AN ODYSSEY INTO LOCAL REFINEMENT AND MULTILEVEL PRECONDITIONING II: STABILIZING HIERARCHICAL BASIS METHODS *

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Abstract. In this article, we examine the wavelet modified (or *stabilized*) hierarchical basis (WHB) methods of Vassilevski and Wang, and extend their original quasiuniformity-based framework and results to local 2D and 3D red-green refinement procedures. The concept of a stable Riesz basis plays a critical role in the original work on WHB, and in the design of efficient multilevel preconditioners in general. We carefully examine the impact of local mesh refinement on Riesz bases and matrix conditioning. In the analysis of WHB methods, a critical first step is to establish that the BPX preconditioner is optimal for the refinement procedures under consideration. Therefore, the first article in this series was devoted to extending the results of Dahmen and Kunoth on the optimality of BPX for 2D local red-green refinement to 3D local red-green refinement procedures introduced by Bornemann-Erdmann-Kornhuber (BEK). These results from the first article, together with the local refinement extension of the WHB analysis framework presented here, allow us to establish optimality of the WHB preconditioner on locally refined meshes in both 2D and 3D. In particular, with the minimal smoothness assumption that the PDE coefficients are in L_∞, we establish optimality for the additive WHB preconditioner on locally refined 2D and 3D meshes. An interesting implication of the optimality of WHB preconditioner is the *a priori* H^1-stability of the L_2-projection. The existing *a posteriori* approaches in the literature dictate a reconstruction of the mesh if such conditions cannot be satisfied. The proof techniques employed throughout the paper allow extension of the optimality results, the H^1-stability of L_2-projection results, and the various supporting results to arbitrary spatial dimension $d \geq 1$.

Key words. finite element approximation theory, multilevel preconditioning, hierarchical bases, wavelets, two and three dimensions, local mesh refinement, red-green refinement.

AMS subject classifications. 65M55, 65N55, 65N22, 65F10

1. Introduction.

In this article, we analyze the impact of local adaptive mesh refinement on the stability of multilevel finite element spaces and on the optimality (linear space and time complexity) of multilevel preconditioners. Adaptive refinement techniques have become a crucial tool for many applications, and access to optimal or near-optimal multilevel preconditioners for locally refined mesh situations is of primary concern to computational scientists. The preconditioners which can be expected to have somewhat favorable space and time complexity in such local refinement scenarios are the hierarchical basis (HB) method, the Bramble-Pasciak-Xu (BPX) preconditioner, and the wavelet modified (or stabilized) hierarchical basis (WHB) method. While there are optimality results for both the BPX and WHB preconditioners in the literature, these are primarily for quasiuniform meshes and/or two space dimensions (with some exceptions noted below). In particular, there are few hard results in the literature on the optimality of these methods for various realistic local mesh refinement hierarchies, especially in three space dimensions. We assemble

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a number of such results in this article, which is the second in a series of three articles [2, 3] on local refinement and multilevel preconditioners (the material forming this trilogy is based on the first author's Ph.D. dissertation [1]). This second article focuses on the WHB methods; the first article [3] developed some results for the BPX preconditioner.

The problem class we focus on here is linear second order partial differential equations (PDE) of the form:

$$-\nabla \cdot (p \; \nabla u) + q \; u = f, \quad u \in H_0^1(\Omega). \tag{1.1}$$

Here, $f \in L_2(\Omega)$, $p, q \in L_\infty(\Omega)$, $p : \Omega \to L(\mathbb{R}^d, \mathbb{R}^d)$, $q : \Omega \to \mathbb{R}$, where p is a symmetric positive definite matrix function, and where q is a nonnegative function. Let $\mathcal{T}_0$ be a shape regular and quasiuniform initial partition of Ω into a finite number of d simplices, and generate $\mathcal{T}_1, \mathcal{T}_2, \dots$ by refining the initial partition using the standard red-green refinement procedure in $d = 2$ or $d = 3$ spatial dimensions. The local 2D red-green refinement of interest is as described in [10], whereas the 3D version of interest is as described by Bornemann-Erdmann-Kornhuber [7]. Denote as $\mathcal{S}_j$ the simplicial linear C^0 finite element space corresponding to $\mathcal{T}_j$ equipped with zero boundary values. The set of nodal basis functions for $\mathcal{S}_j$ is denoted by $\Phi^{(j)} = \{\phi_i^{(j)}\}_{i=1}^{N_j}$ where $N_j = \dim \mathcal{S}_j$ is equal to the number of interior nodes in $\mathcal{T}_j$. Successively refined finite element spaces will form the following nested sequence:

$$\mathcal{S}_0 \subset \mathcal{S}_1 \subset \ldots \subset \mathcal{S}_j \subset \ldots \subset H_0^1(\Omega). \tag{1.2}$$

Let the bilinear form and the functional associated with the weak formulation of (1.1) be denoted as

$$a(u, v) = \int_\Omega p \; \nabla u \cdot \nabla v + q \; u \; v \; dx, \quad b(v) = \int_\Omega f \; v \; dx, \quad u, v \in H_0^1(\Omega).$$

We consider primarily the following Galerkin formulation: Find $u \in \mathcal{S}_j$, such that

$$a(u, v) = b(v), \quad \forall v \in \mathcal{S}_j. \tag{1.3}$$

The finite element approximation in $\mathcal{S}_j$ has the form $u^{(j)} = \sum_{i=1}^{N_j} u_i \phi_i^{(j)}$, where $u = (u_1, \dots, u_{N_j})^T$ denotes the coefficients of $u^{(j)}$ with respect to $\Phi^{(j)}$. The resulting *discretization operator* $A^{(j)} = \{a(\phi_k^{(j)}, \phi_l^{(j)})\}_{k,l=1}^{N_j}$ determines the interaction of basis functions with respect to $a(\cdot, \cdot)$ and must be inverted numerically to determine the coefficients u from the linear system:

$$A^{(j)} u = F^{(j)}, \tag{1.4}$$

where $F^{(j)} = \{b(\phi_l^{(j)})\}_{l=1}^{N_j}$. Our task is to solve (1.4) with optimal (linear) complexity in both storage and computation, where the finite element spaces $\mathcal{S}_j$ are built on locally refined meshes.

HB methods are particularly attractive in the local refinement setting because (by construction) each iteration has linear (optimal) computational and storage complexity. Unfortunately, the resulting preconditioner is not optimal due to condition number growth: in two dimensions the growth is slow, and the method is quite effective (nearly optimal), but in three dimensions the condition number grows much more rapidly with the number of unknowns. To address this instability, one can employ

L_2-orthonormal wavelets in place of the hierarchical basis; such wavelets form a stable Riesz bases in H^1, thereby giving rise to an optimal preconditioner [11]. However, the complicated nature of traditional wavelet bases, in particular the non-local support of the basis functions and problematic treatment of boundary conditions, severely limits computational feasibility. WHB methods have been developed [21, 22] as an alternative, and they can be interpreted as a wavelet modification (or *stabilization*) of the hierarchical basis. These methods have been shown to optimally stabilize the condition number of the systems arising from hierarchical basis methods on quasiuniform meshes in both two and three space dimensions, and retain a comparable cost per iteration.

The framework developed in [21, 22] for the analysis of stabilizations of the hierarchical basis on quasiuniform meshes relies on establishing an optimal BPX preconditioner. In this article, we adopt the modern framework which exploits estimates related to depth of the hierarchy rather than the element size (i.e. 2^{-j} versus h). This framework enables the extension to the local refinement setting. To use the extended framework, one again begins by establishing optimality of the BPX preconditioner, but now for the particular local refinement procedures of interest.

Outline of the paper. In §2, we give a theoretical framework for constructing optimal multilevel preconditioners through decompositions of finite element spaces. There are two main players in this framework: the slice operator and the HB operator. We outline the link between the two. In §3, we introduce the the WHB preconditioner as well as the operator used in its definition. In §4, we review the relationship between condition numbers of matrices and stable Riesz bases. In §5, condition number bounds for the HB and WHB preconditioners are given by establishing explicit Riesz basis stability bounds, and we show that H^1-stability of the slice operators π_j is a necessary condition for obtaining a H^1-stable Riesz basis (or equivalently, an optimal preconditioner). In §6, we briefly describe 3D red-green local refinement of two- and three-dimensional simplex meshes, and list a number of critical geometrical results for the resulting refined meshes that were established in [3]. In §7, we set up the main theoretical results in the paper, state the fundamental assumption for establishing basis stability and WHB preconditioner optimality. We establish the main result, namely, the optimality of the WHB preconditioner in the 2D and 3D local red-green refinements. The optimality result is established for general PDE coefficients $p \in L_\infty(\Omega)$. The results in §7 rest completely on the BPX results from the companion article [3] and on Bernstein estimates, the latter of which rest on the geometrical results established in §6. Appendix §9 is dedicated to showing that the fine-fine interaction operator $A_{22}^{(j)}$ is well-conditioned.

Finally, we note that as optimality of the WHB preconditioner implies H^1-stability of the W_j operator restricted to finite element spaces under the same class of local refinement algorithms, likewise a surprising implication of the optimality of the BPX preconditioner is H^1-stability of L_2-projection. This question has been actively studied in the finite element community due to its relationship to multilevel preconditioning. The existing theoretical results, mainly due to Carstensen [9] and Bramble-Pasciak-Steinbach [8] involve *a posteriori* verification of somewhat complicated mesh conditions after refinement has taken place. If such mesh conditions are not satisfied, one has to redefine the mesh. However, the stability result we obtain in §5.1 appears to be the first *a priori* H^1-stability result for L_2-projection on the finite element spaces produced.

2. Multilevel preconditioning framework. The primary goal of this work is to describe an approximation theory framework for constructing and analyzing multilevel preconditioners, and then to use the framework to show that the WHB preconditioner is optimal for 2D and 3D local red-green refinement procedures. Multilevel preconditioning exploits the underlying multilevel hierarchical structure. Let $\mathcal{N}_j^f$ denote the newly introduced (fine) nodes in a locally refined mesh, then the following decomposition at level j is naturally introduced:

$$\mathcal{N}_j = \mathcal{N}_{j-1} \cup \mathcal{N}_j^f. \tag{2.1}$$

The key point is to reflect the hierarchical ordering of nodes (2.1) in the corresponding nodal basis functions, thereby reaching a hierarchical splitting:

$$\mathcal{S}_j = \mathcal{S}_{j-1} \oplus \mathcal{S}_j^f, \tag{2.2}$$

where $\mathcal{S}_j^f$ is called a *slice space* (superscript f stands for *fine* and later c will stand for *coarse*). The two-level decomposition is central to HB methods [5]. In this process the slice space $\mathcal{S}_j^f$ is selected as a hierarchical complement of $\mathcal{S}_{j-1}$ in $\mathcal{S}_j$. Namely,

$$\mathcal{S}_j^f = (\pi_j - \pi_{j-1})\mathcal{S}_j, \tag{2.3}$$

where $\pi_j : L_2 \to \mathcal{S}_j$ is a linear operator with the following three properties:

$$\pi_j \mid_{\mathcal{S}_j} = I, \tag{2.4}$$

$$\pi_j \pi_k = \pi_{\min\{j,k\}}, \tag{2.5}$$

$$\|(\pi_j - \pi_{j-1})u^{(j)}\|_{L_2} \approx \|u^{(j)}\|_{L_2}, \quad u^{(j)} \in (I_j - I_{j-1})\mathcal{S}_j, \tag{2.6}$$

where $I_j : L_2(\Omega) \to \mathcal{S}_j$ denotes the finite element interpolation operator. Here, we should clarify that by *stable* splitting, we mean that the bounds in the corresponding norm equivalence (2.16) are favorable, in the best case, *optimal* [16, 17].

Since the decomposition $\mathcal{S}_j = \mathcal{S}_{j-1} \oplus \mathcal{S}_j^f$ in (2.2) is direct, $A^{(j)}$ can be represented by a two-by-two block form:

$$A^{(j)} = \begin{bmatrix} A^{(j-1)} & A_{12}^{(j)} \\ A_{21}^{(j)} & A_{22}^{(j)} \end{bmatrix} \begin{matrix} \} & \mathcal{S}_{j-1} \\ \} & \mathcal{S}_j^f \end{matrix} \quad, \tag{2.7}$$

where $A^{(j-1)}$, $A_{12}^{(j)}$, $A_{21}^{(j)}$, and $A_{22}^{(j)}$ correspond to coarse-coarse, coarse-fine, fine-coarse, and fine-fine interactions respectively. Applying the two-level decomposition (2.2) may not give a *stable* splitting of $\mathcal{S}_j$. This means that $A^{(j-1)}$ may not be well-conditioned. This difficulty can be overcome by repeating the above procedure so that $\mathcal{S}_J$ can be represented completely by slice spaces:

$$\mathcal{S} = \mathcal{S}_J = \mathcal{S}_0 \oplus \mathcal{S}_1^f \oplus \ldots \oplus \mathcal{S}_J^f. \tag{2.8}$$

Such a splitting will turn out not only to be stable, but as a consequence it will also have the advantage of producing well-conditioned fine-fine interaction operators $A_{22}^{(j)}$ as will be explained in §9. In light of (2.8), multilevel preconditioning can be interpreted as a *stable* splitting of $u \in \mathcal{S}_J$ with $\pi_{-1} = 0$,

$$u = \sum_{j=0}^{J} (\pi_j - \pi_{j-1})u. \tag{2.9}$$

Following the exposition in the companion article [3], we can now define the generic (additive) HB preconditioner:

$$Hu := \sum_{j=0}^{J} 2^{j(d-2)} \sum_{i \in N_j^f} (u, \psi_i^{(j)})\psi_i^{(j)}, \tag{2.10}$$

where $\psi_i^{(j)} = (\pi_j - \pi_{j-1})\phi_i^{(j)}$. We can move to a general framework where (2.10) becomes a special example. From this point on, the following operator will be referred to as the HB preconditioner.

$$Bu := \sum_{j=0}^{J} B_{22}^{(j)^{-1}} (\pi_j - \pi_{j-1})u, \tag{2.11}$$

where $B_{22}^{(j)} : \mathcal{S}_j^f \to \mathcal{S}_j^f$ is the smoother (or an approximation to $A_{22}^{(j)^{-1}}$). Utilizing the splitting (2.9), one can write $u = \sum_{j=0}^{J} u^{(j)^f}$, where $u^{(j)^f} := (\pi_j - \pi_{j-1})u$. Note that B can be written as a diagonal operator using (2.9):

$$B = \mathrm{diag}(B_{22}^{(0)^{-1}}\pi_0, B_{22}^{(1)^{-1}}(\pi_1 - \pi_0), \ldots, B_{22}^{(J)^{-1}}(\pi_J - \pi_{J-1})).$$

Using properties (2.4) and (2.5), one can observe that

$$B_{22}^{(j)}(\pi_j - \pi_{j-1})B_{22}^{(j)^{-1}}(\pi_j - \pi_{j-1})u^{(j)^f} = u^{(j)^f}.$$

Then,

$$B^{-1} = \mathrm{diag}(B_{22}^{(0)}\pi_0, B_{22}^{(1)}(\pi_1 - \pi_0), \ldots, B_{22}^{(J)}(\pi_J - \pi_{J-1})).$$

The ultimate goal is the prove the following norm equivalence

$$(B^{-1}u, u) \eqsim (Au, u). \tag{2.12}$$

To reach this goal, we use an important operator called as the *slice* operator induced by the splitting (2.9).

$$Cu := \sum_{j=0}^{J} 2^{2j}(\pi_j - \pi_{j-1})u. \tag{2.13}$$

In any HB method smoothing is performed on the fine discretization operator $A_{22}^{(j)}$. Hence, existence of approximations $B_{22}^{(j)}$, SPD in $\mathcal{S}_j^f$, to the operators $A_{22}^{(j)}$ becomes critical. Finally, in order to link B^{-1} to C, we will need (9.4) and the following assumption. The reader can find the verification of this assumption in §9.

Assumption 2.1.

$$(A_{22}^{(j)}u^f, u^f) \le (B_{22}^{(j)}u^f, u^f) \le (1 + b_1)(A_{22}^{(j)}u^f, u^f), \quad \forall u^f \in \mathcal{S}_j^f.$$

Now, we can link the two operators.

$$(B^{-1}u, u) = \sum_{j=0}^{J} B_{22}^{(j)}((\pi_j - \pi_{j-1})u, (\pi_j - \pi_{j-1})u) \tag{2.14}$$

$$\eqsim \sum_{j=0}^{J} 2^{2j}\|(\pi_j - \pi_{j-1})u\|_{L_2}^2 = (Cu, u). \tag{2.15}$$

Therefore, we concentrate on proving the following crucial norm equivalence

$$(Cu, u) \eqsim (Au, u). \tag{2.16}$$

3. The WHB preconditioner. Let $Q_j : L_2(\Omega) \rightarrow \mathcal{S}_j$ denote the L_2-projection. We are going to apply this framework to different examples by selecting π_j equal to I_j and Q_j, which will give rise to HB and BPX preconditioners, respectively. In local refinement, HB methods enjoy an optimal complexity of $O(N_j - N_{j-1})$ per iteration per level (resulting in $O(N_J)$ overall complexity per iteration) by only using degrees of freedom (DOF) corresponding to $\mathcal{S}_j^f$ by the virtue of (2.3). However, HB methods suffer from suboptimal iteration counts or equivalently suboptimal condition number. On the other hand, the BPX preconditioner enjoys an optimal condition number in the case of uniform refinement in 2D and 3D. In the companion article [3], we also showed that the optimal condition number extends to 2D/3D red-green refinement procedures. The BPX decomposition $\mathcal{S}_j = \mathcal{S}_{j-1} \oplus (Q_j - Q_{j-1})\mathcal{S}_j$ gives rise to basis functions which are not locally supported, but they decay rapidly outside a local support region. This allows for locally supported approximations, and in addition the WHB methods [21, 22, 23] can be viewed as an approximation of the wavelet basis stemming from the BPX decomposition [11]. A similar wavelet-like multilevel decomposition approach was taken in [20], where the orthogonal decomposition is formed by a discrete L_2-equivalent inner product. This approach utilizes the same BPX two-level decomposition [19, 20].

The WHB preconditioner introduced in [21, 22] is, in some sense, the best of both worlds. While the condition number of the HB preconditioner is stabilized by inserting Q_j in the definition of π_j, somehow employing the operators $I_j - I_{j-1}$ at the same time guarantees optimal computational and storage cost per iteration. The operators which will be seen to meet both goals at the same time are:

$$W_k = \prod_{j=k}^{J-1} I_j + Q_j^a(I_{j+1} - I_j), \tag{3.1}$$

with $W_J = I$. The exact L_2-projection Q_j is replaced by a computationally feasible approximation $Q_j^a : L_2 \rightarrow \mathcal{S}_j$. To control the approximation quality of Q_j^a, a small fixed tolerance γ is introduced:

$$\|(Q_j^a - Q_j)u\|_{L_2} \leq \gamma \|Q_j u\|_{L_2}, \quad \forall u \in L_2(\Omega). \tag{3.2}$$

In the limiting case $\gamma = 0$, W_k reduces to the exact L_2-projection on $\mathcal{S}_J$ by (2.4):

$$W_k = Q_k \quad I_{k+1} Q_{k+1} \ldots I_{J-1} Q_{J-1} \quad I_J = Q_k Q_{k+1} \ldots Q_{J-1} = Q_k.$$

The properties (2.4), (2.5), and (2.6) can be verified for W_k as follows:

- Property (2.4): Let $u^{(k)} \in \mathcal{S}_k$. Since $(I_{j+1} - I_j)u^{(k)} = 0$ and $I_j u^{(k)} = u^{(k)}$ for $k \leq j$, then $[I_j + Q_j^a(I_{j+1} - I_j)](u^{(k)}) = u^{(k)}$, verifying (2.4) for W_k. It also implies

$$W_k^2 = W_k. \tag{3.3}$$

- Property (2.5): Let $k \leq l$, then by (3.3)

$$W_k W_l = [(I_k + Q_k^a(I_{k+1} - I_k)) \ldots (I_{l-1} + Q_{l-1}^a(I_l - I_{l-1})) \; W_l] W_l = W_k. \tag{3.4}$$

Since $W_k u \in \mathcal{S}_k$ and $\mathcal{S}_k \subset \mathcal{S}_l$, then by (2.4) we have

$$W_l(W_k u) = W_k u. \tag{3.5}$$

Finally, (2.5) then follows from (3.4) and (3.5).

• Property (2.6): This is an implication of Lemma 9.1.

The case of interest is when $\pi_j = W_j$, where W_j is as in (3.1). So, the WHB preconditioner is defined as below:

$$Bu := \sum_{j=0}^{J} {B_{22}^{(j)}}^{-1}(W_j - W_{j-1})u. \tag{3.6}$$

The optimality of the WHB preconditioner, namely (2.16), under locally refinement will be the main result of this paper (see Theorem 7.2). For convenience, we will use the following notation

$$(Cu, u) = \|u\|^2_{\text{WHB}} := \sum_{j=0}^{J} 2^{2j}\|(W_j - W_{j-1})u\|^2_{L_2}.$$

For an overview, we list the corresponding slice spaces for the preconditioners of interest:

$$\begin{array}{lcll}
\text{HB:} & \mathcal{S}_j^f & = & (I_j - I_{j-1})\mathcal{S}_j, \\
\text{BPX:} & \mathcal{S}_j^f & = & (Q_j - Q_{j-1})\mathcal{S}_j, \\
\text{WHB:} & \mathcal{S}_j^f & = & (W_j - W_{j-1})\mathcal{S}_j = (I - Q^a_{j-1})(I_j - I_{j-1})\mathcal{S}_j.
\end{array}$$

4. Condition numbers and Riesz Bases. Let H be a separable Hilbert space with a nested sequence of finite dimensional subspaces,

$$H_0 \subset H_1 \subset \ldots \subset H_j \subset \ldots \subset H,$$

where $\dim(H_j) = N_j$. Consider a bounded bilinear form $a(\cdot,\cdot)$ defined on $H \times H$ satisfying the inf-sup condition. Let $u \in H_j$ and let $\Phi^{(j)} = \{\phi_i\}_{i=1}^{N_j}$ be a basis for H_j such that $u = \sum_{i=1}^{N_j} u_i\phi_i$, where $u = (u_1, \ldots, u_{N_j})^T$ denotes the coordinates of u with respect to $\Phi^{(j)}$. Let $A^{(j)} = \{a(\phi_k, \phi_l)\}_{k,l=1}^{N_j}$ denote the discretization operator with respect to $\Phi^{(j)}$. As remarked earlier, we are generally interested in the condition number of $A^{(j)}$ for different choices of bases, such as hierarchical-type bases.

A basis-dependent inner-product in the coefficient space will be used for the calculation of $\kappa_{\Phi^{(j)}}(A^{(j)})$, $\langle u, v\rangle_{\Phi^{(j)}} = \sum_{i=1}^{N_j} u_i v_i$, and the norm induced by $\langle\cdot,\cdot\rangle_{\Phi^{(j)}}$ will be denoted as $\|u\|^2_{\Phi^{(j)}} = \sum_{i=1}^{N_j} u_i^2$. Note that $\kappa_{\Phi^{(j)}}(A^{(j)})$ becomes uniformly bounded if $\Phi^{(j)}$ chosen to be an orthonormal basis with respect to the inner-product $(\cdot,\cdot)_H$ of H. However, it is not practical to assume the existence of an orthonormal basis which is also computationally feasible. In a separable Hilbert space H, the next best thing to an orthonormal basis, in this sense, is an H-stable Riesz basis.

Definition 4.1. *Let $\Phi = \{\phi_i\}_{i=1}^{\infty}$ be a basis for H, and $u = \sum_{i=1}^{\infty} c_i\phi_i$. If there exist two absolute constants σ_1 and σ_2 such that*

$$\sigma_1\|u\|^2_H \le \sum_{i=1}^{\infty} c_i^2 \le \sigma_2\|u\|^2_H, \quad \forall u \in H, \tag{4.1}$$

then Φ is called an H-stable Riesz basis.

The condition (4.1) for finite dimensional H_j can be written as

$$\sigma_1^{(j)} \le \frac{\|u\|^2_{\Phi^{(j)}}}{\|u\|^2_{H_j}} \le \sigma_2^{(j)}, \quad \forall u \in H_j. \tag{4.2}$$

The primary task becomes gaining some control over the ratio $\sigma_2^{(j)}/\sigma_1^{(j)}$.

DEFINITION 4.2. *The family* $\left\{\Phi^{(j)} := \{\phi_i\}_{i=1}^{N_j}\right\}$ *is a uniformly* H_j*-stable family of Riesz bases if there exists* c *independent of* j *such that:* $\quad \sigma_2^{(j)}/\sigma_1^{(j)} \le c, \quad j \to \infty.$

The case of primary interest is when $H_j = \mathcal{S}_j$. The discussion above results in the following theorem.

THEOREM 4.3. *Let* $\Phi^{(j)}$ *be a basis of* $\mathcal{S}_j$ *satisfying (4.2). Then with* c *depending on the norm of the bilinear form and the stability constant from the inf-sup condition;*

$$\kappa_{\Phi^{(j)}}(A^{(j)}) \le c\ \sigma_2^{(j)}/\sigma_1^{(j)}.$$

Note that $\sigma_2^{(j)}/\sigma_1^{(j)}$ is basis-dependent and our motivation is to find H^1-stable Riesz bases so that the condition number is uniformly bounded.

5. H^1-stable Riesz bases and the WHB slice norm. As the multilevel decomposition (2.8) suggests, one can view $\mathcal{S}_J$ as a span of multilevel hierarchical basis (MHB) functions. The MHB can be any computationally feasible basis and it is the nodal basis $\phi_i^{(j)}$ in our context. Modification to the nodal basis can be made by any linear operator π_j satisfying the properties (2.4), (2.5), and (2.6), in particular by the WHB operator W_j given in (3.1).

DEFINITION 5.1. *Let* $\{\phi_i^{(j)}\}_{i=1}^{N_j}$ *be the hierarchical basis for* $\mathcal{S}_j$, $j = 0, \ldots, J$. *Then the wavelet modified multilevel hierarchical basis (WMHB) for* $\mathcal{S}_j$ *is defined as:*

$$\Phi^{(J)} = \bigcup_{j=0}^{J} \left\{(W_j - W_{j-1})\phi_i^{(j)}\right\}_{i=N_{j-1}+1}^{N_j}. \tag{5.1}$$

It can be shown (see Lemma 3.1 in [21]) that the WMHB (5.1) forms a basis for $\mathcal{S}_J$. With this fact at our disposal, let u be represented with respect to the WMHB:

$$u = \sum_{j=0}^{J} \sum_{i=N_{j-1}+1}^{N_j} c_i (W_j - W_{j-1})\phi_i^{(j)}. \tag{5.2}$$

Property (2.5) leads to:

$$W_k u = \sum_{j=0}^{k} \sum_{i=N_{j-1}+1}^{N_j} c_i (W_j - W_{j-1})\phi_i^{(j)}. \tag{5.3}$$

In order to establish Riesz stability, we will need a scaled version of the WMHB in (5.1) given as below:

$$\bar{\Phi}^{(J)} = \bigcup_{j=0}^{J} \left\{(W_j - W_{j-1})\bar{\phi}_i^{(j)}\right\}_{i=N_{j-1}+1}^{N_j}, \tag{5.4}$$

where $u = \sum_{j=0}^{J} \sum_{i=N_{j-1}+1}^{N_j} \bar{c}_i \bar{\phi}_i^{(j)} = \sum_{j=0}^{J} \sum_{i=N_{j-1}+1}^{N_j} c_i \phi_i^{(j)}$ and the following coefficient relationship holds:

$$\bar{c}_i := 2^{j/2(2-d)} c_i, \quad i = N_{j-1}+1, \ldots, N_j, \quad j = 0, \ldots, J. \tag{5.5}$$

The slice norm will now be connected to the cofficient norm $\|u\|^2_{\bar{\Phi}^{(J)}} := \sum_{i=1}^{N_J} \bar{c}_i^2$:

LEMMA 5.2. *Let* $u = \sum_{j=0}^{J} \sum_{i=N_{j-1}+1}^{N_j} \bar{c}_i \ (\pi_j - \pi_{j-1})\bar{\phi}_i^{(j)}$ *and let* π_j *satisfy the properties (2.4), (2.5), and (2.6). Then*

$$(Cu, u) = \sum_{j=0}^{J} 2^{2j} \|(\pi_j - \pi_{j-1})u\|_{L_2}^2 \eqsim \sum_{i=1}^{N_J} \bar{c}_i^2. \tag{5.6}$$

Proof. Using (5.3) and linearity of π_j respectively:

$$(\pi_j - \pi_{j-1})u = \sum_{i=N_{j-1}+1}^{N_j} c_i(\pi_j - \pi_{j-1})\phi_i^{(j)} = (\pi_j - \pi_{j-1}) \sum_{i=N_{j-1}+1}^{N_j} c_i\phi_i^{(j)}.$$

Note that $\sum_{i=N_{j-1}+1}^{N_j} c_i\phi_i^{(j)} \in (I_j - I_{j-1})\mathcal{S}_j$. Then by property (2.6)

$$\|(\pi_j - \pi_{j-1})u\|_{L_2}^2 \eqsim \| \sum_{i=N_{j-1}+1}^{N_j} c_i\phi_i^{(j)} \|_{L_2}^2.$$

The mass matrix is equivalent to its diagonal due to shape regularity and compact support of basis functions. Moreover for $i = N_{j-1}+1, \dots, N_j, \quad j = 0, \dots, J$, the local refinements under consideration promise a quasiuniform support of $\phi_i^{(j)}$ (see (6.3)), hence $\|\phi_i^{(j)}\|_{L_2}^2 \eqsim 2^{-jd}$. Putting these facts together, one gets:

$$\| \sum_{i=N_{j-1}+1}^{N_j} c_i\phi_i^{(j)} \|_{L_2}^2 \eqsim \sum_{i=N_{j-1}+1}^{N_j} c_i^2 \|\phi_i^{(j)}\|_{L_2}^2 \eqsim \sum_{i=N_{j-1}+1}^{N_j} c_i^2 \ 2^{-jd}.$$

Eventually by (5.5),

$$\sum_{j=0}^{J} 2^{2j} \|(\pi_j - \pi_{j-1})u\|_{L_2}^2 \eqsim \sum_{j=0}^{J} 2^{j(2-d)} \sum_{i=N_{j-1}+1}^{N_j} c_i^2 = \sum_{i=1}^{N_J} \bar{c}_i^2.$$

□

There are two important connections here to H^1-stable Riesz bases. First, the equivalence (5.6) implies that constructing an optimal preconditioner is equivalent to forming an H^1-stable Riesz basis $\bar{\Phi}^{(J)}$. The involvement of π_j in both the splitting (2.9) and in the WMHB representation in (5.2) makes it the most crucial element in the stabilization. We then come to the central question: Which choice of π_j can make MHB an H^1-stable Riesz basis? The second connection to H^1-stable Riesz bases is the following theorem, which sets a guideline for picking π_j. It shows that H^1-stability of the π_j is actually a *necessary condition* for obtaining an optimal preconditioner.

THEOREM 5.3. *If* $\bar{\Phi}^{(J)}$ *is an* H^1*-stable Riesz basis for* $\mathcal{S}_J$*, then for all* $u \in \mathcal{S}_J$ *there exists an absolute constant* c *such that*

$$\|\pi_k u\|_{H^1} \leq c \ \|u\|_{H^1}, \quad \forall k \leq J.$$

Proof. Let u be written with respect to (5.4). Then, property (2.5) implies:

$$\pi_k u = \sum_{j=0}^{k} \sum_{i=N_{j-1}+1}^{N_j} \bar{c}_i(\pi_j - \pi_{j-1})\bar{\phi}_i^{(j)}.$$

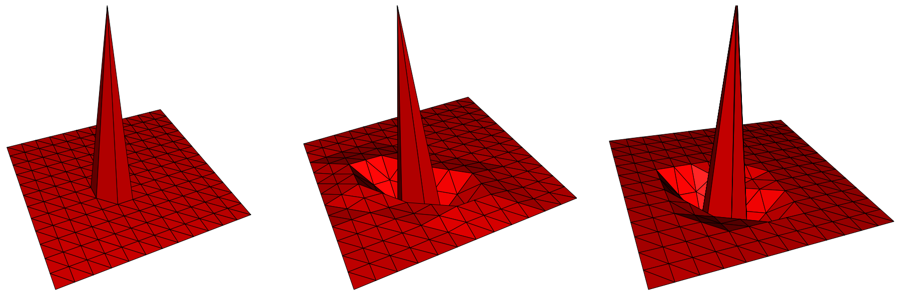

FIG. 5.1. *Left: Hierarchical basis function without modification. Wavelet modified hierarchical basis functions. Middle: One iteration of symmetric Gauss-Seidel approximation. Right: One iteration of Jacobi approximation.*

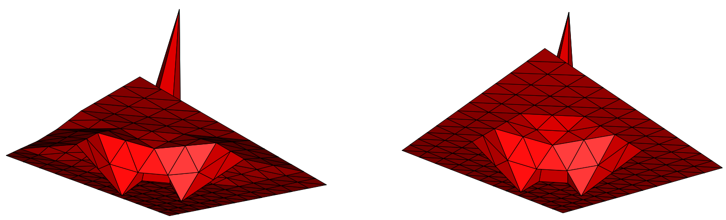

FIG. 5.2. *Lower view of middle and left basis functions in Figure 5.1.*

We assume that $\bar{\Phi}^{(J)}$ is an H^1-stable Riesz basis. Namely, recalling (4.1), there exist two absolute constants σ_1 and σ_2 such that

$$\sigma_1 \|u\|_{H^1}^2 \leq \sum_{j=0}^{J} \sum_{i=N_{j-1}+1}^{N_j} \bar{c}_i^2 \leq \sigma_2 \|u\|_{H^1}^2, \quad \forall u \in \mathcal{S}_J. \tag{5.7}$$

Using (5.7) for $\pi_k u$:

$$\|\pi_k u\|_{H^1}^2 \leq \sigma_2 \sum_{j=0}^{k} \sum_{i=N_{j-1}+1}^{N_j} \bar{c}_i^2 \leq \sigma_2 \sum_{j=0}^{J} \sum_{i=N_{j-1}+1}^{N_j} \bar{c}_i^2 \leq \frac{\sigma_2}{\sigma_1} \|u\|_{H^1}^2.$$

□

The finite element interpolation operator I_j is not bounded in the H^1-norm, and the following explicit tight bounds are well-known [4, 6, 15, 25]:

$$\|I_j u\|_{H^1} \leq c \left\{ \begin{array}{ll} (J-j+1)^{1/2}, & d=2 \\ 2^{(J-j)/2}, & d=3 \end{array} \right\} \|u\|_{H^1}.$$

In the light of Theorem 5.3, the basis in the HB method [6, 24] cannot form an H^1-stable Riesz basis. For the performance analysis of the HB preconditioner, we choose the suitably scaled MHB as in (5.4) and (5.5). Then, by Lemma 5.2,

$$\|u\|_{\mathrm{HB}}^2 = \sum_{j=0}^{J} 2^{2j} \|(I_j - I_{j-1})u\|_{L_2}^2 \eqsim \sum_{j=1}^{N_J} \bar{c}_i^2 = \|u\|_{\bar{\Phi}^{(J)}}^2.$$

The suboptimal bounds for I_j manifest themselves as in the following widely known result [14, 16] about HB.

$$c_1 \left\{ \begin{array}{ll} J^{-2}, & d=2 \\ 2^{-J}, & d=3 \end{array} \right\} \|u\|_{\mathrm{HB}}^2 \le \|u\|_{H^1}^2 \le c_2 \|u\|_{\mathrm{HB}}^2.$$

Therefore, the HB preconditioner is not optimal, and its performance severely deteriorates in dimension $d = 3$. Furthermore, Theorem 4.3 implies that the discretization operator $\bar{A}^{(J)} = \{a(\bar{\phi}_k^{(J)}, \bar{\phi}_l^{(J)})\}_{k,l=1}^{N_J}$ with respect to the scaled HB cannot be well-conditioned with the following tight bounds:

$$\kappa_{\bar{\Phi}^{(J)}}(\bar{A}^{(J)}) \le c \left\{ \begin{array}{ll} J^2, & d=2 \\ 2^J, & d=3 \end{array} \right\}.$$

On the other hand, Theorem 7.2 indicates that the WMHB in (5.1) forms an H^1-stable Riesz basis (see Corollary 7.3). Hence, by Theorem 4.3, the discretization operator relative to the scaled WMHB in (5.4) is well-conditioned: $\kappa_{\bar{\Phi}^{(J)}}(\bar{A}^{(J)}) \le c$. Riesz stability is attained through wavelet modifications. In particular, the modification is made by subtracting from each HB function $\phi_i^{(j)} \in \mathcal{S}_j^f$ its approximate L_2-projection $Q_{j-1}^a \phi_i^{(j)}$ onto the coarse level $j-1$. Such modifications are depicted in Figures 5.1 and 5.2. Note that modification with symmetric Gauss Seidel approximation gives rise to basis functions with larger supports than the ones modified with Jacobi approximation.

5.1. H^1-stable L_2-projection. We present a crucial consequence of Theorem 5.3.

Corollary 5.4. *L_2-projection, $Q_j|_{\mathcal{S}_j} : L_2 \to \mathcal{S}_j$, restricted to $\mathcal{S}_j$ is H^1-stable on 2D and 3D locally refined meshes by red-green refinement procedures.*

Proof. Optimality of the BPX preconditioner on the above locally refined meshes is established in the companion article [3]. Application of Theorem 5.3 with Q_j proves the result. Alternatively, the same result can be obtained through Theorem 5.3 applied to the WHB framework. Theorem 7.2 will establish the optimality of the WHB preconditioner for the local refinement procedures. Hence, the operator W_j restricted to $\mathcal{S}_j$ is H^1-stable. Since W_j is none other than Q_j in the limiting case, we can also conclude the H^1-stability of the L_2-projection. ∎

Our stability result appears to be the first *a priori* H^1-stability for the L_2-projection on these classes of locally refined meshes. H^1-stability of L_2-projection is guaranteed for the subset $\mathcal{S}_j$ of $L_2(\Omega)$, not for all of $L_2(\Omega)$. This question is currently undergoing intensive study in the finite element and approximation theory community. The existing theoretical results, mainly in [8, 9], involve *a posteriori* verification of somewhat complicated mesh conditions after refinement has taken place. If such mesh conditions are not satisfied, one has to redefine the mesh. The mesh conditions mentioned require that the simplex sizes do not change drastically between regions of refinement. In this context, quasiuniformity in the support of a basis function becomes crucial. This type of local quasiuniformity is usually called as *patchwise* *quasiuniformity*. Local quasiuniformity requires neighbor generation relations as in (6.1), neighbor size relations, and shape regularity of the mesh. It was shown in [1] that patchwise quasiuniformity holds also for 3D marked tetrahedron bisection [12] and for 2D newest vertex bisection [13, 18]. Then these are promising refinement procedures for which H^1-stability of the L_2-projection can be established.

6. Red-green refinement. We present only the highlights of the 3D local red-green refinement procedure introduced by Bornemann, Erdmann, and Kornhuber [7]; more technical detail can be found in the preceding article [3]. The level of a simplex $\tau \in \mathcal{T}_j$ is defined as

$$L(\tau) = \min\{j : \tau \in \mathcal{T}_j\}.$$

Let us denote the support of basis functions corresponding to $\mathcal{N}_j^f$ as Ω_j^f. Due to nested refinement, we will have a quasiuniform triangulation on Ω_j^f. One can analogously introduce a triangulation hierarchy

$$\mathcal{T}_j^f := \{\tau \in \mathcal{T}_j : L(\tau) = j\} = \mathcal{T}_j|_{\Omega_j^f}.$$

Simplices in $\mathcal{T}_j^f$ are exposed to uniform refinement, hence $\mathcal{T}_j^f$ becomes a quasiuniform tetrahedralization.

The following generation bound for neighbor simplices, established rigorously in [3], will be the foundation for the approximation theory estimates. Let τ and τ' be two d simplices in $\mathcal{T}_j$ sharing common d vertices. Then

$$|L(\tau) - L(\tau')| \leq 1. \tag{6.1}$$

The generation bound (6.1) give rise to a L_2-stable Riesz basis in the following way [1, 3, 10]: Let the properly scaled nodal basis function be denoted as

$$\hat{\phi}_i^{(j)} = 2^{d/2 L_{j,i}}\ \phi_i^{(j)}, \quad \hat{u}_i = 2^{-d/2 L_{j,i}}\ u_i, \quad x_i \in \mathcal{N}_j,$$

where $L_{j,i} = \min\{L(\tau) : \tau \in \mathcal{T}_j,\ x_i \in \tau\}$. Then $\bigcup_{j=0}^{J} \{\hat{\phi}_i^{(j)}\}_{i=N_{j-1}+1}^{N_j}$ becomes a L_2-stable Riesz basis [3]:

$$\| \sum_{x_i \in \mathcal{N}_j} \hat{u}_i \hat{\phi}_i^{(j)} \|_{L_2(\Omega)} \eqsim \| \{\hat{u}_i\}_{x_i \in \mathcal{N}_j} \|_{l_2}. \tag{6.2}$$

Then (6.2) forms the sufficient condition to establish the Bernstein estimate:

$$\omega_2(u,t)_2 \leq c\ (\min\{1, t2^J\})^{3/2} \|u\|_{L_2}, \quad u \in \mathcal{S}_J, \tag{6.3}$$

where $\omega_2(u,t)_2$ denotes second moduli of smoothness of u in L_2 with step size t. The constant c is independent of u and J. This crucial property helps us to prove Theorem 7.2.

7. The fundamental WHB assumption, optimal preconditioners, and basis stability. As in the BPX splitting, the main ingredient in the WHB splitting is the L_2-projection. Hence, the stability of the BPX splitting is still important in the WHB splitting. The lower bound in the BPX norm equivalence is the *fundamental assumption* for the WHB preconditioner. Namely, there exists a constant σ independent of J satisfying:

ASSUMPTION 7.1.

$$\sum_{j=0}^{J} 2^{2j} \|(Q_j - Q_{j-1})u\|_{L_2}^2 \leq \sigma \|u\|_{H^1}^2, \quad \forall u \in \mathcal{S}_J.$$

Utilizing a local projection $\tilde{Q}_j$, A.7.1 was verified by the authors [1, 3] for 3D local red-green refinement procedure. The same result easily holds for the projection Q_j. Dahmen and Kunoth [10] verified A.7.1 for the 2D red-green refinement procedures.

Before getting to the stability result we remark that the existing perturbation analysis of WHB is one of the primary insights in [21, 22]. Although not observed in [21, 22], the result does not require substantial modification for locally refined meshes. Let $e_j := (W_j - Q_j)u$ be the error, then the following holds.

Lemma 7.1. *Let γ be as in (3.2). There exists an absolute c satisfying:*

$$\sum_{j=0}^{J} 2^{2j} \|e_j\|_{L_2}^2 \leq c\gamma^2 \sum_{j=0}^{J} 2^{2j} \|(Q_j - Q_{j-1})u\|_{L_2}^2, \quad \forall u \in \mathcal{S}_J. \tag{7.1}$$

Proof. See Lemma 5.1 and page 119 in [21] or Lemma 1 in [22]. ◻

We arrive now at the primary result, which indicates that the WHB slice norm is optimal on the class of locally refined meshes under consideration.

Theorem 7.2. *If there exists sufficiently small γ_0 such that (3.2) is satisfied for $\gamma \in [0, \gamma_0)$, then*

$$\|u\|_{\text{WHB}}^2 = \sum_{j=0}^{J} 2^{2j} \|(W_j - W_{j-1})u\|_{L_2}^2 \eqsim \|u\|_{H^1}^2, \quad u \in \mathcal{S}_J. \tag{7.2}$$

Proof. Observe that

$$\begin{aligned} (W_j - W_{j-1})u &= (W_j - Q_j)u - (W_{j-1} - Q_{j-1})u + (Q_j - Q_{j-1})u \\ &= e_j - e_{j-1} + (Q_j - Q_{j-1})u. \end{aligned} \tag{7.3}$$

This gives

$$\begin{aligned} \sum_{j=0}^{J} 2^{2j} \|(W_j - W_{j-1})u\|_{L_2}^2 &\leq c \sum_{j=0}^{J} 2^{2j} \|(Q_j - Q_{j-1})u\|_{L_2}^2 + c \sum_{j=0}^{J} 2^{2j} \|e_j\|_{L_2}^2 \\ &\leq c(1+\gamma^2) \sum_{j=0}^{J} 2^{2j} \|(Q_j - Q_{j-1})u\|_{L_2}^2 \quad \text{(using (7.1))} \\ &\leq c \|u\|_{H^1}^2 \quad \text{(using A.7.1)}. \end{aligned}$$

Let us now proceed with the upper bound. The Bernstein estimate (6.3) holds for $\mathcal{S}_j$ [1, 3, 10] for the local refinement procedures. Hence we are going to utilize an inequality involving the Besov norm $\|\cdot\|_{B_{2,2}^1}$ which naturally fits our framework when the moduli of smoothness is considered in (6.3). The following important inequality holds, provided that (6.3) holds (see page 39 in [17]):

$$\|u\|_{B_{2,2}^1}^2 \leq c \sum_{j=0}^{J} 2^{2j} \|u^{(j)}\|_{L_2}^2, \tag{7.4}$$

for any decomposition such that $u = \sum_{j=0}^{J} u^{(j)}, \quad u^{(j)} \in \mathcal{S}_j$, in particular for $u^{(j)} = (W_j - W_{j-1})u$. Then the upper bound holds due to $H^1(\Omega) \cong B_{2,2}^1(\Omega)$. ◻

Remark 7.1. *The following equivalence is used for the upper bound in the proof* *of Theorem 7.2 on uniformly refined meshes (cf. Lemma 4 in [22]).*

$$c_1\|u\|^2_{H^1} \leq \inf_{u=\sum_{j=0}^J u^{(j)},\ u^{(j)}\in\mathcal{S}_j} \sum_{j=0}^{J} 2^{2j}\|u^{(j)}\|^2_{L_2} \leq c_2\|u\|^2_{H^1}.$$

Let us emphasize that the left hand side holds in the presence of the Bernstein estimate (6.3), and the right hand side holds in the simultaneous presence of Bernstein and Jackson estimates. However, the Jackson estimate cannot hold under local refinement procedures (cf. counter example in section 8 in [3]). That is why we can utilize only the left hand side of the above equivalence as in (7.4).

The WHB slice norm optimality will be connected to Riesz basis and the scaled WMHB will now be a H^1-stable Riesz basis by Lemma 5.2 and Theorem 7.2.

COROLLARY 7.3. *Let u be represented with respect to $\bar{\Phi}^{(J)}$ in (5.4). If there exists $\gamma \in [0, \gamma_0)$ such that (3.2) holds, then $\bar{\Phi}^{(J)}$ forms an H^1-stable Riesz basis:*

$$\|u\|^2_{\bar{\Phi}^{(J)}} = \sum_{i=1}^{N_J} \bar{c}_i^2 \eqsim \|u\|^2_{H^1}.$$

Now, we have all the required estimates at our disposal to establish the optimality of WHB preconditioner for 2D/3D red-green refinement procedures for $p \in L_\infty(\Omega)$. We would like to emphasize that our framework supports any spatial dimension $d \geq 1$, provided that the necessary geometrical abstractions are in place.

THEOREM 7.4. *If A.7.1 holds and if there exists sufficiently small γ_0 such that (3.2) is satisfied for $\gamma \in (0, \gamma_0)$, then*

$$(Bu, u) \eqsim (Au, u).$$

Proof. By A.2.1, $B_{22}^{(j)}$ is spectrally equivalent to $A_{22}^{(j)}$. Since $A_{22}^{(j)}$ is a well-conditioned matrix, using (9.4) it is spectrally equivalent to $2^{2j}I$. The result follows from Theorem 7.2 and (2.15). □

An extension to multiplicative WHB preconditioner is also possible under additional assumptions. These results will not be reported here.

8. Conclusion. In this article we examined the wavelet modified hierarchical basis (WHB) methods of Vassilevski and Wang, and extended their original quasiuniformity-based framework and results to local 2D and 3D red-green refinement procedures. A critical step in the extension involved establishing that the BPX preconditioner is optimal for the local refinement procedures under consideration. The first article of this series was devoted to extending the results of Dahmen and Kunoth on the optimality of BPX for 2D local red-green refinement to 3D local red-green refinement procedures introduced by Bornemann-Erdmann-Kornhuber (BEK). The results from the first article, together with the local refinement extension of the WHB analysis framework presented here, allowed us to establish optimality of the WHB preconditioner on locally refined meshes in both 2D and 3D under the minimal regularity assumptions required for well-posendess. An interesting implication of the optimality of WHB preconditioner was the *a priori* H^1-stability of the L_2-projection. Existing *a posteriori* approaches in the literature dictate a reconstruction of the mesh if such conditions cannot be satisfied. The proof techniques employed throughout the paper allow extension of the optimality results, the H^1-stability of L_2-projection results, and the various supporting results to arbitrary spatial dimension $d \geq 1$.

9. Appendix: Well-conditioned $A_{22}^{(j)}$. The lemma below is essential to extend the existing results for quasiuniform meshes (cf. Lemma 6.1 in [21] or Lemma 2 in [22]) to the locally refined ones. $\mathcal{S}_j^{(f)} = (I_j - I_{j-1})\mathcal{S}_j$ denotes the HB slice space.

LEMMA 9.1. *Let $\mathcal{T}_j$ be constructed by the local refinements under consideration. Let $\mathcal{S}_j^f = (I - \pi_{j-1})\mathcal{S}_j^{(f)}$ be the modified hierarchical subspace where π_{j-1} is any L_2-bounded operator. Then, there are constants c_1 and c_2 independent of j such that*

$$c_1\|\phi^f\|_X^2 \leq \|\psi^f\|_X^2 \leq c_2\|\phi^f\|_X^2, \quad X = H^1, L_2, \tag{9.1}$$

holds for any $\psi^f = (I - \pi_{j-1})\phi^f \in \mathcal{S}_j^f$ with $\phi^f \in \mathcal{S}_j^{(f)}$.

Proof. The Cauchy-Schwarz like inequality [5] is central to the proof: There exists $\delta \in (0, 1)$ independent of the mesh size or level j such that

$$(1-\delta^2)(\nabla\phi^f, \nabla\phi^f) \leq (\nabla(\phi^c + \phi^f), \nabla(\phi^c + \phi^f)), \quad \forall \phi^c \in \mathcal{S}_{j-1}, \phi^f \in \mathcal{S}_j^{(f)}. \tag{9.2}$$

$$(1-\delta^2)\|\phi^f\|_{L_2}^2 \leq c|\phi^c + \phi^f|_{H^1}^2 \quad \text{(by Poincare inequality and (9.2))}. \tag{9.3}$$

Combining (9.2) and (9.3): $(1-\delta^2)\|\phi^f\|_{H^1}^2 \leq \|\phi^c + \phi^f\|_{H^1}^2$. Choosing $\phi^c = -\pi_{j-1}\phi^f$, we get the lower bound: $(1-\delta^2)\|\phi^f\|_{H^1}^2 \leq \|\psi^f\|_{H^1}^2$. To derive the upper bound: The inverse inequality holds for $\mathcal{S}_j^f$ because of the quasiuniformity of $\mathcal{T}_j^f$. The right scaling is obtained by father-son size relation.

Using the inverse inequalities and L_2-boundedness of π_{j-1}, one gets

$$\|\psi^f\|_{H^1}^2 \leq c_0 2^{2j}\|\psi^f\|_{L_2}^2 \leq c_0 2^{2j}\left(1 + \|\pi_{j-1}\|_{L_2}\right)^2 \|\phi^f\|_{L_2}^2 \leq c 2^{2j}\|\phi^f\|_{L_2}^2.$$

The slice space $\mathcal{S}_j^{(f)}$ is oscillatory. Then there exists c such that $\|\phi^f\|_{L_2}^2 \leq c2^{-2j}\|\phi^f\|_{H^1}^2$. Hence, $\|\psi^f\|_{H^1}^2 \leq c\|\phi^f\|_{H^1}^2$. The case for $X = L_2$ can be established similarly. ∎

Using the above tools, one can establish that $A_{22}^{(j)}$ is well-conditioned. Namely,

$$c_1 2^{2j} \leq \lambda_{j,\min}^f \leq \lambda_{j,\max}^f \leq c_2 2^{2j}, \tag{9.4}$$

where $\lambda_{j,\min}^f$ and $\lambda_{j,\max}^f$ are the smallest and largest eigenvalues of $A_{22}^{(j)}$, and c_1 are and c_2 both independent of j. For details see Lemma 4.3 in [21] or Lemma 3 in [22].

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