Math 171B: Numerical Optimization: Nonlinear Problems

Instructor: Michael Holst

Spring Quarter 2015

Solutions for Homework Assignment #5

Exercise 5.1. *Consider the nonlinearly constrained problem*

$$\begin{aligned} &\underset{x\in\mathcal{R}^2}{\text{minimize}}\ \ 3x_2 + x_1^2 + x_2^2 \\ &\text{subject to}\ \ x_1^2 + (x_2+1)^2 - 1 = 0. \end{aligned} \tag{5.1}$$

(a) *Show that $x(\alpha) = (\sin\alpha, \cos\alpha - 1)^T$ is a feasible path for the nonlinear constraint $x_1^2 + (x_2+1)^2 - 1 = 0$ of problem (5.1). Compute the tangent to the feasible path at $\bar{x} = (0,0)^T$.*

We need to check three conditions:

(1) $\underline{x(0) = x}: (\sin x_1, \cos x_2 - 1)^T = (0,0)^T$.

(2) $\underline{c(x(\alpha)) = 0}: c(x(\alpha)) = (\sin\alpha)^2 + (\cos\alpha - 1 + 1)^2 - 1 = 1 - 1 = 0$.

(3) $\underline{\frac{dx\alpha}{d\alpha}|_{\alpha=0} \neq 0}: (\cos\alpha, -\sin\alpha)|_{\alpha=0} = (1,0)^T \neq (0,0)^T$.

So $x(\alpha)$ is a feasible path. The tangent to this path at $\bar{x}$ is

$$\begin{aligned} p &= \frac{d}{d\alpha}x(\alpha)|_{\alpha=0} \\ &= (\cos\alpha, -\sin\alpha)^T|_{\alpha=0} \\ &= (1,0)^T. \end{aligned}$$

(b) *If $f(x)$ denotes the objective function of problem (5.1), find an expression for $f(x(\alpha))$ and compute $f(x(0))$.*

$f(x(\alpha)) = 3(\cos\alpha - 1) + (\sin\alpha)^2 + (\cos\alpha - 1)^2 = \cos\alpha - 1$.

$f(x(0)) = 0$.

(c) *Define the Lagrangian function $L(x,\lambda)$ and constraint Jacobian $J(x)$ for problem (5.1). Derive $\nabla L(x,\lambda)$, the gradient of the Lagrangian, and $\nabla^2_{xx}L(x,\lambda)$, the Hessian of the Lagrangian with respect to x.*

The Jacobian is

$$J(x)^T = \nabla c(x) = \begin{pmatrix} 2x_1 \\ 2(x_2+1) \end{pmatrix}.$$

The Lagrangian is

$$L(x,\lambda) = f(x) - \lambda^T c(x).$$

so

$$\nabla L(x,\lambda) = \begin{pmatrix} 2x_1(1-\lambda) \\ 2x_2(1-\lambda) - 2\lambda + 3 \\ -(x_1^2 + (x_2+1)^2 - 1) \end{pmatrix},$$

$$\nabla^2_{xx}L(x,\lambda) = \begin{pmatrix} 2-2\lambda & 0 \\ 0 & 2-2\lambda \end{pmatrix}.$$

(d) *Determine whether or not the point $\bar{x} = (0,0)^T$ is a constrained minimizer of problem (5.1).*

We need to check three conditions:

(1) $\underline{\bar{x}\text{ is feasible}}$: $c((0,0)^T) = 0$. So $\bar{x}$ is feasible.

(2) There exists λ^* s.t. $g(\bar{x}) - J(\bar{x})^T\lambda^* = 0$: $g(\bar{x}) - J(\bar{x})^T\lambda^* = \begin{pmatrix} 0 \\ -2\lambda^* + 3 \end{pmatrix}$. So for $\lambda^* = 3/2$, there exists λ^* s.t. $g(\bar{x}) - J(\bar{x})^T\lambda^* = 0$.

(3) $p^T H(\bar{x}, \lambda^*)p \geq 0$ for every p satisfying $J(\bar{x})p = 0$: Take $p = (1, 0)^T \in \text{null}(J(\bar{x}))$. Then $p^T H(\bar{x}, \lambda^*)p = p^T \nabla^2_{xx} L(\bar{x}, \lambda^*)p = -1 < 0$.

Since (3) fails, the second-order necessary conditions do not hold, so $\bar{x}$ is not a minimizer.

Exercise 5.2. *Consider the problem*

$$\begin{aligned} &\underset{x \in \mathcal{R}^2}{\text{minimize}} \;\; x_1^2 + 2x_2^2 \\ &\text{subject to} \;\; x_1 + x_2 - 1 = 0. \end{aligned}$$

(a) *Find a point satisfying the KKT conditions. Verify that it is indeed an optimal point.*

We need a x^* that is feasible and a λ^* such that $g(x^*) - J(x^*)^T\lambda^* = 0$. Given $L(x, \lambda) = x_1^2 + 2x_2^2 - \lambda(x_1 + x_2 - 1)$:

$$\nabla L = \begin{pmatrix} 2x_1 - \lambda \\ 4x_2 - \lambda \\ -x_1 - x_2 + 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solving this system gives $x^* = (2/3, 1/3)^T$, $\lambda^* = 4/3$.

This x^* and λ^* satisfies (1) and (2) (from part (d) in the above problem). We still need to check (3). Since $H(x^*, \lambda^*)$ is positive definite, $p^T H(x^*, \lambda^*)p \geq 0$ for every p satisfying $J(x^*)p = 0$. So this point is optimal.

(b) *Repeat Part (a) with the objective replaced by $x_1^3 + x_2^3$.*

$$\nabla L = \begin{pmatrix} 3x_1^2 - \lambda \\ 3x_2^2 - \lambda \\ -x_1 - x_2 + 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solving this system gives $x^* = (1/2, 1/2)^T$, $\lambda^* = 3/4$. Since $H(x^*, \lambda^*)$ is positive definite, this point is optimal.

Exercise 5.3.* *Write a* MATLAB *function that will compute $c(x)$ and $J(x)$ for the constraint function*

$$c(x) = x_1 + x_2 - x_1x_2 - \frac{3}{2}.$$

Use your function to find $c(x)$ and $J(x)$ at $x = (.1 - .5)^T$, $x = (.5, -1)^T$ and $x = (1.18249728, -1.73976692)^T$. At each of these points, discuss the optimality of the constrained minimization problem:

$$\begin{aligned} &\underset{x \in \mathcal{R}^2}{\text{minimize}} \;\; e^{x_1}(4x_1^2 + 2x_2^2 + 4x_1x_2 + 2x_2 + 1) \\ &\text{subject to} \;\; x_1 + x_2 - x_1x_2 - \frac{3}{2} = 0. \end{aligned}$$

The first part of the exercise is similar to the previous MATLAB exercises. The discussion on optimality is similar to the exercise above.

Exercise 5.4.* *The m-file newbat.m, implementing a NEWton with Backtracking for the problem $F(x) = 0$, can be found on the class webpage.*

(a) *Starting at $x_0 = (2, \frac{1}{2})^T$, $\lambda_0 = 0$, use the implementation to solve the problem in Exercise 5.3.*

(b) *Repeat part (a), but change the constraint to $4x_1 - x_2 - 6 = 0$.*

(c) *Repeat part (b) but start at $x_0 = (1, -2)^T$.*

We want newbat.m to solve the system $F(x, \lambda) = 0$, where

$$F(x, \lambda) = \begin{pmatrix} g(x) - J(x)^T\lambda \\ c(x) \end{pmatrix},$$

i.e. you need to provide this F as well as its Jacobian –newbat.m will do the rest. See TA if you have further questions.