}^0)\right] = \sum\{&[D_1L_d(q_k^0,q_k^1)+p_k]\delta q_k^0\\ &-[q_k^1 - q_{k+1}^0]\delta p_{k+1} + [D_2L_d(q_k^0,q_k^1) - p_{k+1}]\delta q_k^1\}\end{aligned} \tag{7.2}$$

from which we obtain the *implicit discrete Euler–Lagrange equations*,

$$p_k = -D_1L_d(q_k^0,q_k^1), \qquad p_{k+1} = D_2L_d(q_k^0,q_k^1), \qquad q_k^1 = q_{k+1}^0. \tag{7.3}$$

7.1. **Intrinsic Discrete Hamilton–Pontryagin Principle.**

We now formulate the discrete Hamilton–Pontryagin principle intrinsically, by obtaining the intrinsic expression for the discrete Hamilton–Pontryagin sum on $((Q\times Q)\oplus T^*Q)\times((Q\times Q)\oplus T^*Q)$. Prior to doing this, we introduce the natural projection maps $\tau_{(Q\times Q)\oplus\mathcal{H}_\pm} : ((Q\times Q)\oplus T^*Q)\times((Q\times Q)\oplus T^*Q) \to ((Q\times Q)\oplus\mathcal{H}_\pm)$, which are given by

$$\tau_{(Q\times Q)\oplus\mathcal{H}_+} : \left((q_k^0,q_k^1)\oplus(q_k^0,p_k),(q_{k+1}^0,q_{k+1}^1)\oplus(q_{k+1}^0,p_{k+1})\right)\mapsto(q_k^0,q_k^1)\oplus(q_k^0,p_{k+1}), \tag{7.4}$$

$$\tau_{(Q\times Q)\oplus\mathcal{H}_-} : \left((q_{k-1}^0,q_{k-1}^1)\oplus(q_{k-1}^1,p_k),(q_k^0,q_k^1)\oplus(q_k^1,p_{k+1})\right)\mapsto(q_k^0,q_k^1)\oplus(p_k,q_k^1), \tag{7.5}$$

and $pr^{d\pm}_{T^*Q}: (Q\times Q)\oplus T^*Q \to T^*Q$ given by

$$pr^{d+}_{T^*Q}: (q^0_k, q^1_k)\oplus(q^0_k, p_k) \mapsto (q^0_k, p_k), \tag{7.6}$$
$$pr^{d-}_{T^*Q}: (q^0_{k-1}, q^1_{k-1})\oplus(q^1_{k-1}, p_k) \mapsto (q^1_{k-1}, p_k). \tag{7.7}$$

Recall the projections $\rho^{d\pm}_{(T^*Q)^2}: T^*Q\times T^*Q \to (Q\times Q)\oplus\mathcal{H}_\pm$ given by

$$\rho^{d+}_{(T^*Q)^2}: \big((q^0_k, p_k), (q^0_{k+1}, p_{k+1})\big) \mapsto (q^0_k, q^0_{k+1})\oplus(q^0_k, p_{k+1}), \tag{7.8}$$
$$\rho^{d-}_{(T^*Q)^2}: \big((q^1_{k-1}, p_k), (q^1_k, p_{k+1})\big) \mapsto (q^1_{k-1}, q^1_k)\oplus(p_k, q^1_k). \tag{7.9}$$

The projections $\rho^{d\pm}_{(T^*Q)^2}$ and $pr^{d\pm}_{T^*Q}\times pr^{d\pm}_{T^*Q}$ can be combined to yield a second projection from $((Q\times Q)\oplus T^*Q)\times((Q\times Q)\oplus T^*Q)$ to $(Q\times Q)\oplus\mathcal{H}_\pm$ that is distinct from $\tau_{(Q\times Q)\oplus\mathcal{H}_\pm}$, namely,

$$\rho^{d+}_{(T^*Q)^2}\circ(pr^{d+}_{T^*Q}\times pr^{d+}_{T^*Q}): \big((q^0_k, q^1_k)\oplus(q^0_k, p_k), (q^0_{k+1}, q^1_{k+1})\oplus(q^0_{k+1}, p_{k+1})\big) \mapsto (q^0_k, q^0_{k+1})\oplus(q^0_k, p_{k+1}), \tag{7.10}$$

$$\rho^{d-}_{(T^*Q)^2}\circ(pr^{d-}_{T^*Q}\times pr^{d-}_{T^*Q}): \big((q^0_{k-1}, q^1_{k-1})\oplus(q^1_{k-1}, p_k), (q^0_k, q^1_k)\oplus(q^1_k, p_{k+1})\big) \mapsto (q^1_{k-1}, q^1_k)\oplus(p_k, q^1_k). \tag{7.11}$$

Now, recall the definitions of the $(\pm)$-discrete momentum functions from (6.9) and (6.10),

$$G_{d+}\left((q_0, q_1)\oplus(q_0, p_1)\right) = p_1 q_1, \tag{7.12}$$
$$G_{d-}\left((q_0, q_1)\oplus(p_0, q_1)\right) = -p_0 q_0, \tag{7.13}$$

and the $(\pm)$-discrete generalized energies from (6.13) and (6.14),

$$E_{d+}\left((q_0, q_1)\oplus(q_0, p_1)\right) = p_1 q_1 - L_d(q_0, q_1), \tag{7.14}$$
$$E_{d-}\left((q_0, q_1)\oplus(p_0, q_1)\right) = -p_0 q_0 - L_d(q_0, q_1). \tag{7.15}$$

We can now intrinsically characterize the $(\pm)$-discrete Hamilton–Pontryagin sums in terms of the above quantities. By direct computation, one can verify that the $(+)$-discrete Hamilton–Pontryagin sum can be expressed as a functional on

$$(x^+_k, x^+_{k+1}) = \big((q^0_k, q^1_k)\oplus(q^0_k, p_k), (q^0_{k+1}, q^1_{k+1})\oplus(q^0_{k+1}, p_{k+1})\big) \in ((Q\times Q)\oplus T^*Q)\times((Q\times Q)\oplus T^*Q), \tag{7.16}$$

which is given by

$$\begin{aligned}\sum L_d(q^0_k, q^1_k) - p_{k+1}(q^1_k - q^0_{k+1}) &= \sum p_{k+1}q^0_{k+1} - \big(p_{k+1}q^1_k - L_d(q^0_k, q^1_k)\big)\\ &= \sum\Big(G_{d+}\circ\rho^{d+}_{(T^*Q)^2}\circ(pr^{d+}_{T^*Q}\times pr^{d+}_{T^*Q}) - E_{d+}\circ\tau_{(Q\times Q)\oplus\mathcal{H}_+}\Big)(x^+_k, x^+_{k+1}).\end{aligned} \tag{7.17}$$

Similarly, the $(-)$-discrete Hamilton–Pontryagin sum can be expressed as a functional on

$$(x^-_{k-1}, x^-_k) = \big((q^0_{k-1}, q^1_{k-1})\oplus(q^1_{k-1}, p_k), (q^0_k, q^1_k)\oplus(q^1_k, p_{k+1})\big) \in ((Q\times Q)\oplus T^*Q)\times((Q\times Q)\oplus T^*Q), \tag{7.18}$$

which is given by

$$\begin{aligned}\sum L_d(q^0_k, q^1_k) - p_k(q^1_{k-1} - q^0_k) &= \sum -p_k q^1_{k-1} - \big(-p_k q^0_k - L_d(q^0_k, q^1_k)\big)\\ &= \sum\Big(G_{d-}\circ\rho^{d-}_{(T^*Q)^2}\circ(pr^{d-}_{T^*Q}\times pr^{d-}_{T^*Q}) - E_{d-}\circ\tau_{(Q\times Q)\oplus\mathcal{H}_-}\Big)(x^-_{k-1}, x^-_k).\end{aligned} \tag{7.19}$$

The *intrinsic implicit $(\pm)$-discrete Euler–Lagrange equations* are given by

$$\Big(pr^{d\pm}_{T^*Q}\times pr^{d\pm}_{T^*Q}\Big)^*\chi_{d\pm} = \tau^*_{(Q\times Q)\oplus\mathcal{H}_\pm} dE_{d\pm}. \tag{7.20}$$

Since we have previously shown that the discrete Hamilton–Pontryagin principle yields the implicit discrete Euler–Lagrange equations, it remains to show that (7.20) in coordinates recover (7.3). We compute the one-forms on the left hand side of each equation,

$$d\sum p_{k+1}q^0_{k+1} = \sum\left[p_{k+1}dq^0_{k+1} + q^0_{k+1}dp_{k+1}\right]$$

$$
\begin{aligned}
&= \sum \left[p_k dq_k^0 + q_{k+1}^0 dp_{k+1}\right] + \sum \left[p_{k+1} dq_{k+1}^0 - p_k dq_k^0\right] \\
&= \sum \left[p_k dq_k^0 + q_{k+1}^0 dp_{k+1}\right] + p_N dq_N^0 - p_0 dq_0^0, \qquad (7.21)
\end{aligned}
$$

and

$$
\begin{aligned}
d\sum \left[-p_k q_{k-1}^1\right] &= \sum \left[-p_k dq_{k-1}^1 - q_{k-1}^1 dp_k\right] \\
&= \sum \left[-p_{k+1} dq_k^1 - q_{k-1}^1 dp_k\right] + \sum \left[p_{k+1} dq_k^1 - p_k dq_{k-1}^1\right] \\
&= \sum \left[-p_{k+1} dq_k^1 - q_{k-1}^1 dp_k\right] + p_N dq_{N-1}^1 - p_0 dq_{-1}^1. \qquad (7.22)
\end{aligned}
$$

Since we only evaluate these one-forms on discrete curves with fixed endpoints, the boundary terms vanish, and it is sufficient to show that the expressions in the brackets of (7.21) and (7.22) agree with the left hand side of (7.20). By direct computation, we obtain

$$
\begin{aligned}
\left(pr_{T^*Q}^{d+} \times pr_{T^*Q}^{d+}\right)^* \chi_{d+}(x_k^+, x_{k+1}^+) &= \left(\chi_{d+} \circ \left(pr_{T^*Q}^{d+} \times pr_{T^*Q}^{d+}\right)\right)(x_k^+, x_{k+1}^+) \\
&= \chi_{d+}\left(pr_{T^*Q}^{d+}(x_k^+), pr_{T^*Q}^{d+}(x_{k+1}^+)\right) \\
&= \chi_{d+}\left((q_k^0, p_k), (q_{k+1}^0, p_{k+1})\right) \\
&= p_k dq_k^0 + q_{k+1}^0 dp_{k+1}, \qquad (7.23)
\end{aligned}
$$

and

$$
\begin{aligned}
\left(pr_{T^*Q}^{d-} \times pr_{T^*Q}^{d-}\right)^* \chi_{d-}(x_{k-1}^-, x_k^-) &= \left(\chi_{d-} \circ \left(pr_{T^*Q}^{d-} \times pr_{T^*Q}^{d-}\right)\right)(x_{k-1}^-, x_k^-) \\
&= \chi_{d-}\left(pr_{T^*Q}^{d-}(x_{k-1}^-), pr_{T^*Q}^{d-}(x_k^-)\right) \\
&= \chi_{d-}\left((q_{k-1}^1, p_k), (q_k^1, p_{k+1})\right) \\
&= -p_{k+1} dq_k^1 - q_{k-1}^1 dp_k. \qquad (7.24)
\end{aligned}
$$

Computing the right hand side of (7.20) yields

$$
\begin{aligned}
\left(\tau_{(Q\times Q)\oplus\mathcal{H}_+}^* dE_{d+}\right)(x_k^+, x_{k+1}^+) &= d(E_{d+} \circ \tau_{(Q\times Q)\oplus\mathcal{H}_+})(x_k^+, x_{k+1}^+) \\
&= d\left(E_{d+}\left((q_k^0, q_k^1) \oplus (q_k^0, p_{k+1})\right)\right) \\
&= d\left(p_{k+1} q_k^1 - L_d(q_k^0, q_k^1)\right) \\
&= p_{k+1} dq_k^1 + q_k^1 dp_{k+1} - D_1 L_d(q_k^0, q_k^1) dq_k^0 - D_2 L_d(q_k^0, q_k^1) dq_k^1 \\
&= -D_1 L_d(q_k^0, q_k^1) dq_k^0 + \left(p_{k+1} - D_2 L_d(q_k^0, q_k^1)\right) dq_k^1 + q_k^1 dp_{k+1}, \qquad (7.25)
\end{aligned}
$$

and

$$
\begin{aligned}
\left(\tau_{(Q\times Q)\oplus\mathcal{H}_-}^* dE_{d-}\right)(x_{k-1}^-, x_k^-) &= d\left(E_{d-} \circ \tau_{(Q\times Q)\oplus\mathcal{H}_-}\right)(x_{k-1}^-, x_k^-) \\
&= d\left(E_{d-}\left((q_k^0, q_k^1) \oplus (p_k, q_k^1)\right)\right) \\
&= d\left(-p_k q_k^0 - L_d(q_k^0, q_k^1)\right) \\
&= -p_k dq_k^0 - q_k^0 dp_k - D_1 L_d(q_k^0, q_k^1) dq_k^0 - D_2 L_d(q_k^0, q_k^1) dq_k^1 \\
&= \left(-p_k - D_1 L_d(q_k^0, q_k^1)\right) dq_k^0 - D_2 L_d(q_k^0, q_k^1) dq_k^1 - q_k^0 dp_k. \qquad (7.26)
\end{aligned}
$$

Equating (7.23) with (7.25), and (7.24) with (7.26) yield

$$
p_k dq_k^0 + q_{k+1}^0 dp_{k+1} = -D_1 L_d(q_k^0, q_k^1) dq_k^0 + \left(p_{k+1} - D_2 L_d(q_k^0, q_k^1)\right) dq_k^1 + q_k^1 dp_{k+1}, \qquad (7.27)
$$

and

$$
-p_{k+1} dq_k^1 - q_{k-1}^1 dp_k = \left(-p_k - D_1 L_d(q_k^0, q_k^1)\right) dq_k^0 - D_2 L_d(q_k^0, q_k^1) dq_k^1 - q_k^0 dp_k, \qquad (7.28)
$$

which both recover the implicit discrete Euler–Lagrange equations,

$$p_{k+1} = D_2 L_d(q_k^0, q_k^1), \qquad p_k = -D_1 L_d(q_k^0, q_k^1), \qquad q_k^1 = q_{k+1}^0. \tag{7.29}$$

8. Discrete Lagrange–d'Alembert–Pontryagin principle

The (+)-discrete Lagrange–d'Alembert–Pontryagin principle is given by

$$\begin{aligned}\delta \sum \left[L_d(q_k^0, q_k^1) - p_{k+1}(q_k^1 - q_{k+1}^0) \right] \\ = \sum \left\{ \left[D_1 L_d(q_k^0, q_k^1) + p_k \right] \delta q_k^0 + \left[D_2 L_d(q_k^0, q_k^1) - p_{k+1} \right] \delta q_k^1 + \left[q_k^1 - q_{k+1}^0 \right] \delta p_{k+1} \right\} = 0, \end{aligned} \tag{8.1}$$

for fixed endpoints q_0^0 and q_N^0, and variations $(\delta q_k^0, \delta q_k^1, \delta p_k)$ of $(q_k^0, q_k^1, p_k) \in (Q \times Q) \oplus T^*Q$ such that $\delta q_k^0 \in \Delta_Q(q_k^0)$, $\delta q_k^1 \in \Delta_Q(q_k^1)$, and the discrete constraint $(q_k^0, q_k^1) \in \Delta_Q^d$.

Similarly, the (−)-discrete Lagrange–d'Alembert–Pontryagin principle is given by

$$\begin{aligned}\delta \sum \left[L_d(q_k^0, q_k^1) - p_k(q_{k-1}^1 - q_k^0) \right] \\ = \sum \left\{ \left[D_1 L_d(q_k^0, q_k^1) + p_k \right] \delta q_k^0 + \left[D_2 L_d(q_k^0, q_k^1) - p_{k+1} \right] \delta q_k^1 + \left[q_{k-1}^1 - q_k^0 \right] \delta p_k \right\} = 0, \end{aligned} \tag{8.2}$$

for fixed endpoints q_{-1}^1 and q_{N-1}^1 and variations $(\delta q_k^0, \delta q_k^1, \delta p_k)$ of $(q_k^0, q_k^1, p_k) \in (Q \times Q) \oplus \mathcal{H}_-$ such that $\delta q_k^0 \in \Delta_Q(q_k^0)$, $\delta q_k^1 \in \Delta_Q(q_k^1)$, and the discrete constraint $(q_k^0, q_k^1) \in \Delta_Q^d$.

Proposition 8.1. *The (±)-discrete Lagrange–d'Alembert–Pontryagin principles for the discrete curve $\{x_k^\pm\}$, $x_k^\pm \in (Q \times Q) \oplus T^*Q$, yield discrete equations that are given in coordinates by*

$$D_1 L_d(q_k^0, q_k^1) + p_k \in \Delta_Q^\circ(q_k^0), \qquad D_2 L_d(q_k^0, q_k^1) - p_{k+1} \in \Delta_Q^\circ(q_k^1), \qquad q_k^1 = q_{k+1}^0, \qquad (q_k^0, q_k^1) \in \Delta_Q^d. \tag{8.3}$$

These equations are equivalent to the coordinate expressions for implicit discrete Lagrangian systems.

8.1. Constraint distributions on $((Q \times Q) \oplus T^*Q) \times ((Q \times Q) \oplus T^*Q)$.

We will introduce constraint distributions for the intrinsic form of the discrete Lagrange–d'Alembert–Pontryagin principle.

Consider the discrete constraint distribution $\Delta_Q^d \subset Q \times Q$. Recall from Section 4.2.3 that this induces a continuous constraint distribution $\Delta_Q \subset TQ$. In addition, Δ_Q^d also induces the submanifolds $\mathcal{K}_{d\pm} \subset (Q \times Q) \oplus \mathcal{H}_\pm$ given by (6.17) and (6.19), which were defined in Section 6.4 as

$$\mathcal{K}_{d+} = \left\{ (q_0, q_1, p_1) \in (Q \times Q) \oplus \mathcal{H}_+ \mid (q_0, q_1) \in \Delta_Q^d, p_1 = D_2 L_d(q_0, q_1) \right\}, \tag{8.4}$$

and

$$\mathcal{K}_{d-} = \left\{ (q_0, q_1, p_0) \in (Q \times Q) \oplus \mathcal{H}_- \mid (q_0, q_1) \in \Delta_Q^d, p_0 = -D_1 L_d(q_0, q_1) \right\}. \tag{8.5}$$

Recall also the projection map $\tau_{(Q\times Q)\oplus\mathcal{H}_\pm} : ((Q \times Q) \oplus T^*Q) \times ((Q \times Q) \oplus T^*Q) \to ((Q \times Q) \oplus \mathcal{H}_\pm)$, which induces a discrete constraint distribution $\mathcal{A}_{d\pm}$ on $(Q \times Q) \oplus T^*Q$ by

$$\mathcal{A}_{d\pm} := \left(\tau_{(Q\times Q)\oplus\mathcal{H}_\pm} \right)^{-1} (\mathcal{K}_{d\pm}), \tag{8.6}$$

that encode $(q_k^0, q_k^1) \in \Delta_Q^d$, $(q_{k+1}^0, q_{k+1}^1) \in \Delta_Q^d$, and the appropriate (±)-discrete momentum constraints.

We introduce the natural projection $pr_Q^d : (Q \times Q) \oplus T^*Q \to Q$, given by

$$pr_Q^d : (q_k^0, q_k^1) \oplus (q_k^0, p_{k+1}) \mapsto q_k^0, \tag{8.7}$$

from which we obtain the discrete constraint distribution $\mathcal{B}_d \subset ((Q \times Q) \oplus T^*Q) \times ((Q \times Q) \oplus T^*Q)$ by

$$\mathcal{B}_d := \left(pr_Q^d \times pr_Q^d \right)^{-1} (\Delta_Q^d), \tag{8.8}$$

which encodes the constraint $(q_k^0, q_{k+1}^0) \in \Delta_Q^d$.

Let $\mathcal{C}_{d\pm}$ be the intersection of the discrete constraint distribution $\mathcal{B}_d$ with $\mathcal{A}_{d\pm}$, which is given by

$$\mathcal{C}_{d\pm} := \mathcal{B}_d \cap \mathcal{A}_{d\pm} \subset ((Q \times Q) \oplus T^*Q) \times ((Q \times Q) \oplus T^*Q). \tag{8.9}$$

We introduce another pair of natural projections

$$pr_{(TQ)^4}^{d\pm} : T\left(((Q \times Q) \oplus T^*Q) \times ((Q \times Q) \oplus T^*Q) \right) \to TQ \times TQ \times TQ \times TQ, \tag{8.10}$$

given by

$$pr^{d+}_{(TQ)^4} : \left(\left((q^0_k, q^1_k, p_k), (q^0_{k+1}, q^1_{k+1}, p_{k+1})\right), \left((\delta q^0_k, \delta q^1_k, \delta p_k), (\delta q^0_{k+1}, \delta q^1_{k+1}, \delta p_{k+1})\right)\right)$$
$$\mapsto \left((q^0_k, \delta q^0_k), (q^1_k, \delta q^1_k), (q^0_{k+1}, \delta q^0_{k+1}), (q^1_{k+1}, \delta q^1_{k+1})\right), \tag{8.11}$$
$$pr^{d-}_{(TQ)^4} : \left(\left((q^0_k, q^1_k, p_{k+1}), (q^0_{k+1}, q^1_{k+1}, p_{k+2})\right), \left((\delta q^0_k, \delta q^1_k, \delta p_{k+1}), (\delta q^0_{k+1}, \delta q^1_{k+1}, \delta p_{k+2})\right)\right)$$
$$\mapsto \left((q^0_k, \delta q^0_k), (q^1_k, \delta q^1_k), (q^0_{k+1}, \delta q^0_{k+1}), (q^1_{k+1}, \delta q^1_{k+1})\right), \tag{8.12}$$

which lift $\Delta_Q \times \Delta_Q \times \Delta_Q \times \Delta_Q$ to continuous distributions $\mathcal{F}_\pm$ on $((Q \times Q) \oplus T^*Q) \times ((Q \times Q) \oplus T^*Q)$

$$\mathcal{F}_\pm := \left(pr^{d\pm}_{(TQ)^4}\right)^{-1} (\Delta_Q \times \Delta_Q \times \Delta_Q \times \Delta_Q) \subset T\left(((Q \times Q) \oplus T^*Q) \times ((Q \times Q) \oplus T^*Q)\right), \tag{8.13}$$

that encode the constraints $\delta q^0_k \in \Delta_Q(q^0_k)$, $\delta q^1_k \in \Delta_Q(q^1_k)$, $\delta q^0_{k+1} \in \Delta_Q(q^0_{k+1})$, and $\delta q^1_{k+1} \in \Delta_Q(q^1_{k+1})$.

Let $\mathcal{G}_\pm$ be continuous distributions on $((Q \times Q) \oplus T^*Q) \times ((Q \times Q) \oplus T^*Q)$ that are obtained by restricting $\mathcal{F}_\pm$ to $\mathcal{C}_{d\pm}$,

$$\mathcal{G}_\pm := \mathcal{F}_\pm \cap T\mathcal{C}_{d\pm} \subset T\left(((Q \times Q) \oplus T^*Q) \times ((Q \times Q) \oplus T^*Q)\right). \tag{8.14}$$

As we will see, these constraint distributions arise in the intrinsic formulation of the discrete Lagrange–d'Alembert–Pontryagin principle.

8.2. **Intrinsic discrete Lagrange–d'Alembert–Pontryagin principle.** The (+)-discrete Lagrange–d'Alembert–Pontryagin principle for a discrete curve $\{x^+_k\}$ on $(Q \times Q) \oplus T^*Q$ with fixed endpoints is given by

$$\delta \sum \left(G_{d+} \circ \rho^{d+}_{(T^*Q)^2} \circ (pr^{d+}_{T^*Q} \times pr^{d+}_{T^*Q}) - E_{d+} \circ \tau_{(Q\times Q)\oplus\mathcal{H}_+}\right)(x^+_k, x^+_{k+1})$$
$$= \sum \left[\left(pr^{d+}_{T^*Q} \times pr^{d+}_{T^*Q}\right)^* \chi_{d+} - \tau^*_{(Q\times Q)\oplus\mathcal{H}_+} dE_{d+}\right](x^+_k, x^+_{k+1}) \cdot (w^+_{k,k+1}), \tag{8.15}$$

which holds for all $w^+_{k,k+1} \in \mathcal{G}_+\left(x^+_k, x^+_{k+1}\right) \subset T\left(((Q \times Q) \oplus T^*Q) \times ((Q \times Q) \oplus T^*Q)\right)$.

Similarly, the (−)-discrete Lagrange–d'Alembert–Pontryagin principle for a discrete curve $\{x^-_k\}$ on $(Q \times Q) \oplus T^*Q$ with fixed endpoints is given by

$$\delta \sum \left(G_{d-} \circ \rho^{d-}_{(T^*Q)^2} \circ (pr^{d-}_{T^*Q} \times pr^{d-}_{T^*Q}) - E_{d-} \circ \tau_{(Q\times Q)\oplus\mathcal{H}_-}\right)(x^-_{k-1}, x^-_k)$$
$$= \sum \left[\left(pr^{d-}_{T^*Q} \times pr^{d-}_{T^*Q}\right)^* \chi_{d-} - \tau^*_{(Q\times Q)\oplus\mathcal{H}_-} dE_{d-}\right](x^-_{k-1}, x^-_k) \cdot (w^-_{k-1,k}), \tag{8.16}$$

which holds for all $w^-_{k-1,k} \in \mathcal{G}_-\left(x^-_{k-1}, x^-_k\right) \subset T\left(((Q \times Q) \oplus T^*Q) \times ((Q \times Q) \oplus T^*Q)\right)$.

Proposition 8.2. *The* (±)*-discrete Lagrange–d'Alembert–Pontryagin principles are equivalent to the equations,*

$$\left(pr^{d+}_{T^*Q} \times pr^{d+}_{T^*Q}\right)^* \chi_{d+}\left(x^+_k, x^+_{k+1}\right) \cdot \left(w^+_{k,k+1}\right) = \tau^*_{(Q\times Q)\oplus\mathcal{H}_+} dE_{d+}\left(x^+_k, x^+_{k+1}\right) \cdot \left(w^+_{k,k+1}\right), \tag{8.17}$$

for all $w^+_{k,k+1} \in \mathcal{G}_+\left(x^+_k, x^+_{k+1}\right)$*, and*

$$\left(pr^{d-}_{T^*Q} \times pr^{d-}_{T^*Q}\right)^* \chi_{d-}\left(x^-_{k-1}, x^-_k\right) \cdot \left(w^-_{k-1,k}\right) = \tau^*_{(Q\times Q)\oplus\mathcal{H}_-} dE_{d-}\left(x^-_{k-1}, x^-_k\right) \cdot \left(w^-_{k-1,k}\right), \tag{8.18}$$

for all $w^-_{k-1,k} \in \mathcal{G}_-\left(x^-_{k-1}, x^-_k\right)$*, respectively.*

8.3. **Intrinsic form of implicit discrete Lagrangian systems.** We first introduce (±)-discrete vector fields on M, denoted by $\mathfrak{X}_{d\pm}(M)$, to be maps $X_{d\pm} : M \to M \times M$, such that $\pi^0_M \circ X_{d+} = \mathbf{1}_M$ and $\pi^1_M \circ X_{d-} = \mathbf{1}_M$, where $\pi^i_M : M \times M \to M; (m_0, m_1) \mapsto m_i$.

Let $X_{d\pm}$ be (±)-discrete vector fields on $P_\pm$, where $P_\pm = \mathbb{F}L_{d\pm}(\Delta^d_Q)$, and let $\widetilde{X}_{d\pm}$ be (±)-discrete vector fields whose images are in $\mathcal{A}_{d\pm} = \left(\tau_{(Q\times Q)\oplus\mathcal{H}_\pm}\right)^{-1}(\mathcal{K}_{d\pm})$. Consider the discrete integral curves $\{z^\pm_k\}$ on T^*Q of $X_{d\pm}$, where $z^\pm_k = (q^\pm_k, p^\pm_k)$, i.e., $(z^+_k, z^+_{k+1}) = X_{d+}(z^+_k)$ and $(z^-_{k-1}, z^-_k) = X_{d-}(z^-_k)$. We consider discrete curves $\{x_k^{\pm}\}$ on $(Q\times Q)\oplus T^*Q$, where $x_k^+=(q_k^0,q_k^1)\oplus(q_k^0,p_k)$, and $x_k^-=(q_k^0,q_k^1)\oplus(q_k^1,p_{k+1})$, such that the second component is given by the curves $\{z_k^{\pm}\}$ on T^*Q. Explicitly, we have

$$x_k^+=(q_k^+,q_k^1)\oplus(q_k^+,p_k^+), \tag{8.19}$$
$$x_k^-=(q_k^0,q_{k+1}^-)\oplus(q_{k+1}^-,p_{k+1}^-). \tag{8.20}$$

Using the natural projections $pr_{T^*Q}^{d\pm}:(Q\times Q)\oplus T^*Q\to T^*Q$, which we previously introduced, we impose pointwise constraints on the first component,

$$(pr_{T^*Q}^{d\pm}\times pr_{T^*Q}^{d\pm})\left(\widetilde{X}_{d\pm}(x_0^{\pm})\right)=X_{d\pm}\left(pr_{T^*Q}^{d\pm}(x_0^{\pm})\right), \tag{8.21}$$

where $\widetilde{X}_{d+}(x_0^+)=(x_0^+,x_1^+)$, $\widetilde{X}_{d-}(x_0^-)=(x_{-1}^-,x_0^-)$, but the curve is otherwise arbitrary. Since we restrict the image of $\widetilde{X}_{d\pm}$ to $\mathcal{A}_{d\pm}$, we have the property,

$$(pr_{T^*Q}^{d\pm}\times pr_{T^*Q}^{d\pm})\left(\widetilde{X}_{d\pm}(x_k^{\pm})\right)=X_{d\pm}\left(pr_{T^*Q}^{d\pm}(x_k^{\pm})\right). \tag{8.22}$$

Notice that while (8.21) give pointwise conditions at $k=0$, when we restrict our lifted $(\pm)$-discrete vector fields to $\mathcal{A}_{d\pm}$, we obtain (8.22) that holds globally. If $\{x_k^{\pm}\}$, $x_k^+=(q_k^0,q_k^1)\oplus(q_k^0,p_k)\in(Q\times Q)\oplus T^*Q$, $x_k^-=(q_{k-1}^0,q_{k-1}^1)\oplus(q_{k-1}^1,p_k)\in(Q\times Q)\oplus T^*Q$, are discrete integral curves of $\widetilde{X}_{d\pm}$, then it follows that

$$\widetilde{X}_{d+}(x_k^+)=\left((q_k^0,q_k^1)\oplus(q_k^0,p_k),(q_{k+1}^0,q_{k+1}^1)\oplus(q_{k+1}^0,p_{k+1})\right), \tag{8.23}$$
$$\widetilde{X}_{d-}(x_k^-)=\left((q_{k-1}^0,q_{k-1}^1)\oplus(q_{k-1}^1,p_k),(q_k^0,q_k^1)\oplus(q_k^1,p_{k+1})\right). \tag{8.24}$$

Proposition 8.3. *Let $\{x_k^{\pm}\}$ be discrete integral curves of the $(\pm)$-discrete vector fields $\widetilde{X}_{d\pm}$ on $(Q\times Q)\oplus T^*Q$ that are naturally induced from $(\pm)$-discrete vector fields $X_{d\pm}$ on T^*Q. If $\{x_k^{\pm}\}$ are discrete solution curves of the $(\pm)$-discrete Lagrange–d'Alembert–Pontryagin principles, then they satisfy*

$$\left(pr_{T^*Q}^{d+}\times pr_{T^*Q}^{d+}\right)^*\chi_{d+}\left(\widetilde{X}_{d+}(x_k^+)\right)\cdot\left(w_{k,k+1}^+\right)=\tau^*_{(Q\times Q)\oplus\mathcal{H}_+}dE_{d+}\left(\widetilde{X}_{d+}(x_k^+)\right)\cdot\left(w_{k,k+1}^+\right), \tag{8.25}$$

for all $w_{k,k+1}^+\in\mathcal{G}_+\left(\widetilde{X}_{d+}(x_k^+)\right)$, and

$$\left(pr_{T^*Q}^{d-}\times pr_{T^*Q}^{d-}\right)^*\chi_{d-}\left(\widetilde{X}_{d-}(x_k^-)\right)\cdot\left(w_{k-1,k}^-\right)=\tau^*_{(Q\times Q)\oplus\mathcal{H}_-}dE_{d-}\left(\widetilde{X}_{d-}(x_k^-)\right)\cdot\left(w_{k-1,k}^-\right), \tag{8.26}$$

for all $w_{k-1,k}^-\in\mathcal{G}_-\left(\left(\widetilde{X}_{d-}(x_k^-)\right)\right)$.

Proof. Since $\{x_k^{\pm}\}$ are discrete integral curves of $\widetilde{X}_{d\pm}$, respectively, we obtain (8.23) and (8.24). Substituting the respective equations into (8.17) and (8.18) yields the desired result. □

We may now summarize our results in the following theorem.

Theorem 8.4. *Let L_d be a discrete Lagrangian on $Q\times Q$ and Δ_Q^d be a discrete constraint distribution on Q. Let $X_{d\pm}$ be $(\pm)$-discrete vector fields on $P_\pm=\mathbb{F}L_{d\pm}(\Delta_Q^d)$, such that $(L_d,\Delta_Q^d,X_{d\pm})$ are implicit $(\pm)$-discrete Lagrangian systems. Denote discrete curves on $(Q\times Q)\oplus T^*Q$ by $\{x_k^{\pm}\}$, where $x_k^+=(q_k^0,q_k^1,p_k^+)\in(Q\times Q)\oplus T^*Q$, $x_k^-=(q_k^0,q_k^1,p_{k+1}^-)\in(Q\times Q)\oplus T^*Q$. Then, the following are equivalent:*

(i) *$\{x_k^{\pm}\}$ are discrete solution curves of the implicit $(\pm)$-discrete Lagrangian systems $(L_d,\Delta_Q^d,X_{d\pm})$.*
(ii) *$\{x_k^{\pm}\}$ satisfy the $(\pm)$-discrete Lagrange–d'Alembert–Pontryagin principles.*
(iii) *$\{x_k^{\pm}\}$ are the discrete integral curves of $(\pm)$-discrete vector fields $\widetilde{X}_{d\pm}$ on $(Q\times Q)\oplus\mathcal{H}_\pm$ that are naturally induced from $X_{d\pm}$.*

9. Discrete Hamilton's phase space principle

By taking variations with respect to q_k^1 and q_k^0 in the discrete Hamilton–Pontryagin principle, we obtain

$$D_2 L_d(q_k^0, q_k^1) - p_{k+1} = 0, \tag{9.1}$$

$$D_1 L_d(q_k^0, q_k^1) + p_k = 0, \tag{9.2}$$

respectively. These equations define the discrete fiber derivatives, $\mathbb{F}L_d^{\pm} : Q \times Q \to T^*Q$, which are the discrete analogues of the Legendre transform, and are given by

$$\mathbb{F}L_d^+ : (q_k^0, q_k^1) \mapsto \left(q_k^1, D_2 L_d(q_k^0, q_k^1)\right), \tag{9.3}$$

$$\mathbb{F}L_d^- : (q_k^0, q_k^1) \mapsto \left(q_k^0, -D_1 L_d(q_k^0, q_k^1)\right). \tag{9.4}$$

When the discrete fiber derivatives are invertible, we may construct the lift $\sigma_{\mathcal{K}}^{d\pm} : \mathcal{H}_\pm \to \mathcal{K}_{d\pm}$, where we let $\Delta_Q^d = Q \times Q$. Since $\mathcal{K}_{d\pm} \subset (Q \times Q) \oplus \mathcal{H}_\pm$, we may, by a slight abuse of notation, let $\sigma_{\mathcal{K}}^{d\pm} : \mathcal{H}_\pm \to (Q \times Q) \oplus \mathcal{H}_\pm$.

We introduce the $(\pm)$-*discrete Hamiltonian*, $H_{d\pm} : \mathcal{H}_\pm \to \mathbb{R}$, which we define in terms of the discrete generalized energy $E_{d\pm} : (Q \times Q) \oplus \mathcal{H}_\pm \to \mathbb{R}$ to be

$$H_{d\pm} = E_{d\pm} \circ \sigma_{\mathcal{K}}^{d\pm}, \tag{9.5}$$

which agrees with the usual definition of the $(\pm)$-discrete Hamiltonians given by

$$H_{d+}(q_k^0, p_{k+1}) = p_{k+1} q_k^1 - L_d(q_k^0, q_k^1)\big|_{p_{k+1} = D_2 L_d(q_k^0, q_k^1)}, \tag{9.6}$$

$$H_{d-}(p_k, q_{k+1}^0) = -p_k q_k^0 - L_d(q_k^0, q_k^1)\big|_{p_k = -D_1 L_d(q_k^0, q_k^1)}, \tag{9.7}$$

or equivalently,

$$H_{d+}(q_k, p_{k+1}) = p_{k+1} q_{k+1} - L_d(q_k, q_{k+1})\big|_{p_{k+1} = D_2 L_d(q_k, q_{k+1})}, \tag{9.8}$$

$$H_{d-}(p_k, q_{k+1}) = -p_k q_k - L_d(q_k, q_{k+1})\big|_{p_k = -D_1 L_d(q_k, q_{k+1})}. \tag{9.9}$$

From the definition of the discrete Hamiltonian, we can express the discrete variational principle as

$$\delta \sum [p_{k+1} q_{k+1} - H_{d+}(q_k, p_{k+1})] = 0, \tag{9.10}$$

$$\delta \sum [-p_k q_k - H_{d-}(p_k, q_{k+1})] = 0, \tag{9.11}$$

where we keep the endpoints q_0 and q_N fixed. This gives the $(\pm)$-*discrete Hamilton's principles in phase space*.

9.1. Discrete Hamilton's equations.

By considering the $(\pm)$-discrete Hamilton's principle in phase space, we obtain,

$$\begin{aligned} 0 &= \delta \sum [p_{k+1} q_{k+1} - H_{d+}(q_k, p_{k+1})] \\ &= \sum \left\{ [q_{k+1} - D_2 H_{d+}(q_k, p_{k+1})]\, \delta p_{k+1} + [p_k - D_1 H_{d+}(q_k, p_{k+1})]\, \delta q_k \right\}, \end{aligned} \tag{9.12}$$

and

$$\begin{aligned} 0 &= \delta \sum [-p_k q_k - H_{d-}(p_k, q_{k+1})] \\ &= \sum \left\{ [-q_k - D_1 H_{d-}(p_k, q_{k+1})]\, \delta p_k + [-p_{k+1} - D_2 H_{d-}(p_k, q_{k+1})]\, \delta q_{k+1} \right\}, \end{aligned} \tag{9.13}$$

From which we obtain the $(+)$-*discrete Hamilton's equations*,

$$q_{k+1} = D_2 H_{d+}(q_k, p_{k+1}), \qquad p_k = D_1 H_{d+}(q_k, p_{k+1}), \tag{9.14}$$

and the $(-)$-*discrete Hamilton's equations*

$$q_k = -D_1 H_{d-}(p_k, q_{k+1}), \qquad p_{k+1} = -D_2 H_{d-}(p_k, q_{k+1}). \tag{9.15}$$

9.2. **Intrinsic discrete Hamilton's phase space principle.** We recall the two pairs of maps, $\rho^{d\pm}_{(T^*Q)^2} : T^*Q \times T^*Q \to (Q \times Q) \oplus \mathcal{H}_\pm$, and $\tau_{\mathcal{H}_\pm} : T^*Q \times T^*Q \to \mathcal{H}_\pm$. The intrinsic (+)-discrete Poincaré–Cartan sum is given by

$$\sum \left[p_{k+1}q_{k+1} - H_{d+}(q_k, p_{k+1})\right] = \sum \left[G_{d+} \circ \rho^{d+}_{(T^*Q)^2} - H_{d+} \circ \tau_{\mathcal{H}_+}\right](z_k, z_{k+1}), \tag{9.16}$$

and the intrinsic (−)-discrete Poincaré–Cartan sum is given by

$$\sum \left[-p_k q_k - H_{d-}(p_k, q_{k+1})\right] = \sum \left[G_{d-} \circ \rho^{d-}_{(T^*Q)^2} - H_{d-} \circ \tau_{\mathcal{H}_-}\right](z_k, z_{k+1}), \tag{9.17}$$

where $z_k = (q_k, p_k)$. We view both of these sums as functionals on $(z_k, z_{k+1}) \in T^*Q \times T^*Q$.

The intrinsic (±)-discrete Hamilton's equations are given by

$$\chi_{d\pm} = \tau^*_{\mathcal{H}_\pm} dH_{d\pm}. \tag{9.18}$$

Computing the exterior derivative of the discrete Poincaré–Cartan sums yield

$$\begin{aligned}
&d \sum \left[p_{k+1}q_{k+1} - H_{d+}(q_k, p_{k+1})\right] \\
&\quad = \sum \left[p_{k+1}dq_{k+1} + q_{k+1}dp_{k+1} - dH_{d+} \circ \tau_{\mathcal{H}_+}(z_k, z_{k+1})\right] \\
&\quad = \sum \left[p_k dq_k + q_{k+1}dp_{k+1}\right] - \sum \tau^*_{\mathcal{H}_+} dH_{d+}(z_k, z_{k+1}) + \sum \left[p_{k+1}dq_{k+1} - p_k dq_k\right] \\
&\quad = \sum \left[\chi_{d+}(z_k, z_{k+1})\right] - \sum \tau^*_{\mathcal{H}_+} dH_{d+}(z_k, z_{k+1}) + \left[p_N dq_N - p_0 dq_0\right] \\
&\quad = \sum \left[\chi_{d+} - \tau^*_{\mathcal{H}_+} dH_{d+}\right](z_k, z_{k+1}) + \left[p_N dq_N - p_0 dq_0\right],
\end{aligned} \tag{9.19}$$

and

$$\begin{aligned}
&d \sum \left[-p_k q_k - H_{d-}(p_k, q_{k+1})\right] \\
&\quad = \sum \left[-p_k dq_k - q_k dp_k - dH_{d-} \circ \tau_{\mathcal{H}_-}(z_k, z_{k+1})\right] \\
&\quad = \sum \left[-p_{k+1}dq_{k+1} - q_k dp_k\right] - \sum \tau^*_{\mathcal{H}_-} dH_{d-}(z_k, z_{k+1}) + \sum \left[p_{k+1}dq_{k+1} - p_k dq_k\right] \\
&\quad = \sum \left[\chi_{d-}(z_k, z_{k+1})\right] - \sum \tau^*_{\mathcal{H}_-} dH_{d-}(z_k, z_{k+1}) + \left[p_N dq_N - p_0 dq_0\right] \\
&\quad = \sum \left[\chi_{d-} \tau^*_{\mathcal{H}_-} dH_{d-}\right](z_k, z_{k+1}) + \left[p_N dq_N - p_0 dq_0\right].
\end{aligned} \tag{9.20}$$

Since we only evaluate these one-forms on discrete curves with fixed endpoints, the boundary terms vanish, and we recover (9.18).

10. Discrete Hamilton–d'Alembert principle in phase space

We show how the implicit discrete Hamiltonian systems can be derived from a generalization of the discrete Hamilton's principle in phase space, which we refer to as the *discrete Hamilton–d'Alembert principle in phase space.*

The (+)-discrete Hamilton–d'Alembert principle in phase space for a discrete curve $\{(q_k, p_k)\}$ in T^*Q is given by,

$$\delta \sum \left[p_{k+1}q_{k+1} - H_{d+}(q_k, p_{k+1})\right] = 0, \tag{10.1}$$

where we require that $(q_k, q_{k+1}) \in \Delta^d_Q$, and the variation, when the endpoints are kept fixed, is given by

$$\begin{aligned}
&\delta \sum \left[p_{k+1}q_{k+1} - H_{d+}(q_k, p_{k+1})\right] \\
&\quad = \sum \left\{\left[q_{k+1} - D_2 H_{d+}(q_k, p_{k+1})\right]\delta p_{k+1} + \left[p_k - D_1 H_{d+}(q_k, p_{k+1})\right]\delta q_k\right\},
\end{aligned} \tag{10.2}$$

where we choose constrained variations $\delta q_k \in \Delta_Q(q_k)$.

The (−)-discrete Hamilton–d'Alembert principle in phase space for a discrete curve $\{(q_k, p_k)\}$ in T^*Q is given by,

$$\delta \sum \left[-p_k q_k - H_{d-}(q_k, p_{k+1})\right] = 0, \tag{10.3}$$

where we require that $(q_k, q_{k+1}) \in \Delta_Q^d$, and the variation, when the endpoints are kept fixed, is given by

$$\begin{aligned}\delta \sum [-p_k q_k - H_{d-}(p_k, q_{k+1})] \\ = \sum \{[-q_k - D_1 H_{d-}(p_k, q_{k+1})]\, \delta p_k + [-p_{k+1} - D_2 H_{d-}(p_k, q_{k+1})]\, \delta q_{k+1}\}\,, \quad (10.4)\end{aligned}$$

where we choose constrained variations $\delta q_k \in \Delta_Q(q_k)$.

Note that if one starts with the $(\pm)$-discrete Lagrange–d'Alembert–Pontryagin principle, and optimize over q_k^1 and q_k^0, respectively, one obtains the $(\pm)$-discrete Hamilton–d'Alembert principle in phase space.

Proposition 10.1. *The $(\pm)$-discrete Hamilton–d'Alembert principle in phase space for a discrete curve $\{(q_k, p_k)\}$ in T^*Q give the implicit $(\pm)$-discrete Hamiltonian systems in (4.32) and (4.49), respectively.*

Proof. From (10.2), the $(+)$-discrete Hamilton–Pontryagin principle is equivalent to

$$[q_{k+1} - D_2 H_{d+}(q_k, p_{k+1})]\, \delta p_{k+1} + [p_k - D_1 H_{d+}(q_k, p_{k+1})]\, \delta q_k = 0, \tag{10.5}$$

for all $\delta q_k \in \Delta_Q(q_k)$, all δp_{k+1} and all $(q_k, q_{k+1}) \in \Delta_Q^d$. Thus, we obtain (4.35) which are (4.32) expressed in coordinates. Similarly, from (10.4), the $(-)$-discrete Hamilton–Pontryagin principle is equivalent to

$$[-q_k - D_1 H_{d-}(p_k, q_{k+1})]\, \delta p_k + [-p_{k+1} - D_2 H_{d-}(p_k, q_{k+1})]\, \delta q_{k+1} = 0, \tag{10.6}$$

for all $\delta q_{k+1} \in \Delta_Q(q_{k+1})$, all δp_k, and all $(q_k, q_{k+1}) \in \Delta_Q^d$. Thus, we obtain (4.52) which are (4.49) expressed in coordinates. □

10.1. **Constraint distribution on $T^*Q \times T^*Q$.** As with the intrinsic formulation of the discrete Lagrange–d'Alembert–Pontryagin principle, we require a constraint distribution on $T^*Q \times T^*Q$ to formulate the discrete Hamilton–d'Alembert principle intrinsically.

Define a discrete constraint distribution on T^*Q by

$$\Delta_{T^*Q}^d := (\pi_Q \times \pi_Q)^{-1} \left(\Delta_Q^d\right) \subset T^*Q \times T^*Q, \tag{10.7}$$

where $\pi_Q : T^*Q \to Q$. Recall the discrete fiber derivatives, $\mathbb{F}L_d^{\pm} : Q \times Q \to T^*Q$. Then, we let $P_{\pm}$ be the image of the discrete constraint distribution Δ_Q^d under the discrete fiber derivatives, i.e., $P_{\pm} = \mathbb{F}L_d^{\pm}\left(\Delta_Q^d\right) \subset T^*Q$. Let $\Delta_{P_{\pm}}^d$ be the restriction of $\Delta_{T^*Q}^d$ to $P_{\pm}$, which is given by

$$\Delta_{P_{\pm}}^d := \Delta_{T^*Q}^d \cap (P_{\pm} \times P_{\pm}) \subset T^*Q \times T^*Q. \tag{10.8}$$

We canonically identify $T(Q \times Q)$ with $TQ \times TQ$ and, by a slight abuse of notation, write $T(\pi_Q \times \pi_Q) : T(T^*Q \times T^*Q) \to TQ \times TQ$. Define the continuous constraint distribution on $T^*Q \times T^*Q$ by

$$\mathcal{I} := (T(\pi_Q \times \pi_Q))^{-1} (\Delta_Q \times \Delta_Q) \subset T(T^*Q \times T^*Q)\,. \tag{10.9}$$

Then, we let $\mathcal{J}_{\pm}$ be the restriction of $\mathcal{I}$ to $\Delta_{P_{\pm}}^d$, given by

$$\mathcal{J}_{\pm} := \mathcal{I} \cap T\Delta_{P_{\pm}}^d \subset T(T^*Q \times T^*Q)\,. \tag{10.10}$$

10.2. **Intrinsic discrete Hamilton–d'Alembert principle.** The $(\pm)$-discrete Hamilton–d'Alembert principles for discrete curves $\{z_k^{\pm}\}$ on T^*Q with fixed endpoints are given by

$$\delta \sum \left[G_{d\pm} \circ \rho_{(T^*Q)^2}^{d\pm} - H_{d\pm} \circ \tau_{\mathcal{H}_{\pm}}\right](z_k^{\pm}, z_{k+1}^{\pm}) = \sum \left[\chi_{d\pm} - \tau_{\mathcal{H}_{\pm}}^* dH_{d\pm}\right](z_k^{\pm}, z_{k+1}^{\pm}) \cdot \left(s_{k,k+1}^{\pm}\right), \tag{10.11}$$

which holds for all $s_{k,k+1}^{\pm} \in \mathcal{J}_{\pm}(z_k^{\pm}, z_{k+1}^{\pm}) \subset T(T^*Q \times T^*Q)$. This yields the following two propositions.

Proposition 10.2. *The $(\pm)$-discrete Hamilton–d'Alembert principles are equivalent to the equations,*

$$\chi_{d\pm}\left(z_k^{\pm}, z_{k+1}^{\pm}\right) \cdot \left(s_{k,k+1}^{\pm}\right) = \tau_{\mathcal{H}_{\pm}}^* dH_{d\pm}\left(z_k^{\pm}, z_{k+1}^{\pm}\right) \cdot \left(s_{k,k+1}^{\pm}\right), \tag{10.12}$$

for all $s_{k,k+1}^{\pm} \in \mathcal{J}_{\pm}(z_k^{\pm}, z_{k+1}^{\pm})$.

Proposition 10.3. *Let $\{z_k^{\pm}\}$ be discrete integral curves of the $(\pm)$-discrete vector fields $X_{d\pm}$ on T^*Q. If $\{z_k^{\pm}\}$ are discrete solution curves of the $(\pm)$-discrete Hamilton–Pontryagin principles, then they satisfy*

$$\chi_{d+}\left(X_{d+}(z_k^+)\right)\cdot\left(s_{k,k+1}^+\right)=\tau_{\mathcal{H}_+}^* dH_{d+}\left(X_{d+}(z_k^+)\right)\cdot\left(s_{k,k+1}^+\right), \tag{10.13}$$

for all $s_{k,k+1}^+\in\mathcal{J}_+\left(X_{d+}(z_k^+)\right)$, and

$$\chi_{d-}\left(X_{d-}(z_{k+1}^-)\right)\cdot\left(s_{k,k+1}^-\right)=\tau_{\mathcal{H}_-}^* dH_{d-}\left(X_{d-}(z_{k+1}^-)\right)\cdot\left(s_{k,k+1}^-\right), \tag{10.14}$$

for all $s_{k,k+1}^-\in\mathcal{J}_-\left(X_{d-}(z_{k+1}^-)\right)$, respectively.

Proof. Since $\{z_k^{\pm}\}$ are discrete integral curves of $X_{d\pm}$, we obtain, $X_{d+}(z_k^+)=(z_k^+,z_{k+1}^-)$ and $X_{d-}(z_{k+1}^-)=(z_k^-,z_{k+1}^-)$, which together with (10.12) yield the desired results. □

We may now summarize our results in the following theorem.

Theorem 10.4. *Consider $(\pm)$-discrete Hamiltonians on $\mathcal{H}_\pm$ and a given discrete constraint distribution Δ_Q^d on Q. Let $X_{d\pm}$ be $(\pm)$-discrete vector fields on T^*Q, such that $(H_{d\pm},\Delta_Q^d,X_{d\pm})$ are implicit $(\pm)$-discrete Hamiltonian systems. Let $\{z_k^{\pm}\}=\{(q_k^{\pm},p_k^{\pm})\}$ be discrete curves on T^*Q. Then, the following are equivalent:*

(i) *$\{z_k^{\pm}\}$ are discrete solution curves of the implicit $(\pm)$-discrete Hamiltonian systems $(H_{d\pm},\Delta_Q^d,X_{d\pm})$.*
(ii) *$\{z_k^{\pm}\}$ satisfy the $(\pm)$-discrete Hamilton–d'Alembert principles in phase space.*
(iii) *$\{z_k^{\pm}\}$ are the discrete integral curves of the $(\pm)$-discrete vector fields $X_{d\pm}$ on T^*Q.*

11. Conclusion

In this paper, we developed the theoretical foundations of discrete variational mechanics arising from the discrete Hamilton–Pontryagin principle, which is a discrete variational principle that provides a unified treatment of discrete Lagrangian and Hamiltonian mechanics, as well as the discrete analogue of Dirac structures that unifies symplectic and Poisson structures. We demonstrate that just as variational integrators preserve a discrete symplectic structure, Hamilton–Pontryagin integrators preserve a discrete Dirac structure.

Discrete Dirac structures are intimately related to the geometry of Lagrangian submanifolds, which arise in the geometric description of the Hamilton–Jacobi equation and the theory of canonical transformations. Indeed, our construction of a discrete Dirac structure is given in terms of the maps $\kappa_Q^d: T^*Q\times T^*Q\to T^*(Q\times Q)$ and $\Omega_{d\pm}^\flat: T^*Q\times T^*Q\to T^*\mathcal{H}_\pm$, that are obtained naturally from the geometric description of generating functions of symplectic maps. This yields a discrete geometric formulation of implicit discrete Lagrangian and Hamiltonian systems, thereby providing a unified theoretical foundation for developing geometric numerical integrators for degenerate, interconnected, and nonholonomic Lagrangian and Hamiltonian systems.

We introduced an extended discrete variational principle, called the discrete Hamilton–Pontryagin principle that is valid for all discrete Lagrangians, and provide an intrinsic description that is valid semi-globally. Discrete constraints have been incorporated by the introduction of the discrete Lagrange–d'Alembert–Pontryagin principle, and the corresponding discrete Hamiltonian description is given in terms of the discrete Hamilton's principle in phase space, and the discrete Hamilton–d'Alembert principle in phase space. These discrete variational principles establish a link between discrete variational structures, discrete Dirac structures, and implicit discrete Lagrangian and Hamiltonian systems.

While this paper is motivated by the desire to understand the geometry of Dirac integrators in the context of geometric numerical integration, it is worthwhile to note that the discrete Lagrangians and Hamiltonians in our theory are simply generating functions of Types 1, 2, 3, and as such, the resulting theory of discrete Dirac mechanics provides a general characterization of near-identity Dirac maps.

Several interesting topics for future work are suggested by the theoretical developments introduced in this paper:

- Discrete Dirac formulation of the discrete Hamilton–Jacobi equation [12], with a view towards developing a discrete analogue of the Hamilton–Jacobi theory for nonholonomic systems [15]. Since discrete Dirac structures are related to Lagrangian submanifolds, which in turn describe the geometry of the Hamilton–Jacobi equation, it is natural to explore the Dirac description of Hamilton–Jacobi theory, as Dirac structure can incorporate nonholonomic constraints and thereby provide a unified treatment of both the classical and nonholonomic Hamilton–Jacobi theory.

- Galerkin variational Hamiltonian integrators for degenerate systems, by a careful application of the generalized discrete Legendre transformation to Galerkin variational Lagrangian integrators. The Galerkin approach to constructing symplectic methods on the Lagrangian side leverages results in approximation theory to obtain generalizations of symplectic methods that incorporate adaptive, multiscale, and spectral techniques (Chapter 5 of [22]). By embedding the theory of Galerkin variational integrators into discrete Dirac mechanics, and considering the Hamiltonian analogue, we will obtain a general framework for constructing symplectic methods with prescribed numerical approximation properties for degenerate Hamiltonian systems, such as point vortices [26].
- Discrete reduction theory for discrete Dirac mechanics with symmetry. The Dirac formulation of reduction provides a means of unifying symplectic, Poisson, nonholonomic, Lagrangian, and Hamiltonian reduction theory, as well as addressing the issue of reduction by stages. The discrete analogue of Dirac reduction will proceed by considering the issue of quotient discrete Dirac structures, and constructing a category containing discrete Dirac structures, that is closed under quotients.
- Discrete multi-Dirac mechanics for Hamiltonian partial differential equations. Dirac generalizations of multisymplectic field theory, and their corresponding discretizations will provide important insights into the construction of geometric numerical methods for degenerate field theories, such as the Einstein equations of general relativity.

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