Classification of hyperbolicity and stability preservers

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A linear operator T on $\mathbb{C}[z]$ is called *hyperbolicity-preserving* or an *HPO* for short if $T(P)$ is hyperbolic whenever $P \in \mathbb{C}[z]$ is hyperbolic, i.e., it has all real zeros. One of the main challenges in the theory of univariate complex polynomials is to describe the monoid $\mathcal{A}_{HP}$ of all HPOs. This outstanding open problem goes back to Pólya-Schur's well-known characterization of multiplier sequences of the first kind, that is, HPOs which are diagonal in the standard monomial basis of $\mathbb{C}[z]$. Pólya-Schur's 1914 result generated a vast literature on this subject and related topics at the interface between analysis, operator theory and algebra but so far only partial results under rather restrictive conditions have been obtained. In this talk I will report on the progress towards a complete solution of both this problem and its analog for (Hurwitz) stable polynomials as well as their multivariate versions made in an ongoing series of papers jointly with Petter Brändén and Boris Shapiro.

The concepts of hyperbolicity and stability have natural multivariate extensions: a polynomial $f \in \mathbb{C}[z_1, \ldots, z_n]$ is *stable* if $f(z_1, \ldots, z_n) \neq 0$ whenever $\Im(z_j) > 0$, $1 \leq j \leq n$. A stable polynomial with real coefficients is called *real stable*. Hence a univariate real stable polynomial is hyperbolic in the above sense. We generalize the notion of multiplier sequences to multivariate polynomials and give a complete characterization of higher-dimensional multiplier sequences. We then classify all operators in the Weyl algebra $\mathcal{A}_n$ of differential operators that preserve stability and show that real stability preservers in n variables are generated by real stable polynomials in $2n$ variables via the symbol map. One of the key ingredients in the proofs is a natural duality theorem for the Fischer-Fock space in n dimensions $\mathcal{F}_n$ that we establish in the process: an operator in $\mathcal{A}_n$ preserves stability if and only its Fischer-Fock adjoint does. This is a powerful generalization of the classical Hermite-Poulain-Jensen theorem in the univariate case as well as a natural multivariate extension of the latter. For $n = 1$ we thus obtain complete algebraic and geometric descriptions of the monoid $\mathcal{A}_{HP} \cap \mathcal{A}_1$ and we further describe all monotone HPOs on $\mathbb{C}[z]$, which solves the aforementioned problem in essentially all nondegenerate cases. Moreover, we prove an analog of the Lax conjecture for real stable polynomials and as a consequence we deduce a (third) determinantal characterization of operators in $\mathcal{A}_{HP} \cap \mathcal{A}_1$. These results have particularly interesting applications to the spectral theory of exactly solvable operators and the Heine-Stieltjes problem for differential equations of Lamé type.