MATH 171B: Numerical Optimization: Nonlinear Problems

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Homework Assignment #2
Due (See Class Webpage for Due Date)

Our goal in this homework is gain a better understanding of Newton's method for solving nonlinear equations involving vector-valued functions of several real variable, to learn about the convergence properties of the method, and to understand how to implement the method on a computer. We will also explore backtracking (damping) in Newton's method.

The starred exercises are those that require the use of MATLAB. *You must do the* MATLAB *problems to get credit for the homework.*

Exercise 2.1.* Sketch $F(x) = 1/x - a$ for $a = 2$. Then derive a Newton iteration for computing the reciprocal of a positive real number a without performing division. Create a MATLAB m-file to implement the algorithm, and use it with $x_0 = 0.3$ to approximate e^{-1}, where $e = 2.7182818284$, by calculating x_1, x_2 and x_3. Repeat for $x_0 = 1.0$.

Exercise 2.2.* A completed m-file newton.m has been placed on the class webpage.

(a) Download this function from the class webpage for use in this homework, and read it carefully so that you understand how it works.

(b) Use newton.m to find a zero of the function

$$F(x) = \begin{pmatrix} x_1^2 + x_2^2 - 2 \\ (x_1 - \frac{1}{2})^2 + (x_2 - 1)^2 - \frac{9}{4} \end{pmatrix},$$

starting at the points $x_0 = (2,\, 3)^T$, $x_0 = (1,\, 3)^T$ and $x_0 = (1,\, 2 + 10^{-8})^T$. What rate of convergence do you observe? Comment on your results.

Exercise 2.3.* An *eigenvector* x of an $n \times n$ matrix A satisfies $Ax = \lambda x$ for some scalar λ. The scalar λ is known as the *eigenvalue* of A corresponding to the eigenvector x.

(a) If x is an eigenvector of A, show that βx is also an eigenvector. What is the associated eigenvalue? Hence show that the unit vector $x/\|x\|$ is an eigenvector of A.

(b) Define an iteration of Newton's method for the zero of the $n+1$ equations

$$(A - \lambda I)x = 0, \qquad x^T x = 1,$$

in the $n+1$ unknowns (x, λ). Use the m-file newton.m to find an eigenvector and eigenvalue for the matrix

$$A = \begin{pmatrix} 4 & 2 & 1 \\ 2 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \text{starting at} x_0 = \begin{pmatrix} \frac{1}{5} \\ -\frac{1}{5} \\ \frac{4}{5} \\ 1 \end{pmatrix}.$$

Exercise 2.4. One way to make Newton's method more robust for a given nonlinear equation $F(x) = 0$, where $F : \mathbb{R}^n \mapsto \mathbb{R}^n$, is to construct an associated minimization problem such that the minimum occurs at the solution to the original nonlinear equation. A standard choice for the function to minimize is:

$$f(x) = \frac{1}{2}\|F(x)\|^2 = \frac{1}{2}F(x)^T F(x).$$

This is a real-valued function of several variables, and we can form its derivatives, as long as $F(x)$ is differentiable.

(a) Show that the following is true:

$$\nabla f(x) = F'(x)^T F(x).$$

(b) Now, recall that a *direction of decrease* y at x for such a real-valued function satisfies

$$f(x + \lambda y) < f(x),$$

for some $\lambda > 0$. More over, a *descent direction* y at x always satisfies $y^T \nabla f(x) < 0$. Show that a descent direction is always a direction of decrease. (*Hint: Taylor series.*)

(c) Prove that the Newton direction $y = -F'(x)^{-1}F(x)$ at x is always a direction of decrease for $f(x)$ at x.
(*Hint: Show y is a descent direction and use (b).*)

(d) If you don't solve the Newton equations exactly (e.g., you are left with some residual $r = -F(x) - F'(x)\delta \neq 0$), how does this effect the situation?

Exercise 2.5. Consider a vector-valued function F. The back-tracking step length criterion discussed in Professor Gill's notes (pages 62-63) enforces *sufficient descrease* in the norm of the nonlinear function. It requires that the reduction in $\|F(x)\|$ is not worse than μ times the reduction in $\|L_k(x)\|$, i.e.,

$$\frac{\|F(x_k)\| - \|F(x_k + \alpha p_k)\|}{\|L_k(x_k)\| - \|L_k(x_k + \alpha p_k)\|} \geq \mu,$$

where μ is a pre-assigned constant such that $0 < \mu < 1$. Let p_k be the Newton step associated with a point x_k at which $F'(x_k)$ is nonsingular. Show that this condition on α_k is equivalent to the condition

$$\|F(x_k + \alpha_k p_k)\| \leq (1 - \alpha_k \mu)\|F(x_k)\|. \tag{2.1}$$

Hint: Read pages 59-63 of Professor Gill's notes!

Exercise 2.6.* Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable in $\mathbb{R}^n$. Write a MATLAB m-file that finds the zero of a function F using Newton's method with backtracking. I.e., you should modify the m-file `newton.m` from the class webpage so that it implements Algorithm 2.6.1 in Professor Gill's notes (page 63). Use $\mu = \frac{1}{4}$ to define the sufficient-decrease criterion, equation (2.1), in the backtracking algorithm. The step length α_k is chosen as the first member of the sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$, that satisfies (2.1). The algorithm should be terminated when either $\|F(x_k)\| \leq 10^{-8}$, or 50 iterations are performed. The backtracking "`while`" or "`for`" loop must include a test that will terminate the loop if it is executed more than 20 times (this will keep you for burning a lot of CPU time if something goes wrong...) At each iteration, print k, α_k and $\|F(x_k)\|$.

Use your m-file to find a zero the function

$$F(x) = \begin{pmatrix} e^{x_1}(4x_1^2 + 2x_2^2 + 4x_1x_2 + 6x_2 + 8x_1 + 1) \\ e^{x_1}(4x_2 + 4x_1 + 2) \end{pmatrix},$$

starting at $x_0 = (3,\ 0)^T$.